ONSAGER'S CONJECTURE ALMOST EVERYWHERE IN TIME

TRISTAN BUCKMASTER

Abstract. In recent work by P. Isett [15], and later by Buckmaster, De Lellis and Székelyhidi Jr. [2], iterative schemes where presented for constructing solutions belonging to the Hölder class $C^{1/5-\varepsilon}$ of the 3D incompressible Euler equations which do not conserve energy. The cited work is partially motivated by a conjecture of Lars Onsager in 1949 relating to the existence of $C^{1/3-\varepsilon}$ solutions to the Euler equations which dissipate energy. In this note we show how the later scheme can be adapted in order to prove the existence of non-trivial Hölder continuous solutions which for almost every time belong to the critical Onsager Hölder regularity $C^{1/3-\varepsilon}$ and have compact temporal support. The solutions constructed display characteristics reminiscent to the concept of intermittency found in literature related to highly turbulent flows.

0. Introduction

In what follows $\mathbb{T}^3$ denotes the 3-dimensional torus, i.e. $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Formally, we say (v, p) solves the *incompressible Euler equations* if

$$\begin{cases} \partial_t v + \operatorname{div} v \otimes v + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases} . \tag{1}$$

Suppose v is such a solution, then we define its *kinetic energy*, as

$$E(t) := \frac{1}{2} \int |v(x,t)|^2 \, dx.$$

A simple calculation applying integration by parts yields that for any classical solution of (1) the kinetic energy is in fact conserved in time. This formal calculation does not however hold for distributional solutions to Euler (cf. [19, 20, 5, 6, 21, 8]).

In fact in the context of 3-dimensional turbulence, flows *dissipating* energy in time have long been considered. A key postulate of Kolmogorov's K41 theory [16] is that for homogeneous, isotropic turbulence, the dissipation rate is non-vanishing in the inviscid limit. In particular, defining the *structure functions* for homogeneous, isotropic turbulence

$$S_p(\ell) := \left\langle \left[(v(x+\hat{\ell}) - v(x)) \cdot \frac{\hat{\ell}}{\ell} \right]^p \right\rangle,$$

where $\hat{\ell}$ denotes a spatial vector of length ℓ, Kolmogorov's famous four-fifths law can be stated as

$$S_3(\ell) = -\frac{4}{5}\varepsilon_d \ell, \tag{2}$$

where here ε_d denotes the mean energy dissipation per unit mass. More generally, Kolmogorov's scaling laws can be stated as

$$S_p(\ell) = C_p \varepsilon_d^{p/3} \ell^{p/3}, \tag{3}$$

for any positive integer p.

A well known consequence of the above scaling laws is the Kolmogorov spectrum, which postulates a scaling relation on the 'energy spectrum' of a turbulent flow (cf. [14, 12]). It was this observation that provided motivation for Onsager to conjecture in his famous note [18] on statistical hydrodynamics, the following dichotomy:

(a) Any weak solution v belonging to the Hölder space C^θ for $\theta > \frac{1}{3}$ conserves the energy.
(b) For any $\theta < \frac{1}{3}$ there exist weak solutions $v \in C^\theta$ which do not conserve the energy.

Part (a) of this conjecture has since been resolved: it was first considered by Eyink in [11] following Onsager's original calculations and later proven by Constantin, E and Titi in [4]. Subsequently, this later result was strengthened by showing that under weakened assumptions on v (in terms of Besov spaces) kinetic energy is conserved [10, 3].

Part (b) remains an open conjecture and is the subject of this note. The first constructions of non-conservative Hölder-continuous ($C^{1/10-\varepsilon}$) weak solutions appeared in work of De Lellis and Székelyhidi Jr. [7], which itself was based on their earlier seminal work [9] where continuous weak solutions were constructed. Furthermore, it was shown in the mentioned work that such solutions can be constructed obeying any prescribed smooth non-vanishing energy profile. In recent work [15], P. Isett introduced a number of new ideas in order to construct non-trivial $1/5 - \varepsilon$ Hölder-continuous weak solutions with compact temporal support. This construction was later improved by Buckmaster, De Lellis and Székelyhidi Jr. [2], following more closely the earlier work [9, 7], in order construct $1/5 - \varepsilon$ Hölder-continuous weak solution obeying a given energy profile.

In this note we give a proof of the following theorem.

Theorem 0.1. *There exists is a non-trivial continuous vector field* $v \in C^{1/5-\varepsilon}(\mathbb{T}^3 \times (-1,1), \mathbb{R}^3)$ *with compact support in time and a continuous scalar field* $p \in C^{2/5-2\varepsilon}(\mathbb{T}^3 \times (-1,1))$ *with the following properties:*

(i) *The pair* (v, p) *solves the incompressible Euler equations* (1) *in the sense of distributions.*

(ii) *There exists a set $\Omega \subset (-1,1)$ of Hausdorff dimension strictly less than 1 such that if $t \notin \Omega$ then $v(\cdot,t)$ is Hölder $C^{1/3-\varepsilon}$ continuous and p is Hölder $C^{2/3-2\varepsilon}$ continuous*[1].

Relation to intermittency. The theory of intermittency is born of an effort to explain the experimental and numerical evidence (e.g. [1]) of measurable discrepancies from the scaling laws (3) (cf. [13]). In this direction, Mandelbrot conjectured [17] that at the inviscid limit, turbulence concentrates (*in space*) on a fractal set of Hausdorff dimension strictly less than 3.

It is interesting to note that the solutions constructed in order to prove the above theorem have a fractal structure in *time*: namely, the set of times for which v is *not* Hölder $C^{1/3-\varepsilon}$ continuous is contained in a Cantor-like set with Hausdorff dimension strictly less than 1. Since the phenomena observed does not relate to the structure functions from which intermittency was originally postulated – being temporal in nature rather than spatial – it is clearly far-fetched to label such a phenomena as intermittency. Nevertheless, it is the opinion of the author that the parallels to the notion of intermittency remain of interest.

0.1. **Euler-Reynolds system and the convex integration scheme.** In order to prove Theorem 0.1 we construct an iteration scheme in the style of [2], which is itself based on the schemes presented in [9, 7]. At each step $q \in \mathbb{N}$ we construct a triple $(v_q, p_q, \mathring{R}_q)$ solving the Euler-Reynolds system (see Definition 2.1 in [9]):

$$\begin{cases} \partial_t v_q + \operatorname{div}(v_q \otimes v_q) + \nabla p_q = \operatorname{div} \mathring{R}_q \\ \operatorname{div} v_q = 0\,. \end{cases} \tag{4}$$

The initial triple $(v_0, p_0, \mathring{R}_0)$ will be non-trivial with compact support in time; all triples thereafter will be defined inductively as perturbations of the proceeding triples. The perturbation

$$w_q := v_q - v_{q-1},$$

will be composed of weakly interacting perturbed Beltrami flows (see Section 1) oscillating at *frequency* λ_q, defined in such a way to correct for the previous Reynolds error $\mathring{R}_{q-1}$.

[1]More precisely, the Hausdorff dimension d is such that $1 - d > C\varepsilon^2$ for some positive constant C.

In order to ensure convergence of the sequence v_q to a continuous weak $C^{1/5-\varepsilon}$ solution of Euler, we will require the following estimates to be satisfied

$$\|w_q\|_0 + \frac{1}{\lambda_q}\|\partial_t w_q\|_0 + \frac{1}{\lambda_q}\|w_q\|_1 \leq \lambda_q^{-1/5+\varepsilon_0} \tag{5}$$

$$\|p_q - p_{q-1}\|_0 + \frac{1}{\lambda_q}\|\partial_t(p_q - p_{q-1})\|_0 + \frac{1}{\lambda_q^2}\|p_q - p_{q-1}\|_2 \leq \lambda_q^{-2/5+2\varepsilon_0} \tag{6}$$

$$\left\|\mathring{R}_q\right\|_0 + \frac{1}{\lambda_q}\left\|\mathring{R}_q\right\|_1 \leq \lambda_{q+1}^{-2/5+2\varepsilon_0} \tag{7}$$

for some $\varepsilon_0 > 0$ strictly smaller than ε. Here and throughout the article, $\|\cdot\|_\beta$ for $\beta = m + \kappa$, $\beta \in \mathbb{N}$ and $\kappa \in [0, 1)$ will denote the usual *spatial* Hölder $C^{m,\kappa}$ norm. As a minor point of deviation from [2], we keep track of second order spatial derivative estimates of $p_q - p_{q-1}$, whereas in [2] first order estimates – which in the present work are implicit by interpolation – were sufficient. These second order estimates will be used in order to obtain slightly improved bounds on the Reynolds stress (see Section 5).

It is perhaps worth noting that aside from the second order estimate on $p_q - p_{q-1}$, up to a constant multiple, the above estimates are consistent with the estimates given in [2][2].

In order to ensure that our sequence convergences to a non-trivial solution, we will impose the addition requirement that

$$\sum_{q=1}^{\infty} \|w_q\|_0 < \frac{1}{2}\|v_0\|_0 , \tag{8}$$

for times $t \in [-1/8, 1/8]$.

