

W.Quapp and J.M.Bofill

Sliding Paths for Series of Frenkel – Kontorova Models –  
A contribution to the concept of 1D – superlubricity

Supplementary Material 1

MF 2/3      N = 6, ... 24

```
(* ##### *)

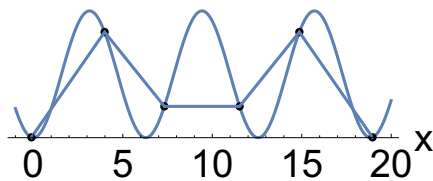
(* N=6 global minimum *)
setxyz = {{x, 2 Pi - 1.23, 2 Pi - 0.82}, {y, 2 Pi + 0.82, 2 Pi + 1.23},
  {z, 4 Pi - 0.2, 4 Pi + 0.5}, {v, 6 Pi - 1.51, 6 Pi - 0.4}, {w, 6 Pi + 1.2, 6 Pi + 1.75},
  {k, 8 Pi - 0.2, 8 Pi + 0.5}}
Potener6 = 0.5 * ((y - 4./3.*Pi - x)^2 + (z - 4./3.*Pi - y)^2 +
  (v - 4./3.*Pi - z)^2 + (w - 4./3.*Pi - v)^2 + (k - 4./3.*Pi - w)^2) + 6. -
  Cos[x] - Cos[y] - Cos[z] - Cos[v] - Cos[w] - Cos[k]
NMinimize[Potener6, setxyz]
{5.06179,
  {x -> 6.18015, y -> 10.2661, z -> 13.6065, v -> 17.8094, w -> 21.1498, k -> 25.2358}}

SetXX /. {x -> 6.180152907583693, y -> 10.266093010034709, z -> 13.606512961102199,
  v -> 17.809409474876826, w -> 21.149827946164496, k -> 25.235770601302292}

SetXX = {6.180152907583693, 10.266093010034709, 13.606512961102199,
  17.809409474876826, 21.149827946164496, 25.235770601302292} - 2 * Pi
{-0.103032, 3.98291, 7.32333, 11.5262, 14.8666, 18.9526}

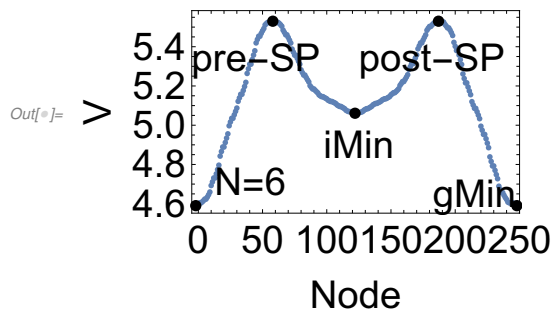
SetAA00 = Riffle[SetXX, 1 - Cos[SetXX]];
XXX = Partition[SetAA00, 2];
Bild = Show[Graphics[{PointSize[0.02135], Black, Point[XXX]}],
  FormatType -> StandardForm, PlotRange -> All];
Bildline = ListLinePlot[XXX, PlotRange -> All]
sidePot = Plot[
  1. - Cos[x], {x, -1.0, 7.*Pi - 2.}, Axes -> {True, False},
  AxesStyle -> Directive[20], AxesLabel -> {"x", "SidePot"}, AspectRatio -> 0.36];
min6D = Show[sidePot, Bild, Bildline]

energy6D = ListPlot[NT6D, PlotRange -> All,
  PlotStyle -> {PointSize[0.017]}, ImageMargins -> {{0, 0}, {0, 50}},
  ImageSize -> Automatic, FrameStyle -> Directive[20],
  FrameLabel -> {"Node", "V"}, Axes -> False, Frame -> True]
```



```
Show[Graphics[{PointSize[Large], Black, Point[{6.429805, 1.0 - Cos[6.429805]}],
  PointSize[Large], Black, Point[{10.82246770, 1.0 - Cos[10.82246770]}],
  PointSize[Large], Black, Point[{14.2300757, 1.0 - Cos[14.2300757]}],
  PointSize[Large], Black, Point[{18.633370, 1.0 - Cos[18.63337095]}]}],

xxDim6 = Graphics[{Text[Style["N=6", FontSize → 20], {40.5, 4.71}],
  Text[Style["iMin", FontSize → 20], {125, 4.9}],
  Text[Style["pre-SP", FontSize → 20], {45, 5.341}],
