in general, the direction of the gradient and of the tangent to a GE form a finite angle.

To understand point (2) we look at $\mathrm{GE}(x, y) = 0$ as an implicit definition of a curve $y = \mathrm{ge}(x)$, and we assume a displacement (dx, dy) along the GE. We get from Eq. (3), $\mathrm{GE}(x, y) = 0$, that

$$\mathrm{GE}_x\, dx + \mathrm{GE}_y\, dy = 0,$$

or

$$\frac{dy}{dx} = \mathrm{ge}'(x) = -\mathrm{GE}_x/\mathrm{GE}_y \qquad (4)$$

(if $\mathrm{GE}_y \neq 0$). It seems (without any proof here) easy to believe that the two vectors (U_x, U_y) and $(-\mathrm{GE}_y, \mathrm{GE}_x)$ point in different directions because Eq. (4) has third order derivatives of U cf [1].

To discuss further details we look for some simple model surfaces. First we give a model containing three GE's with a very normal behaviour.

Example 2.

$$U(x, y) = \tfrac{1}{2}(xy + 2)(y - x). \qquad (5)$$

We find three valleys and three ridges meeting in a central region. We denote valleys by v_1, v_2, v_3 and ridges by r_1, r_2 and r_3, clockwise. From Eq. (3) we find the formula

$$\mathrm{GE}(x, y) = (x + y)(2 + xy - (y - x)((y - x)^2 - 2)^{1/2}) \times (2 + xy + (y - x)((y - x)^2 - 2)^{1/2}) = 0,$$

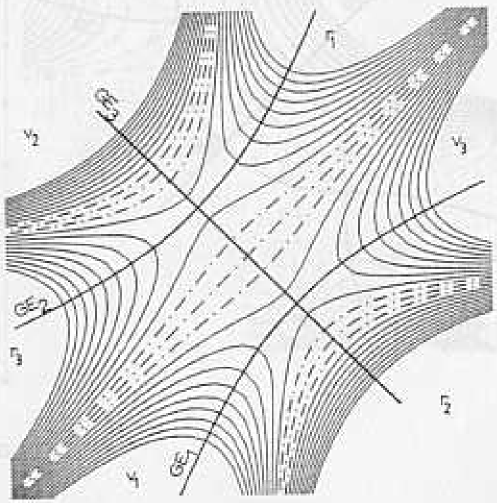


Fig. 1. Model surface $U(x, y) = \frac{1}{2}(xy + 2)(y - x)$, where v_i is a valley, r_i a ridge, GE a gradient extremal (*fat curves*). Contour lines, except for those at −0.5, 0 and 0.5 (*dot-dashed*), are solid

where the three factors give three different branches of the solution of the GE condition. The valleys v_1 and v_3 are connected by a GE over SP_2 numbered GE_1, the ridges r_3 and r_1 are connected by GE No. 2 over SP_1. (Of course, the GE definition equation (2) makes no difference between the floor line of a valley and the crest of a ridge. In both cases σ is minimal.) Note that SP_2 is lower than SP_1. The third straight GE_3 $y = -x$ coming uphill out of v_2 meets the SP_1, then goes downhill through point (0, 0) to SP_2 and at last it goes again up the ridge r_2, see Fig. 2. In the HNR classification [1] the GE_3 in v_2 traces a cirque and on r_2 it traces a cliff. The point (0, 0) is a so-called valley-ridge inflection point [6], but we choose to call it cirque-cliff inflection (CCI) point as this is consistent with [1], see Fig. 3. The three GE's of Fig. 1 meet exactly in the two SP's and no further bifurcation emerges. The CCI point is not a bifurcation point. Trajectories orthogonal to the contour lines are possible in the six sections of the plane divided by the GE's. Thus, except for GE_3 itself, there exists no steepest descent line from any ridge to its opposite valley.

Example 3.

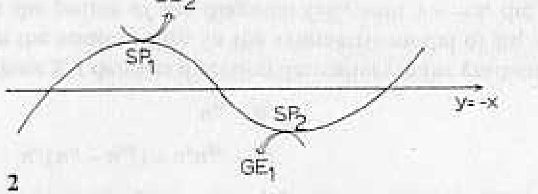
$$U(x, y) = \tfrac{1}{2}(xy - 2)(y - x). \qquad (6)$$

Now, we find the SP's to be on the same contour line, $y = \mathrm{cl}(x) = x$ of height zero, as the CCI point. With a little imagination one can imagine Fig. 4 as a double "monkey" saddle where the one SP is flattened to two SP's and a CCI point in between, cf Fig. 5. One readily ascertains from Eq. (3) the condition

$$\mathrm{GE}(x, y) = (x + y)(2 - xy - (y - x)((y - x)^2 + 2)^{1/2}) \times (2 - xy + (y - x)((y - x)^2 + 2)^{1/2}) = 0$$

which again gives three GE branches. GE_3 from northwest to southeast goes steadily uphill through the CCI point (0, 0). SP_1 connects v_1 and v_2, and SP_2 connects v_3 and v_2. It seems that we find no connection between v_1 and v_3, or r_1 and r_3. Two branches of GE_1 coming from v_1 and v_3 go over SP_1 and SP_2, as expected, and flow into valley v_2. There they meet at the point

$$\mathrm{BP}_1 = (-(\tfrac{2}{5})^{1/2}, (\tfrac{3}{5})^{1/2}).$$



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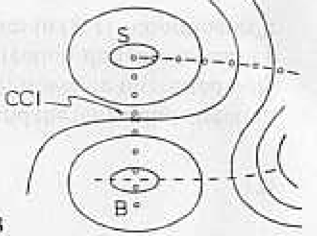

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Fig. 2. Energy profile over GE_3 from northwest to southeast in Fig. 1

Fig. 3. Illustration of contours (*solid lines*); a GE for a valley (*dashed line*); a GE for a ridge (*dot-dashed line*); a GE along cirque; (*dotted*) CCI point; and cliff from bowl B to summit S