The principle new idea of this work is that in addition to the estimates given above, we will keep track of sharper, time localized estimates. As a consequence of these sharper estimates, it can be shown that for any given time $t \in (-1, 1)$ outside a prescribed set Ω of Hausdorff dimension strictly less than 1, there exists a $N = N(t)$ such that

$$\|w_q\|_0 + \frac{1}{\lambda_q}\|\partial_t w_q\|_0 + \frac{1}{\lambda_q}\|w_q\|_1 \leq \lambda_q^{-1/3+\varepsilon_0} \tag{9}$$

$$\|p_q - p_{q-1}\|_0 + \frac{1}{\lambda_q}\|\partial_t(p_q - p_{q-1})\|_0 + \frac{1}{\lambda_q^2}\|p_q - p_{q-1}\|_2 \leq \lambda_q^{-2/3+2\varepsilon_0} \tag{10}$$

$$\left\|\mathring{R}_q\right\|_0 + \frac{1}{\lambda_q}\left\|\mathring{R}_q\right\|_1 \leq \lambda_{q+1}^{-2/3+2\varepsilon_0},\text{[3]} \tag{11}$$

for every $q \geq N$.

[2]In [2] the estimates corresponding to (5)-(7) are written in terms of a sequence of parameters δ_q which in the context of the present paper are defined to be $\delta_q := \lambda_q^{-2/5+2\varepsilon_0}$ (cf. Section 3 and Section 6).

[3]Here and throughout the paper we suppress the dependence on the time variable t.

0.2. **The main iteration proposition and the proof of Theorem 0.1.**

Proposition 0.2. *For every small $\varepsilon_0 > 0$, there exists an $\alpha > 1$, $d < 1$ and a sequence of parameters $\lambda_0, \lambda_1, \dots$ satisfying $1/2\lambda_0^{\alpha^q} < \lambda_q < 2\lambda_0^{\alpha^q}$ such that the following holds. A sequence of triples $(v_q, p_q, \mathring{R}_q)$ can be constructed with temporal support confined to $[-1/2, 1/2]$ solving (4) and satisfying the estimates (5-8). Moreover, for any $\delta > 0$, there exists an integer M such that if Ξ^M denotes the set of times t such that there exists a $q \geq M$ satisfying either*

$$\begin{aligned} \|w_q\|_0 + \frac{1}{\lambda_q}\|w_q\|_1 > \lambda_q^{-1/3+\varepsilon_0}, \quad \text{or} \\ \|p_q - p_{q-1}\|_0 + \frac{1}{\lambda_q}\|p_q - p_{q-1}\|_1 > \lambda_q^{-2/3+2\varepsilon_0}, \end{aligned} \tag{12}$$

then there exists a cover of Ξ^M consisting of a sequence of balls of radius r_i such that

$$\sum r_i^d < \delta. \tag{13}$$

Proof of Theorem 0.1. Fix $\varepsilon_0 = \varepsilon/2$ and let $(v_q, p_q, \mathring{R}_q)$ be a sequence as in Proposition 0.2. It follows then easily that (v_q, p_q) converge uniformly to a pair of continuous functions (v, p) satisfying (1), having compact temporal support. Moreover, by interpolating the inequalities (5) and (6) we obtain that v_q converges in $C^{1/5-\varepsilon}$ and p_q in $C^{2/5-2\varepsilon}$.

In order to prove (ii) we first fix $\delta > 0$ and let M and Ξ^M be as in Proposition 0.2. Hence by assumption if $t \notin \Xi^M$

$$\begin{aligned} \|w_q\|_0 + \frac{1}{\lambda_q}\|w_q\|_1 \leq \lambda_q^{-1/3+\varepsilon_0} \\ \|p_q - p_{q-1}\|_0 + \frac{1}{\lambda_q}\|p_q - p_{q-1}\|_1 \leq \lambda_q^{-2/3+2\varepsilon_0}, \end{aligned} \tag{14}$$

for all $q \geq M$. Thus interpolating the inequalities above we obtain that $v - v_M$ is bounded in $C^{1/3-\varepsilon}$ and $p - p_M$ in $C^{2/3-2\varepsilon}$. By (5) and (6), the pair (v_M, p_M) are C^1 bounded and thus it follows that v and p are bounded in $C^{1/3-\varepsilon}$ and $C^{2/3-2\varepsilon}$ respectively. Letting δ tend to zero we obtain our claim. □

0.3. **Plan of the paper.** After recalling in Section 1 some preliminary notation from the paper [9], in Section 2 we give the precise definition of the sequence of triples $(v_q, p_q, \mathring{R}_q)$. In Section 3 we list a number of inequalities that we will require on the various parameters of our scheme. The Sections 4 and 5 will focus on estimating, respectively, $w_{q+1} = v_{q+1} - v_q$, and $\mathring{R}_{q+1}$. These estimates are then collected in Section 6 where Proposition 0.2 will be finally proved. Throughout the entire article we will rely heavily on the arguments of [2] – in some sense the scheme presented here is a simple variant of that given in [2] – as such the present paper is intentionally structured in a similar manner to [2] in order to aide comparison.

0.4. **Acknowledgments.** I wish to thank Camillo De Lellis and László Székelyhidi Jr. for the enlightening discussions I had with them both. I would also like to thank Antoine Choffrut, Camillo De Lellis, Charles Doering and László Székelyhidi Jr. for their helpful comments and corrections regarding the manuscript.

This work is supported as part of the ERC Grant Agreement No. 277993.

1. Preliminaries

Throughout this paper we denote the 3×3 identity matrix by Id. In this section we state a number of results found in [9] which are fundamental to the present scheme as well its predecessors [9, 7, 2].

1.1. **Geometric preliminaries.** The following two results will form the cornerstone in which to construct the highly oscillating flows required by our scheme.

Proposition 1.1 (Beltrami flows). *Let $\bar\lambda \geq 1$ and let $A_k \in \mathbb{R}^3$ be such that*

$$A_k \cdot k = 0,\ |A_k| = \tfrac{1}{\sqrt{2}},\ A_{-k} = A_k$$

for $k \in \mathbb{Z}^3$ with $|k| = \bar\lambda$. Furthermore, let

$$B_k = A_k + i\frac{k}{|k|} \times A_k \in \mathbb{C}^3.$$

For any choice of $a_k \in \mathbb{C}$ with $\overline{a_k} = a_{-k}$ the vector field

$$W(\xi) = \sum_{|k|=\bar\lambda} a_k B_k e^{ik\cdot\xi} \tag{15}$$

is real-valued, divergence-free and satisfies

$$\operatorname{div}(W \otimes W) = \nabla \frac{|W|^2}{2}. \tag{16}$$

Furthermore

$$\langle W \otimes W \rangle = \fint_{\mathbb{T}^3} W \otimes W\, d\xi = \frac{1}{2} \sum_{|k|=\bar\lambda} |a_k|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \right). \tag{17}$$

Lemma 1.2 (Geometric Lemma). *For every $N \in \mathbb{N}$ we can choose $r_0 > 0$ and $\bar\lambda > 1$ with the following property. There exist pairwise disjoint subsets*

$$\Lambda_j \subset \{k \in \mathbb{Z}^3 : |k| = \bar\lambda\} \qquad j \in \{1, \dots, N\}$$

and smooth positive functions

$$\gamma_k^{(j)} \in C^\infty(B_{r_0}(\mathrm{Id})) \qquad j \in \{1, \dots, N\},\ k \in \Lambda_j,\ ^{4}$$

such that

(a) *$k \in \Lambda_j$ implies $-k \in \Lambda_j$ and $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$;*

[4]Here $B_{r_0}(\mathrm{Id})$ denotes the ball around Id of radius r_0 under the usual matrix operator norm $|A| := \max_{|v|=1} |Av|$.

(b) *For each* $R \in B_{r_0}(\mathrm{Id})$ *we have the identity*

$$R = \frac{1}{2}\sum_{k\in\Lambda_j}\left(\gamma_k^{(j)}(R)\right)^2\left(\mathrm{Id} - \frac{k}{|k|}\otimes\frac{k}{|k|}\right) \qquad \forall R \in B_{r_0}(\mathrm{Id})\,. \tag{18}$$

1.2. The operator $\mathcal{R}$.

The following operator will be used in order to deal the the Reynolds Stresses arising from our iteration scheme.

Definition 1.3. *Let* $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ *be a smooth vector field. We then define* $\mathcal{R}v$ *to be the matrix-valued periodic function*

$$\mathcal{R}v := \frac{1}{4}\left(\nabla\mathcal{P}u + (\nabla\mathcal{P}u)^T\right) + \frac{3}{4}\left(\nabla u + (\nabla u)^T\right) - \frac{1}{2}(\operatorname{div} u)\mathrm{Id},$$

where $u \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ *is the solution of*

$$\Delta u = v - \fint_{\mathbb{T}^3} v \text{ in } \mathbb{T}^3$$

with $\fint_{\mathbb{T}^3} u = 0$ *and* $\mathcal{P}$ *is the Leray projection onto divergence-free fields with zero average.*

Lemma 1.4 ($\mathcal{R} = \operatorname{div}^{-1}$)**.** *For any* $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ *we have*

(a) $\mathcal{R}v(x)$ *is a symmetric trace-free matrix for each* $x \in \mathbb{T}^3$*;*
(b) $\operatorname{div}\mathcal{R}v = v - \fint_{\mathbb{T}^3} v$.

2. The construction of the triples $(v_q, p_q, \mathring{R}_q)$

2.1. The initial triple $(v_0, p_0, \mathring{R}_0)$.

Let χ_0 be a smooth non-negative function, compactly supported on the interval $[-1/4, 1/4]$, bounded above by 1 and identically equal to 1 on $[-1/8, 1/8]$. We now set our initial velocity to be the divergence-free vector field

$$v_0(t,x) := \frac{1}{2}\lambda_0^{-\frac{1}{5}+\varepsilon_0}\chi_0(t)(\cos(\lambda_0 x_3), \sin(\lambda_0 x_3), 0),$$

where here we use the notation $x = (x_1, x_2, x_3)$. The initial pressure p_0 is then defined to be identically zero. Finally if we set

$$\mathring{R}_0 = \frac{1}{2}\lambda_0^{-\frac{6}{5}+\varepsilon_0}\chi_0'(t)\begin{pmatrix} 0 & 0 & \sin(\lambda_0 x_3) \\ 0 & 0 & -\cos(\lambda_0 x_3) \\ \sin(\lambda_0 x_3) & -\cos(\lambda_0 x_3) & 0 \end{pmatrix},$$

we obtain

$$\partial_t v_0 + \operatorname{div}(v_0 \otimes v_0) + \nabla p_0 = \operatorname{div}\mathring{R}_0.$$

Hence the triple $(v_0, p_0, \mathcal{R}_0)$ is a solution to the Euler-Reynolds system (4). Furthermore, it follows immediately that

$$\left\|\mathring{R}_0\right\|_0 + \frac{1}{\lambda_0}\left\|\mathring{R}_0\right\|_1 \le C\lambda_0^{-6/5+\epsilon_0}.$$

Thus if λ_0 is sufficiently large we obtain (5-7) for $q = 0$.

Remark. *The choice of initial triple* $(v_0, p_0, \mathring{R}_0)$ *is not of any great importance: any choice satisfying the conditions set out in Section 0.1 and is such that* $|v_0| \approx \lambda_0^{-1/5+\varepsilon_0}$ *for times* $t \in [-1/8, 1/8]$ *should suffice.*

2.2. **The inductive step.** The procedure of constructing $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ in terms of $(v_q, p_q, \mathring{R}_q)$ follows in the same spirit as that of the scheme outlined in [2] with a few minor modifications in order to satisfy the specific requirements of Proposition 0.2.

We will assume that λ_0 is chosen large enough such that

$$\sum_{j<q} \lambda_j^{2/3} \leq \lambda_q^{2/3}, \quad \sum_{j=0}^{\infty} \lambda_j^{-1/5+\varepsilon_0} \leq 1 \quad \text{and} \sum_{j=1}^{\infty} \lambda_j^{-1/5+\varepsilon_0} < \frac{\lambda_0^{-1/5+\varepsilon_0}}{4}. \tag{19}$$

Notice as a direct consequence (8) follows from (5) and the definition of v_0.

We fix a symmetric non-negative convolution kernel ψ with support confined to $[-1, 1]$.

With a slight abuse of notation, we will use $(v, p, \mathring{R})$ for $(v_q, p_q, \mathring{R}_q)$ and $(v_1, p_1, \mathring{R}_1)$ for $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$.

As was done in [2], we discretize time into intervals of size μ^{-1}. The parameter μ will be chosen explicitly as

$$\mu = \mu_{q+1} = \lambda_q^{1/2(1+\alpha)(-1/5+\varepsilon_0+1)}.$$

We will see in Section 6 that our choice of μ is consistent with the choice made in [2].

The choice of cut-off functions $\chi = \chi^{(q+1)}$ used in this article will differ slightly to that described in [2]. Specifically, we define χ to be a smooth function such that for a small parameter $\varepsilon_1 > 0$ (to be chosen later) χ satisfies the following conditions:

- The support of χ is contained in $\left(-\frac{1}{2} - \frac{\lambda_{q+1}^{-\varepsilon_1}}{4}, \frac{1}{2} + \frac{\lambda_{q+1}^{-\varepsilon_1}}{4}\right)$.
- In the range $\left(-\frac{1}{2} + \frac{\lambda_{q+1}^{-\varepsilon_1}}{4}, \frac{1}{2} - \frac{\lambda_{q+1}^{-\varepsilon_1}}{4}\right)$ we have $\chi \equiv 1$.
- The sequence $\{\chi^2(x - l)\}_{l \in \mathbb{Z}}$ forms a partition of unity of $\mathbb{R}$, i.e.
$$\sum_{l \in \mathbb{Z}} \chi^2(x - l) = 1.$$
- For $N \geq 0$ we have the estimates
$$\left|\partial_x^N \chi\right| \leq C \lambda_{q+1}^{N\varepsilon_1},$$
where the constant C depends only on N – in particular it is independent of q.

Having defined χ, we adopt the notation $\chi_l(t) := \chi(\mu t - l)$. The fundamental difference to choice of χ in [2] is the extra factor $\lambda_{q+1}^{-\varepsilon_1}$ appearing in the definition. A consequence of this modification is that the Lebesgue measure of the set

$$\bigcap_{q=1}^{\infty}\bigcup_{q'=q}^{\infty}\bigcup_{l}\operatorname{support}(\chi'_{q',l})$$

is zero. We will see this will provide us with a key ingredient in order to prove a.e. in time $C^{1/3-\varepsilon}$ convergence of the sequence v_q.

For each l define the amplitude function

$$\rho_l = 2r_0^{-1}\left\|\mathring{R}(\cdot, l\mu^{-1})\right\|_0 .$$

The function ρ_l will play a similar role to the ρ_l found in [2]: the comparatively simpler definition above reflects the fact that we are only interested in correcting for the Reynolds error and are not attempting to construct a solution to Euler with a prescribed energy as was done in [2].

Keeping in mind the new choices of ρ_l and χ_l, the construction of $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$ proceeds in exactly the same manner as that described in [2], with the minor exception that the mollification parameter ℓ will be chosen explicitly to be

$$\ell = \lambda_{q+1}^{-1+\varepsilon_2},$$

where $\varepsilon_2 > 0$ is a small parameter to be chosen later.

For completeness we recall the remaining steps required to construct the triple $(v_{q+1}, p_{q+1}, \mathring{R}_{q+1})$.

Having set

$$R_l(x) := \rho_l \mathrm{Id} - \mathring{R}(x, l\mu^{-1})$$

and $v_\ell = v * \psi_\ell$, we define $R_{\ell,l}$ to be the unique solution to the following transport equation

$$\begin{cases} \partial_t R_{\ell,l} + v_\ell \cdot \nabla R_{\ell,l} = 0 \\ \\ R_{\ell,l}(\frac{l}{\mu}, \cdot) = R_l * \psi_\ell \,. \end{cases}$$

For every integer $l \in [-\mu, \mu]$, we let $\Phi_l : \mathbb{R}^3 \times (-1, 1) \to \mathbb{R}^3$ be the solution of

$$\begin{cases} \partial_t \Phi_l + v_\ell \cdot \nabla \Phi_l = 0 \\ \\ \Phi_l(x, l\mu^{-1}) = x. \end{cases}$$

Applying Lemma 1.2 with $N = 2$, we denote by Λ^e and Λ^o the corresponding families of frequencies in $\mathbb{Z}^3$ and set $\Lambda := \Lambda^o + \Lambda^e$. For each $k \in \Lambda$ and each $l \in \mathbb{Z} \cap [0, \mu]$ we then define

$$a_{kl}(x,t) := \sqrt{\rho_l}\gamma_k\left(\frac{R_{\ell,l}(x,t)}{\rho_l}\right),$$
$$w_{kl}(x,t) := a_{kl}(x,t)\, B_k e^{i\lambda_{q+1}k\cdot\Phi_l(x,t)}.$$

The perturbation w is then defined as the sum of a "principal part" and a "corrector". The "principal part" being the map

$$w_o(x,t) := \sum_{l \text{ odd}, k\in\Lambda^o} \chi_l(t) w_{kl}(x,t) + \sum_{l \text{ even}, k\in\Lambda^e} \chi_l(t) w_{kl}(x,t)\,.$$

The "corrector" w_c is then defined in such a way that the sum $w = w_o + w_c$ is divergence free:

$$w_c = \sum_{kl} \chi_l \Big(\frac{i}{\lambda_{q+1}} \nabla a_{kl} - a_{kl}(D\Phi_l - \mathrm{Id})k \Big) \times \frac{k \times B_k}{|k|^2} e^{i\lambda_{q+1} k\cdot\Phi_l}.$$

The new pressure is defined as

$$p_1 = p - \frac{|w_o|^2}{2} - \frac{1}{3}|w_c|^2 - \frac{2}{3}\langle w_o, w_c\rangle - \frac{2}{3}\langle v - v_\ell, w\rangle\,.$$

and finally we set $\mathring{R}_1 = R^0 + R^1 + R^2 + R^3 + R^4 + R^5$, where

$$R^0 = \mathcal{R}\left(\partial_t w + v_\ell\cdot\nabla w + w\cdot\nabla v_\ell\right) \tag{20}$$

$$R^1 = \mathcal{R}\mathrm{div}\Big(w_o \otimes w_o - \sum_l \chi_l^2 R_{\ell,l} - \tfrac{|w_o|^2}{2}\mathrm{Id}\Big) \tag{21}$$

$$R^2 = w_o\otimes w_c + w_c\otimes w_o + w_c\otimes w_c - \tfrac{|w_c|^2 + 2\langle w_o, w_c\rangle}{3}\mathrm{Id} \tag{22}$$

$$R^3 = w\otimes(v - v_\ell) + (v - v_\ell)\otimes w - \tfrac{2\langle (v-v_\ell), w\rangle}{3}\mathrm{Id} \tag{23}$$

$$R^4 = \mathring{R} - \mathring{R} * \psi_\ell \tag{24}$$

$$R^5 = \sum_l \chi_l^2 (\mathring{R}_{\ell,l} + \mathring{R} * \psi_\ell)\,. \tag{25}$$

2.3. **Compact support in time.** By construction it follows that if the triple $(v,p,\mathring{R})$ is supported in the time interval $[T,T']$ then $(v_1,p_1,\mathring{R}_1)$ is supported in the time interval $[T-\mu^{-1}, T'+\mu^{-1}]$. Therefore since $(v_0,p_0,\mathring{R}_0)$ is supported in the time interval $[-1/4, 1/4]$ it follows by induction that the triple $(v_q,p_q,\mathring{R}_q)$ is supported in the time interval

$$[-1/4 - \sum_{j=0}^{q-1} \lambda_j^{-1/2(1+\alpha)(-1/5+\varepsilon_0+1)}, 1/4 + \sum_{j=0}^{q-1} \lambda_j^{1/2(1+\alpha)(-1/5+\varepsilon_0+1)}] \subset [-1/2, 1/2],$$

assuming λ_0 is chosen to be appropriately large depending on the choice of α and ε_0.

3. Ordering of parameters

In order to better aid comparison to arguments of [2], we introduce a sequence of *strictly decreasing* parameters $\delta_q < 1$. In Section 6 we will provide an explicit definition of δ_q, but for now we restrict ourselves to specifying a number of inequalities that δ_q will need to satisfy. Analogously to [2] we will assume the following estimates

$$\|v_q\|_1 \leq \delta_q^{1/2}\lambda_q \tag{26}$$

$$\|p_q\|_2 \leq \delta_q\lambda_q^2 \tag{27}$$

$$\left\|\mathring{R}_q\right\|_0 + \frac{1}{\lambda_q}\left\|\mathring{R}_q\right\|_1 \leq \frac{1}{C_0}\delta_{q+1} \tag{28}$$

$$\left\|\partial_t + v\cdot\nabla\mathring{R}_q\right\|_0 \leq \delta_{q+1}\delta_q^{1/2}\lambda_q, \tag{29}$$

where $C_0 > 1$ is a large number to be specified in the next section.

Furthermore we will assume in addition that the following parameter inequalities are satisfied

$$\begin{gathered}\sum_{j<q}\delta_j\lambda_j \leq \delta_q\lambda_q, \qquad \frac{\delta_q^{1/2}\lambda_q\ell}{\delta_{q+1}^{1/2}} \leq 1,\\ \frac{\delta_q^{1/2}\lambda_q}{\mu} + \frac{1}{\ell\lambda_{q+1}} \leq \lambda_{q+1}^{-\beta} \quad \text{and} \quad \frac{1}{\lambda_{q+1}} \leq \frac{\delta_{q+1}^{1/2}}{\mu}.\end{gathered} \tag{30}$$

The sequence δ_q will be applied in the context of proving $1/5 - \varepsilon$ convergence of the velocities v_q; however note that unlike the case in [2], the sequence does not appear explicitly in the definition of the triples $(v_q, p_q, \mathring{R}_q)$.

In order to prove a.e. time $1/3 - \varepsilon$ convergence, we will require localized estimates (in time). To this aim we fix a time $t_0 \in (-1, 1)$ and set l_q to be the unique integer such that $\mu_q t_0 \in [-1/2 + l_q, 1/2 + l_q)$. We now introduce a new sequence of strictly decreasing parameters $\delta_{q,t_0} \leq \delta_q$ such that for a given time t satisfying $|\mu_{q+1}t - l_{q+1}| < 1$ we have the following estimates

$$\|v_q\|_1 \leq \delta_{q,t_0}^{1/2}\lambda_q \tag{31}$$

$$\|p_q\|_2 \leq \delta_{q,t_0}\lambda_q^2 \tag{32}$$

$$\left\|\mathring{R}_q\right\|_0 + \frac{1}{\lambda_q}\left\|\mathring{R}_q\right\|_1 \leq \frac{1}{C_0}\delta_{q+1,t_0} \tag{33}$$

$$\left\|\partial_t + v\cdot\nabla\mathring{R}_q\right\|_0 \leq \delta_{q+1,t_0}\delta_{q,t_0}^{1/2}\lambda_q. \tag{34}$$

Analogously to (30) we assume the following inequalities are satisfied

$$\sum_{j<q}\delta_{j,t_0}\lambda_j \leq \delta_{q,t_0}\lambda_q, \quad \frac{\delta_{q,t_0}^{1/2}\lambda_q\ell}{\delta_{q+1,t_0}^{1/2}} \leq 1, \quad \text{and} \quad \frac{\delta_{q,t_0}^{1/2}\lambda_q}{\mu} + \frac{1}{\ell\lambda_{q+1}} \leq \lambda_{q+1}^{-\beta}. \tag{35}$$

The last inequality being a trivial consequence of (30) and the inequality $\delta_{q,t_0} \leq \delta_q$. Observe that we *do not* assume a condition akin to the last inequality of (30). This remark is worth keeping in mind as we will apply the arguments of [2] extensively, where such a condition was present. Luckily, this condition is only really required at one specific point in the paper: the estimation of

$$\left\|\partial_t \mathring{R}_1 + v_q \cdot \nabla \mathring{R}_1\right\|_0 .$$

This condition was also used in a few isolated cases in [2] in order to simplify a number of terms arising from estimates, however this was primarily done for aesthetic reasons.

4. Estimates on the perturbation

In order to bound the perturbation, we apply nearly identical arguments used in Section 3 of [2].

We recall the following notation from [2]

$$\phi_{kl}(x,t) := e^{i\lambda_{q+1}k\cdot[\Phi_l(x,t)-x]},$$
$$L_{kl} := a_{kl}B_k + \Big(\frac{i}{\lambda_{q+1}}\nabla a_{kl} - a_{kl}(D\Phi_l - \mathrm{Id})k\Big) \times \frac{k\times B_k}{|k|^2}.$$

The perturbation w can then be written as

$$w = \sum_{kl} \chi_l\, L_{kl}\, \phi_{kl}\, e^{i\lambda_{q+1}k\cdot x} = \sum_{kl} \chi_l\, L_{kl}\, e^{i\lambda_{q+1}k\cdot \Phi_l} .$$

For reference we note that as a consequence of (5), (6), (19) and (32) we have

$$\left\|v_q\right\|_0 \leq 1 \tag{36}$$
$$\left\|p_q\right\|_1 \leq C\delta_{q,t_0}\lambda_q \tag{37}$$

where the last inequality follows by interpolation.

We also recall that as a consequence of simple convolution inequalities together with the inequalities (31) we have for a fixed t_0, $N \geq 1$ and times t satisfying $|\mu_{q+1}t - l_{q+1}| < 1$

$$\left\|v_\ell\right\|_N \leq \delta_{q,t_o}^{1/2}\lambda_q \ell^{-N+1}. \tag{38}$$

With this notation we now present a minor variant of Lemma 3.1 from [2].

Lemma 4.1. *Fix a time $t_0 \in (-1,1)$ and let l_{q+1} be as before, i.e. the unique integer such that $t_0 \in [-1/2 + l_{q+1}, 1/2 + l_{q+1})$ Assuming the series of inequalities listed in Section 3 hold then we have the following estimates. For t such that $|\mu t - l| < 1$ where $l \in \{l_{q+1}-1, l_{q+1}, l_{q+1}+1\}$ we have*

$$\left\|D\Phi_l\right\|_0 \leq C \tag{39}$$
$$\left\|D\Phi_l - \mathrm{Id}\right\|_0 \leq C\frac{\delta_{q,t_0}^{1/2}\lambda_q}{\mu} \tag{40}$$
$$\left\|D\Phi_l\right\|_N \leq C\frac{\delta_{q,t_0}^{1/2}\lambda_q}{\mu\ell^N}, \qquad N \geq 1 \tag{41}$$

Moreover,

$$\|a_{kl}\|_0 + \|L_{kl}\|_0 \le C\delta_{q+1,t_0}^{1/2} \tag{42}$$

$$\|a_{kl}\|_N \le C\delta_{q+1,t_0}^{1/2}\lambda_q \ell^{1-N}, \qquad N \ge 1 \tag{43}$$

$$\|L_{kl}\|_N \le C\delta_{q+1,t_0}^{1/2}\ell^{-N}, \qquad N \ge 1 \tag{44}$$

$$\|\phi_{kl}\|_N \le C\lambda_{q+1}\frac{\delta_{q,t_0}^{1/2}\lambda_q}{\mu\ell^{N-1}} + C\left(\frac{\delta_{q,t_0}^{1/2}\lambda_q\lambda_{q+1}}{\mu}\right)^N$$

$$\le C\lambda_{q+1}^{N(1-\beta)} \qquad N \ge 1. \tag{45}$$

Consequently, for any $N \ge 0$

$$\|w_c\|_N \le C\delta_{q+1,t_0}^{1/2}\left(\frac{\lambda_q}{\lambda_{q+1}} + \frac{\delta_{q,t_0}^{1/2}\lambda_q}{\mu}\right)\lambda_{q+1}^N \tag{46}$$

$$\le C\delta_{q+1}^{1/2}\frac{\delta_q^{1/2}\lambda_q}{\mu}\lambda_{q+1}^N, \tag{47}$$

$$\|w_o\|_N \le C\delta_{q+1,t_0}^{1/2}\lambda_{q+1}^N \tag{48}$$

$$\le C\delta_{q+1}^{1/2}\lambda_{q+1}^N. \tag{49}$$

The constants appearing in the above estimates depend only on N *and* C_0*. In particular for a fixed* N*, the constants appearing in* (42)-(44) *and* (46)-(49) *can be made arbitrarily small by taking* C_0 *to be sufficiently large. Furthermore, the weaker estimates* (47) *and* (49) *hold uniformly in time.*

The proof of the above lemma follows from essentially exactly the same arguments to those given in the proof of Lemma 3.1 from [2] – making use of our new sequence of parameters δ_{q,t_0}. The only minor point of departure from [2] is the appearance of the term $\frac{\lambda_q}{\lambda_{q+1}}$ in (46). From the definition of w_c we have

$$\|w_c\|_N \le C\sum_{kl}\chi_l\left(\frac{1}{\lambda_{q+1}}\|a_{kl}\|_{N+1} + \|a_{kl}\|_0\|D\Phi_l - \mathrm{Id}\|_N + \|a_{kl}\|_N\|D\Phi_l - \mathrm{Id}\|_0\right)$$
$$+ C\|w_c\|_0\sum_l \chi_l\left(\lambda_{q+1}^N\|D\Phi_l\|_0^N + \lambda_{q+1}\|D\Phi_l\|_{N-1}\right).$$

Hence applying (39)-(43) and applying the inequalities from Section 3 we obtain (46).

We now present a variant of Lemma 3.2 from [2]. We recall from [2] the notation for the *material derivative*: $D_t := \partial_t + v_\ell \cdot \nabla$.

Lemma 4.2. *Under the assumptions of Lemma 4.1 we have*

$$\|D_t v_\ell\|_N \leq C\delta_{q,t_0}\lambda_q(1+\lambda_q\ell^{1-N}) + C\delta_{q+1,t_0}\lambda_q\ell^{-N}\,, \tag{50}$$
$$\leq C\delta_{q,t_0}\lambda_q\ell^{-N} \tag{51}$$
$$\|D_t L_{kl}\|_N \leq C\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q\ell^{-N}\,, \tag{52}$$
$$\|D_t^2 L_{kl}\|_N \leq C\delta_{q+1,t_0}^{1/2}\lambda_q\ell^{-N}(\delta_{q,t_0}\lambda_q + \delta_{q+1,t_0}\ell^{-1})\,, \tag{53}$$
$$\leq C\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}\lambda_q\ell^{-N-1} \tag{54}$$

Consequently for t in the range $|t\mu - l_{q+1}| \leq 1/2(1-\lambda_{q+1}^{-\varepsilon_1})$ we have

$$\|D_t w_c\|_N \leq C\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q\lambda_{q+1}^N\,, \tag{55}$$
$$\|D_t w_o\|_N \equiv 0\,. \tag{56}$$

Moreover we have the following estimates which are valid uniformly in time

$$\|D_t w_c\|_N \leq C\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q\lambda_{q+1}^{N+\varepsilon_1}\,, \tag{57}$$
$$\|D_t w_o\|_N \leq C\delta_{q+1}^{1/2}\mu\lambda_{q+1}^{N+\varepsilon_1}\,. \tag{58}$$

Again, we note that the constants C depend only on our choice of C_0: in particular, the constants appearing in (52)-(58) *can be made arbitrarily small by taking C_0 sufficiently large.*

Proof. First note that (52), (57) and (58) follow by exactly the same arguments as those given in Lemma 3.2 of [2]. However in contrast to [2], time derivatives falling on χ_l for some l pick up an additional factor of $\lambda_{q+1}^{\varepsilon_1}$, which explains this additional factor appearing in (57) and (58).

In order to prove (51), we note that by the arguments of [2] we obtain that

$$\|D_t v_\ell\|_N \leq \|\nabla p * \psi_\ell\|_N + \|\operatorname{div}\mathring{R} * \psi_\ell\|_N + C\lambda_q^2\ell^{1-N}\delta_{q,t_0}$$

Then from the estimates on p, $\mathring{R}$ together with standard convolution estimates we obtain (50).

We now consider the estimate (53).

$$D_t^2 L_{kl} = \Big(-\frac{i}{\lambda_{q+1}}(D_t D v_\ell)^T\nabla a_{kl} + \frac{i}{\lambda_{q+1}}Dv_\ell^T Dv_\ell^T\nabla a_{kl} + \\ - a_{kl}D\Phi_l Dv_\ell Dv_\ell k + a_{kl}D\Phi_l D_t Dv_\ell k\Big) \times \frac{k\times B_k}{|k|^2}.$$

Note that $D_t Dv_\ell = DD_t v_\ell - Dv_\ell Dv_\ell$, so that

$$\|D_t Dv_\ell\|_N \leq \|D_t v_\ell\|_{N+1} + C\|Dv_\ell\|_N\|Dv_\ell\|_0$$
$$\leq C(\delta_{q,t_0}\lambda_q^2\ell^{-N} + \delta_{q+1,t_0}\lambda_q\ell^{-N-1})\,(1+\lambda_q\ell)$$
$$\leq C\delta_{q,t_0}\lambda_q^2\ell^{-N} + C\delta_{q+1,t_0}\lambda_q\ell^{-N-1}.$$

Hence utilizing the estimates in Lemma 4.1 we obtain

$$\begin{aligned}\|D_t^2 L_{kl}\|_N &\le C\delta_{q+1,t_0}^{1/2}\lambda_q\ell^{-N}\left(\delta_{q,t_0}\lambda_q+\frac{\delta_{q+1,t_0}}{\ell}\right)\left(1+\frac{\lambda_q}{\lambda_{q+1}}+\frac{\delta_{q,t_0}^{1/2}\lambda_q}{\mu}\right)\\ &\quad\cdot(1+(\lambda_q\ell)^3)\\ &\le C\delta_{q+1,t_0}^{1/2}\lambda_q\ell^{-N}(\delta_{q,t_0}\lambda_q+\delta_{q+1,t_0}\ell^{-1}).\end{aligned}$$

Thus we obtain (53). The estimate (55) follows also as a trivial consequence. □

We note in passing that keeping track of the second derivative of the pressure is critical in obtaining the sharper estimates (50) and (53).

5. Estimates on the Reynolds stress

In this section we describe the estimates on Reynolds stress which follow by applying the arguments of Section 5 of [2] to the present scheme.

Let $\epsilon > 0$ be such that $\varepsilon_1 \le \epsilon/2$. Then by applying nearly identical arguments to that of Proposition 5.1 of [2] we obtain the following two estimates:

$$\|\mathring{R}_1\|_0+\frac{1}{\lambda_{q+1}}\|\mathring{R}_1\|_1+\frac{1}{\mu}\|D_t\mathring{R}_1\|_0\le C\left(\frac{\delta_{q+1}^{1/2}\mu}{\lambda_{q+1}^{1-\epsilon}}+\frac{\delta_{q+1}\delta_q^{1/2}\lambda_q\lambda_{q+1}^{\epsilon}}{\mu}+\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q\ell\right), \tag{59}$$

$$\|\partial_t\mathring{R}_1+v_1\cdot\nabla\mathring{R}_1\|_0\le C\delta_{q+1}^{1/2}\lambda_{q+1}\left(\frac{\delta_{q+1}^{1/2}\mu}{\lambda_{q+1}^{1-\epsilon}}+\frac{\delta_{q+1}\delta_q^{1/2}\lambda_q\lambda_{q+1}^{\epsilon}}{\mu}+\delta_{q+1}^{1/2}\delta_q^{1/2}\lambda_q\ell\right). \tag{60}$$

The careful reader will note that unlike [2], no term of the type

$$\frac{\delta_{q+1}^{1/2}\delta_q\lambda_q}{\lambda_{q+1}^{1-\varepsilon}\mu\ell}, \tag{61}$$

appears within the brackets of the right hand side of (59) and (60). This is related to the fact that in [2] the authors did not keep track of second derivatives of the pressure. We come back to this point at the end of this section.

A second point of difference to [2] is that in the present scheme derivatives falling on χ_l pick up an extra factor of $\lambda_{q+1}^{\varepsilon_1}$. However as we assumed $\varepsilon_1 \le \epsilon/2$, we may simply apply the arguments of Proposition 5.1 of [2] with ϵ replaced with $\epsilon/2$ in order to absorb this extra error.

Localizing in time by fixing a time t_0, we now consider the case that t lies in the range $|t\mu - l_{q+1}| \le 1/2 - \lambda_{q+1}^{-\varepsilon_1}$ where as before l_{q+1} is the unique integer such that $\mu t_0 \in [-1/2 + l_{q+1}, 1/2 + l_{q+1})$. The following estimates will be in terms our new sequence of parameters δ_{q,t_0}. Again, the arguments will be a minor variation to those found in Proposition 5.1 of [2], the key differences being:

(A) Since for all l we have χ_l' is identically zero in this range, no positive powers of μ will appear as a consequence of differentiating in time.

(B) As previously mentioned in Section 3, in contrast to the case in [2] we *do not* have an estimate of the type

$$\frac{1}{\lambda_{q+1}} \leq \frac{\delta_{q+1,t_0}^{1/2}}{\mu} \tag{62}$$

at our disposal.

(C) In many of the material derivative estimates in [2] the estimate $\delta_q^{1/2}\lambda_q \leq \mu$ was used in order to simplify terms: we will avoid employing such an estimate, although in its place we will often use the estimate $\delta_{q,t_0}^{1/2}\lambda_q \leq \delta_{q+1,t_0}^{1/2}\lambda_{q+1,t_0}$.

Proposition 5.1. *Fix t in the range $|t\mu - l_{q+1}| < 1/2(1-\lambda_{q+1}^{-\varepsilon_1})$. For any choice of small positive numbers ϵ, β there is a constant C such that, if δ_q, δ_{q+1}, μ, λ_{q+1} and ℓ satisfy the conditions in Section 3, then we have*

$$\|R^0\|_0 + \frac{1}{\lambda_{q+1}}\|R^0\|_1 + \frac{1}{\delta_{q+1,t_0}^{1/2}\lambda_{q+1}}\|D_tR^0\|_0 \leq C\frac{\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q}{\lambda_{q+1}^{1-\epsilon}} \tag{63}$$

$$\|R^1\|_0 + \frac{1}{\lambda_{q+1}}\|R^1\|_1 + \frac{1}{\delta_{q+1,t_0}^{1/2}\lambda_{q+1}}\|D_tR^1\|_0 \leq C\frac{\delta_{q+1,t_0}\delta_{q,t_0}^{1/2}\lambda_q\lambda_{q+1}^{\epsilon}}{\mu} + C\frac{\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q}{\lambda_{q+1}^{1-\epsilon}} \tag{64}$$

$$\|R^2\|_0 + \frac{1}{\lambda_{q+1}}\|R^2\|_1 + \frac{1}{\delta_{q+1,t_0}^{1/2}\lambda_{q+1}}\|D_tR^2\|_0 \leq C\frac{\delta_{q+1,t_0}\delta_{q,t_0}^{1/2}\lambda_q\lambda_{q+1}^{\epsilon}}{\mu} + C\frac{\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q}{\lambda_{q+1}^{1-\epsilon}} \tag{65}$$

$$\|R^3\|_0 + \frac{1}{\lambda_{q+1}}\|R^3\|_1 + \frac{1}{\delta_{q+1,t_0}^{1/2}\lambda_{q+1}}\|D_tR^3\|_0 \leq C\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q\ell \tag{66}$$

$$\|R^4\|_0 + \frac{1}{\lambda_{q+1}}\|R^4\|_1 + \frac{1}{\delta_{q+1,t_0}^{1/2}\lambda_{q+1}}\|D_tR^4\|_0 \leq C\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q\ell \tag{67}$$

$$\|R^5\|_0 + \frac{1}{\lambda_{q+1}}\|R^5\|_1 + \frac{1}{\delta_{q+1,t_0}^{1/2}\lambda_{q+1}}\|D_tR^5\|_0 \leq C\frac{\delta_{q+1,t_0}\delta_{q,t_0}^{1/2}\lambda_q}{\mu} +$$

$$C\frac{\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q}{\lambda_{q+1}^{1-\epsilon}}\,. \tag{68}$$

Thus

$$\|\mathring{R}_1\|_0+\frac{1}{\lambda_{q+1}}\|\mathring{R}_1\|_1+\frac{1}{\delta_{q+1,t_0}^{1/2}\lambda_q}\|D_t\mathring{R}_0\|_0\leq$$
$$C\left(\frac{\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q}{\lambda_{q+1}^{1-\epsilon}}+\frac{\delta_{q+1,t_0}\delta_{q,t_0}^{1/2}\lambda_q\lambda_{q+1}^{\epsilon}}{\mu}+\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q\ell\right)\,, \tag{69}$$

$$\|\partial_t\mathring{R}_1+v_1\cdot\nabla\mathring{R}_1\|_0\leq$$
$$C\delta_{q+1,t_0}^{1/2}\lambda_{q+1}\left(\frac{\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q}{\lambda_{q+1}^{1-\epsilon}}+\frac{\delta_{q+1,t_0}\delta_{q,t_0}^{1/2}\lambda_q\lambda_{q+1}^{\epsilon}}{\mu}+\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q\ell\right)\,. \tag{70}$$

Proof. Keeping in mind the observations (A), (B) and (C) above, the proof of (65)-(68) follows by applying nearly identical arguments to that found in Proposition 5.1 of [2]. The estimate (69) easily follows as a consequence of (63)-(68), and (70) follows from (69) together with the observation

$$\|\partial_t\mathring{R}_1+v_1\cdot\nabla\mathring{R}_1\|_0\leq\|D_t\mathring{R}_1\|_0+\left(\|v-v_\ell\|_0+\|w\|_0\right)\|\mathring{R}_1\|_1\,.$$

Therefore we will restrict ourselves to proving the estimates (63) and (64). For reasons of brevity, in what follows we adopt the abuse of notation $l_1=l_{q+1}$.

Estimates on R^0. Recall that in [2] that a key ingredient to bounding R^0 involved bounding the terms Ω_{kl} such that

$$\partial_t w+v_\ell\cdot\nabla w+w\cdot\nabla v_\ell=\sum_{kl}\Omega_{kl}e^{i\lambda_{q+1}k\cdot x}\,,$$

that is

$$\Omega_{kl}:=\left(\chi_l'L_{kl}+\chi_lD_tL_{kl}+\chi_lL_{kl}\cdot\nabla v_\ell\right)e^{ik\cdot\Phi_l}\,.$$

and the terms Ω_{kl}' such that

$$D_t\left(\partial_t w+v_\ell\cdot\nabla w+w\cdot\nabla v_\ell\right):=\sum_k\Omega_{kl}'e^{i\lambda_{q+1}k\cdot x}, \tag{71}$$

that is

$$\Omega_{kl}':=\Big(\partial_t^2\chi_lL_{kl}+2\partial_t\chi_lD_tL_{kl}+\chi_lD_t^2L_{kl}+$$
$$+\partial_t\chi_lL_{kl}\cdot\nabla v_\ell+\chi_lD_tL_{kl}\cdot\nabla v_\ell+\chi_lL_{kl}\cdot\nabla D_tv_\ell-\chi_lL_{kl}\cdot\nabla v_\ell\cdot\nabla v_\ell\Big)e^{ik\cdot\Phi_l}. \tag{72}$$

Precisely, it was shown in [2] that

$$\|R_0\|_0 \leq \sum_{kl} \left(\lambda_{q+1}^{\varepsilon-1} \|\Omega_{kl}\|_0 + \lambda_{q+1}^{-N+\varepsilon} \|\Omega_{kl}\|_N + \lambda_{q+1}^{-N} \|\Omega_{kl}\|_{N+\varepsilon} \right), \tag{73}$$

$$\begin{aligned}\|R_0\|_1 \leq& \lambda_{q+1} \sum_{kl} \left(\lambda_{q+1}^{\varepsilon-1} \|\Omega_{kl}\|_0 + \lambda_{q+1}^{-N+\varepsilon} \|\Omega_{kl}\|_N + \lambda_{q+1}^{-N} \|\Omega_{kl}\|_{N+\varepsilon} \right) + \\ & \sum_{k} \left(\lambda_{q+1}^{\varepsilon-1} \|\Omega_{kl}\|_1 + \lambda_{q+1}^{-N+\varepsilon} \|\Omega_{kl}\|_{N+1} + \lambda_{q+1}^{-N} \|\Omega_{kl}\|_{N+1+\varepsilon} \right)\end{aligned} \tag{74}$$

and

$$\begin{aligned}\|D_t R_0\|_0 \leq& C \sum_{kl} \Bigg[\frac{\|\Omega'_{kl}\|_0}{\lambda_{q+1}^{1-\varepsilon}} + \frac{\|\Omega'_{kl}\|_N}{\lambda_{q+1}^{N-\varepsilon}} + \frac{\|\Omega'_{kl}\|_{N+\varepsilon}}{\lambda_{q+1}^{N}} \\ & + C \frac{\|\Omega_{kl}\|_{N+1+\varepsilon} \|v_\ell\|_{2+\varepsilon} + \|\Omega_{kl}\|_{3+\varepsilon} \|v_\ell\|_{N+\varepsilon}}{\lambda_{q+1}^{N}} \\ & + C \frac{\|\Omega_{kl}\|_{N+1} \|v_\ell\|_2 + \|\Omega_{kl}\|_3 \|v_\ell\|_N}{\lambda_{q+1}^{N-\varepsilon}} + C \frac{\|\Omega_{kl}\|_1 \|v_\ell\|_1}{\lambda_{q+1}^{2-\varepsilon}} \\ & + C \frac{\|\Omega_{kl}\|_{N+\varepsilon} \|v_\ell\|_{2+\varepsilon} + \|\Omega_{kl}\|_{2+\varepsilon} \|v_\ell\|_{N+\varepsilon}}{\lambda_{q+1}^{N-1}} \\ & + C \frac{\|\Omega_{kl}\|_N \|v_\ell\|_2 + \|\Omega_{kl}\|_2 \|v_\ell\|_N}{\lambda_{q+1}^{N-1-\varepsilon}} + C \frac{\|\Omega_{kl}\|_0 \|v_\ell\|_1}{\lambda_{q+1}^{1-\varepsilon}} \Bigg]\end{aligned} \tag{75}$$

Observe that since we assumed $|t\mu - l_1| < {}^1\!/_2(1 - \lambda_{q+1}^{-\varepsilon_1})$ we have that $\Omega_{kl}, \Omega'_{kl} \equiv 0$ for all $l \neq l_1$. Moreover

$$\Omega_{kl_1} := \left(D_t L_{kl_1} + L_{kl_1} \cdot \nabla v_\ell \right) e^{ik\cdot\Phi_{l_1}},$$

and

$$\Omega_{kl_1} := \Big(D_t^2 L_{kl_1} + D_t L_{kl_1} \cdot \nabla v_\ell + L_{kl_1} \cdot \nabla D_t v_\ell - L_{kl_1} \cdot \nabla v_\ell \cdot \nabla v_\ell \Big) e^{ik\cdot\Phi_{l_1}}.$$

Applying Lemmas 4.1, Lemma 4.2, (38) and (35) we obtain

$$\|\Omega_{kl_1}\|_N \leq C \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \ell^{-N} \leq C \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \lambda_{q+1}^{N(1-\beta)}. \tag{76}$$

Similarly we obtain

$$\begin{aligned}\|\Omega'_{kl_1}\|_N &\leq C \delta_{q+1,t_0}^{1/2} \lambda_q \ell^{-N} (\delta_{q,t_0} \lambda_q + \delta_{q+1,t_0} \ell^{-1}) \\ &\leq C \delta_{q+1,t_0} \delta_{q,t_0}^{1/2} \lambda_q \lambda_{q+1} \ell^{-N} \\ &\leq C \delta_{q+1,t_0} \delta_{q,t_0}^{1/2} \lambda_q \lambda_{q+1}^{1+N(1-\beta)}.\end{aligned} \tag{77}$$

Hence choosing N large enough such that $N\beta \geq 3$, then combining (73)-(77) we obtain (63).

Estimates on R^1. Recall that a key ingredient to the estimation of R^1 involves estimating

$$f_{klk'l'} := \chi_l \chi_{l'} a_{kl} a_{k'l'} \phi_{kl} \phi_{k'l'}$$

and

$$D_t\left(\nabla f_{klk'l'}\, e^{i\lambda_{q+1}(k+k')\cdot x}\right)e^{-i\lambda_{q+1}(k+k')\cdot x}=\Omega''_{klk'l'}.$$

Precisely it was shown in [2] that

$$\|R^1\|_0\leq \sum_{\substack{(k,l),(k',l')\\ k+k'\neq 0}}\left(\lambda_{q+1}^{\varepsilon-1}\|f_{klk'l'}\|_1+\lambda_{q+1}^{-N+\varepsilon}\|f_{klk'l'}\|_{N+1}+\lambda_{q+1}^{-N}[f_{klk'l'}]_{N+1+\varepsilon}\right)$$

$$\begin{aligned}\|R^1\|_1\leq &\lambda_{q+1}\sum_{\substack{(k,l),(k',l')\\ k+k'\neq 0}}\left(\lambda_{q+1}^{\varepsilon-1}\|f_{klk'l'}\|_1+\lambda_{q+1}^{-N+\varepsilon}\|f_{klk'l'}\|_{N+1}+\lambda_{q+1}^{-N}[f_{klk'l'}]_{N+1+\varepsilon}\right)\\ &+\sum_{\substack{(k,l),(k',l')\\ k+k'\neq 0}}\left(\lambda_{q+1}^{\varepsilon-1}\|f_{klk'l'}\|_2+\lambda_{q+1}^{-N+1+\varepsilon}\|f_{klk'l'}\|_{N+2}+\lambda_{q+1}^{-N}[f_{klk'l'}]_{N+2+\varepsilon}\right)\end{aligned}$$

and

$$\begin{aligned}\|D_tR^1\|_0\leq &\sum_{\substack{(k,l),(k',l')\\ k+k'\neq 0}}\left(\lambda_{q+1}^{\varepsilon-1}[\Omega''_{klk'l'}]_0+\lambda_{q+1}^{-N+\varepsilon}[\Omega''_{klk'l'}]_N+\lambda_{q+1}^{-N}[\Omega''_{klk'l'}]_{N+\varepsilon}\right)\\ &+C\frac{\|f_{klk'l'}\|_{N+2+\varepsilon}\|v_\ell\|_{2+\varepsilon}+\|f_{klk'l'}\|_{4+\varepsilon}\|v_\ell\|_{N+\varepsilon}}{\lambda_{q+1}^{N}}\\ &+C\frac{\|f_{klk'l'}\|_{N+2}\|v_\ell\|_{2}+\|f_{klk'l'}\|_{4}\|v_\ell\|_{N}}{\lambda_{q+1}^{N-\varepsilon}}+C\frac{\|f_{klk'l'}\|_{2}\|v_\ell\|_{1}}{\lambda_{q+1}^{2-\varepsilon}}\\ &+C\frac{\|f_{klk'l'}\|_{N+1+\varepsilon}\|v_\ell\|_{2+\varepsilon}+\|f_{klk'l'}\|_{3+\varepsilon}\|v_\ell\|_{N+\varepsilon}}{\lambda_{q+1}^{N-1}}\\ &+C\frac{\|f_{klk'l'}\|_{N+1}\|v_\ell\|_{2}+\|f_{klk'l'}\|_{3}\|v_\ell\|_{N}}{\lambda_{q+1}^{N-1-\varepsilon}}+C\frac{\|f_{klk'l'}\|_{1}\|v_\ell\|_{1}}{\lambda_{q+1}^{1-\varepsilon}}.\end{aligned}$$

Again as a consequence of our assumption $|t\mu-l_1|<1/2(1-\lambda_{q+1}^{-\varepsilon_1})$ we have that if either $l\neq l_1$ or $l'\neq l_1$ then $f_{klk'l'}\equiv 0$ and $\Omega''_{klk'l'}\equiv 0$. Moreover we have

$$\begin{aligned}\Omega''_{kl_1k'l_1}:=&-\left(a_{kl_1}Dv_\ell^T\nabla a_{k'l_1}+a_{k'l_1}Dv_\ell^T\nabla a_{kl_1}\right)\phi_{kl_1}\phi_{k'l_1}\\ &-a_{kl_1}a_{k'l_1}\left(D\Phi_lDv_\ell^Tk+D\Phi_{l_1}Dv_\ell^Tk'\right)\phi_{kl_1}\phi_{k'l_1}.\end{aligned}$$

Estimating $f_{kl_1k'l_1}$ and $\Omega''_{kl_1k'l_1}$ we have from Lemma 4.1 and Lemma 4.2 for $N\geq 1$

$$\|f_{kl_1k'l_1}\|_N\leq C\delta_{q+1}\ell^{1-N}\left(\lambda_q+\frac{\delta_q^{1/2}\lambda_q\lambda_{q+1}}{\mu}\right),\tag{78}$$

and

$$\left\|\Omega''_{kl_1k'l_1}\right\|_0\leq C\delta_{q+1}\delta_{q,t_0}^{1/2}\lambda_q\left(\lambda_q+1\right)\leq C\delta_{q+1}\delta_{q,t_0}^{1/2}\lambda_q^2\tag{79}$$

$$\left\|\Omega''_{kl_1k'l_1}\right\|_N\leq C\delta_{q+1}\delta_{q,t_0}^{1/2}\lambda_q^2\lambda_{q+1}^{N(1-\beta)},\tag{80}$$

for $N\geq 1$.

Combining the above estimates and again selecting N such that $N\beta \geq 3$ we obtain (64). □

We conclude this section by providing an explanation for the absence of terms of the type (61) appearing within the brackets on the right hand side of the inequalities (59) and (60). By carefully examining the arguments of [2] one sees that the only point in which such a term arises is the estimation of $D_t R_0$. More precisely the term arises from estimating

$$\mathcal{R}\left[\sum_{kl}\left(\chi_l D_t^2 L_{kl} + \chi_l L_{kl}\cdot\nabla D_t v_\ell\right)e^{i\lambda_{q+1}k\cdot\Phi_l}\right] := \sum_{kl}\mathcal{R}\left[\Gamma_{kl}e^{i\lambda_{q+1}k\cdot\Phi_l}\right]$$

which appears in the decomposition of $D_t R_0$ used in Proposition 5.1 of [2] (cf. (71) and (72)). Applying Lemma (4.1), Lemma (4.2) together with the inequalities $\delta_{q,t_0} \leq \delta_q$ and $\delta_{q+1,t_0} \leq \delta_{q+1}$ we obtain

$$\|\Gamma_{kl}\|_N \leq C\delta_{q+1}^{1/2}\lambda_q \ell^{-N}(\delta_q\lambda_q + \delta_{q+1}\ell^{-1}), \tag{81}$$

whereas instead applying the weaker estimates of [2] one obtains the worse estimate

$$\|\Gamma_{kl}\|_N \leq C\delta_{q+1}^{1/2}\lambda_q\ell^{-N-1}\delta_q.$$

The improvement is directly related to the fact that in contrast to [2] we keep track of spatial second derivative estimates of the pressure. Applying Proposition E.1 (ii) from [2] together with (81) and (30) we obtain

$$\begin{aligned}\left\|\sum_{kl}\mathcal{R}\left[\Gamma_{kl}e^{i\lambda_{q+1}k\cdot\Phi_l}\right]\right\|_0 &\leq C\sum_{kl}\left[\frac{\|\Gamma_{kl}\|_0}{\lambda_{q+1}^{1-\varepsilon}} + \frac{\|\Gamma_{kl}\|_N}{\lambda_{q+1}^{N-\varepsilon}} + \frac{\|\Gamma_{kl}\|_{N+\varepsilon}}{\lambda_{q+1}^{N}}\right] \\ &\leq C\frac{\delta_{q+1}^{1/2}\lambda_q(\delta_q\lambda_q + \delta_{q+1}\ell^{-1})}{\lambda_{q+1}^{1-\varepsilon}} \\ &\leq C\mu\left(\frac{\delta_{q+1}^{1/2}\mu}{\lambda_{q+1}^{1-\varepsilon}} + \frac{\delta_{q+1}\delta_q^{1/2}\lambda_q\lambda_{q+1}^{\epsilon}}{\mu}\right),\end{aligned}$$

where we assume N is chosen such that $N\beta \geq 1$.

6. Choice of the parameters and conclusion of the proof

We begin by noting that we have not imposed any upper bounds on the choice of λ_0 and thus we are free to choose λ_0 to be as large as need be: in what follows we will use this fact multiple times without further comment.

$1/5 - \varepsilon$ convergence. We now make the following parameter choices

$$\begin{aligned} \alpha &:= 1+\varepsilon_0, & \lambda_q &= \lfloor \lambda_0^{\alpha^q}\rfloor, \\ \epsilon, \varepsilon_2 &:= \frac{\varepsilon_0^2}{8}, & \varepsilon_1 &:= \frac{\varepsilon_0^2}{16}, \\ \delta_q &:= \lambda_q^{-2/5+2\varepsilon_0}, \end{aligned}$$

here $\lfloor a \rfloor$ denotes the largest integer smaller than a.

It is worth noting that with the above choices, our definition of μ agrees with the definition given in [2], i.e.

$$\mu = \delta_q^{1/4}\delta_{q+1}^{1/4}\lambda_q^{1/2}\lambda_{q+1}^{1/2}.$$

Having made the above choices it is clear the inequalities (30) are satisfied. Moreover assuming (26)-(29), it follows as a consequence of Lemma 4.1 that (26) and (27) are satisfied with q replaced by $q+1$. In order to show (28) with q replaced with $q+1$ we note that with our choices of parameters we obtain from (59) that

$$\begin{aligned}\left\|\mathring{R}_q\right\|_0 + \frac{1}{\lambda_q}\left\|\mathring{R}_q\right\|_1 &\leq C\frac{\delta_{q+1}^{1/2}\mu}{\lambda_{q+1}^{1-\varepsilon}} \\ &= C\lambda_q^{-2/5+\frac{6\varepsilon_0}{5}+\frac{13\varepsilon_0^2}{8}+\frac{\varepsilon_0^3}{8}} \\ &\leq C\delta_{q+2}\lambda_q^{-\varepsilon_0^2}.\end{aligned}$$

Thus we obtain both (28), and by similar arguments in combination with (69), also (29) with q replaced by $q+1$. Since the inequalities (26)-(29) hold for $q=0$, we obtain by induction that the inequalities hold for $q \in \mathbb{N}$. The inequalities (5)-(8) then follow as a consequence of Lemma 4.1, Lemma 4.2 and Proposition (5.1). In particular, the estimates on $p_q - p_{q-1}$ follow directly from the estimates on w, w_o and w_c; furthermore, one may derive time derivative estimates on w, w_o and w_c from the simple decomposition $\partial_t = D_t - v_\ell \cdot \nabla$.

$1/3-\varepsilon$ **convergence.** Let us define $U^{(q)}$ to be the set

$$U^{(q)} = \bigcup_{l\in[-\mu_q,\mu_q]} [\mu_q^{-1}(l + 1/2 - \lambda_q^{-\varepsilon_1}), \mu_q^{-1}(l + 1/2 + \lambda_q^{-\varepsilon_1})],$$

i.e. a union of $\sim 2\mu_q$ balls of radius $\lambda_q^{-\varepsilon_1}\mu_q^{-1}$ and define

$$V^{(q)} = \bigcup_{q'=q}^{\infty} U^{(q')}.$$

In particular note that $V^{(q)}$ can be covered by a sequence of balls of radius r_i such that

$$\sum r_i^d \leq 3\sum_{q'=q}^{\infty} \lambda_{q'}^{-d\varepsilon_1}\mu_{q'}^{1-d}. \tag{82}$$

Thus assuming

$$d > \frac{(1+\alpha)(-\frac{1}{5}+\varepsilon_0+1)}{(1+\alpha)(-\frac{1}{5}+\varepsilon_0+1)+2\alpha\varepsilon_1}, \tag{83}$$

it follows that the right hand side of (82) converges to zero as q tends to infinity.

From this point on we assume $d < 1$ is fixed, satisfying (83) – which we note is possible due to the fact the right hand side of (83) is strictly less than 1.

For any time $t_0 \in \bigcap_N V^{(N)}$ we simply set $\delta_{q,t_0} = \delta_q$ for all q.

Now suppose $t_0 \notin V^{(N)}$ for some integer N, furthermore assume N to be the smallest such integer. We now make the following parameter choices

$$\delta_{q+1,t_0} := \begin{cases} \lambda_{q+1}^{-2/5+2\varepsilon_0} & \text{if } q \leq N \\ \max\left(\lambda_q^{-\frac{\varepsilon_0^2}{8}} \delta_{q,t_0}^{\alpha}, \lambda_{q+1}^{-2/3+2\varepsilon_0}\right), & \text{if } q > N \end{cases}$$

It can then be checked that the inequalities (35) are satisfied. Applying Lemma 4.1, Corollary 4.2 and Proposition 5.1 iteratively we see that (31-34) hold for all $q \geq N$. In particular, in order to show (33) for q replaced by $q+1$ we note that by Proposition 5.1 we have for all times t satisfying $|t\mu_{q+1} - l_{q+1}| < 1/2(1 - \lambda_{q+1}^{-\varepsilon_1})$

$$\|\mathring{R}_1\|_0 + \frac{1}{\lambda_{q+1}}\|\mathring{R}_1\|_1 \leq \underbrace{C\frac{\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q}{\lambda_{q+1}^{1-\epsilon}}}_{I} + \underbrace{C\frac{\delta_{q+1,t_0}\delta_{q,t_0}^{1/2}\lambda_q\lambda_{q+1}^{\epsilon}}{\mu}}_{II}. \tag{84}$$

Notice that if $|\mu_{q+2}t - l_{q+2}| < 1$ then

$$\begin{aligned} |t\mu_{q+1} - l_{q+1}| &\leq \frac{\mu_{q+1}}{\mu_{q+2}}|\mu_{q+2}t - l_{q+2}| + \left|\frac{\mu_{q+1}l_{q+2}}{\mu_{q+2}} - l_{q+1}\right| \\ &< \frac{2\mu_{q+1}}{\mu_{q+2}} + |\mu_{q+1}t_0 - l_{q+1}| \\ &< 2\lambda_q^{-1/4\varepsilon_0} + 1/2 - \lambda_{q+1}^{-\varepsilon_1} \\ &< 1/2(1 - \lambda_{q+1}^{-\varepsilon_1}). \end{aligned}$$

Thus (84) holds for times t in the range $|\mu_{q+2}t - l_{q+2}| < 1$.

Taking logarithms of I and II we obtain

$$\ln I \leq \left(1 + \frac{\varepsilon_0}{2}\right)\ln\delta_{q,t_0} + \left(\frac{\varepsilon_0^2}{8} + \frac{\varepsilon_0^3}{8} - \varepsilon_0\right)\ln\lambda_q + C \tag{85}$$

and

$$\ln II \leq \left(\frac{3}{2} + \varepsilon_0\right)\ln\delta_{q,t_0} + \left(\frac{1}{5} - \frac{7\varepsilon_0}{5} - \frac{3\varepsilon_0^2}{8} + O(\varepsilon_0^3)\right)\ln\lambda_q + C. \tag{86}$$

Note by definition we have

$$\ln\delta_{q+2,t} \geq (1+\varepsilon_0)^2\ln\delta_{q,t_0} - \left(\frac{2\varepsilon_0^2}{8} + O(\varepsilon_0^3)\right)\ln\lambda_q. \tag{87}$$

Thus since $\delta_{q,t_0} \geq \lambda_q^{-2/3+2\varepsilon_0}$, combining (85) and (87) we obtain

$$\begin{aligned}\ln\left(\frac{I}{\delta_{q+2,t}}\right) &\leq \left(-\frac{3\varepsilon_0}{2} - \varepsilon_0^2\right)\ln\delta_{q,t_0} + \left(\frac{3\varepsilon_0^2}{8} - \varepsilon_0 + O(\varepsilon_0^3)\right)\ln\lambda_q + C \\ &\leq \left(-\frac{\varepsilon_0^2}{4} + O(\varepsilon_0^3)\right)\ln\lambda_q + C. \qquad (88)\end{aligned}$$

Similarly, since $\delta_{q,t_0} \leq \lambda_q^{-2/5+2\varepsilon_0}$, combining (86) and (87) we obtain

$$\begin{aligned}\ln\left(\frac{II}{\delta_{q+2,t}}\right) &\leq \left(\frac{1}{2} - \varepsilon_0 - \varepsilon_0^2\right)\ln\delta_{q,t_0} + \left(\frac{1}{5} - \frac{7\varepsilon_0}{5} - \frac{\varepsilon_0^2}{8} + O(\varepsilon_0^3)\right)\ln\lambda_q + C \\ &\leq \left(-\varepsilon_0^2 + O(\varepsilon_0^3)\right)\ln\lambda_q + C. \qquad (89)\end{aligned}$$

Hence assuming ε_0 is sufficiently small, from (88) and (89) we obtain (33) for q replaced by $q+1$.

Observe also that there exists an N' such that for all $q \geq N + N'$ we have

$$\delta_{q,t_0} = \lambda_q^{-2/3+2\varepsilon_0},$$

and hence the inequality (12) is never satisfied for $q \geq N + N'$. Thus

$$\Xi^{N+N'} \subset V^N.$$

In particular N' can be chosen universally, independent of N. Fixing $\delta > 0$ and choosing N such that V^N can be covered by a sequence of balls of radius r_i satisfying

$$\sum r_i^d < \delta,$$

we obtain that if we set $M = N + N'$ then (13) is satisfied which concludes the proof of Proposition 0.2.

Remark. *For the sake of completeness we note that analogously to the estimates* (5)-(7)*, the estimates* (9)-(11) *follow as a consequence of Lemma 4.1, Lemma 4.2 and Proposition* (5.1) *– here the set* Ω *can be taken explicitly to be*

$$\Omega := \bigcap_{q=1}^{\infty} V^{(q)}.$$

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Institut für Mathematik, Universität Leipzig, D-04103 Leipzig
E-mail address: tristan.buckmaster@math.uni-leipzig.de