  Text[Style["post-SP", FontSize → 20], {182, 5.341}],
  Text[Style["gMin", FontSize → 20], {215, 4.64}]],
FormatType → StandardForm];
pp6 = Graphics[{PointSize[0.0345], Black, Point[{-2.0, 4.59}],
  PointSize[0.0345], Black, Point[{58.0, 5.53}],
  PointSize[0.0345], Black, Point[{122.0, 5.061}],
  PointSize[0.0345], Black, Point[{187.0, 5.53}],
  PointSize[0.0345], Black, Point[{248.0, 4.59}]}],
FormatType → StandardForm];
ener6D = Show[energy6D, xxDim6, pp6]
```



```
node245 = {6.1828830009, 10.2701688578, 13.6092242091 ×
17.8121248219, 21.1539244811, 25.2385115796};
(* SP1 ENERGY energy=5.53247 *)
node423 = {9.7215483260, 13.6194850935, 18.3863905517,
22.7065137943, 26.3707438893, 30.9801073781} - 2 * Pi
{3.43836, 7.3363, 12.1032, 16.4233, 20.0876, 24.6969}
```

```

(* SP1 ENERGY 5.532369 *)
node67 = {0.4416357566, 5.0465843686, 8.7068584450,
13.0249516503, 17.7857210504, 21.6722666259};
(* new min ENERGY 4.6034786, detHess 7.8581454
Control by MMA procedure for minimum *)
node125 = {0.9758819850, 5.9838777601, 10.6970149250,
14.4543908548, 19.1618718078, 24.1766160733}
k = 1.0 ;
Setxx = node125;
SetAA = Table[A[i], {i, 6}];
SetXX = Setxx + SetAA;
as = 6.283185307180;
a0 = 4.188790204787;
(* The formula for the FK model for N=6 *)
VWV = Sum[1 - Cos[SetXX[[i]]], {i, 6}] +
Sum[k/2.* (SetXX[[i+1]] - SetXX[[i]] - a0)^2, {i, 5}];
GGG = Table[D[VWV, A[i]], {i, 6}];
Set00 = Table[-0.172, {i, 6}];
Set22 = Table[0.172, {i, 6}];
SetAA00 = Riffle[SetAA, Set00];
SetA0 = Partition[SetAA00, 2];
SetAA22 = Riffle[SetA0, Set22];
SetFl = Flatten[SetAA22];
SetFin = Partition[SetFl, 3];
G2G2G2 = Sum[GGG[[i]]^2, {i, 6}];
NMinimize[VWV, SetFin]
HHH = Table[D[GGG, A[j]], {j, 6}];
{4.60339, {A[1] → -0.00990529, A[2] → -0.00650618, A[3] → -0.0093177,
A[4] → -0.00934702, A[5] → -0.00650223, A[6] → -0.00985159}}

SetXX /. {A[1] → -0.009905285752918687, A[2] → -0.0065061796722942255,
A[3] → -0.009317704173561982, A[4] → -0.00934701713731663,
A[5] → -0.006502232232078612, A[6] → -0.00985159143605484}
HHH /. {A[1] → -0.009905285752918687, A[2] → -0.0065061796722942255,
A[3] → -0.009317704173561982, A[4] → -0.00934701713731663,
A[5] → -0.006502232232078612, A[6] → -0.00985159143605484};

SetXX = {0.9659766992470813, 5.9773715804277066, 10.687697220826438,
14.445043837662682, 19.15536957556792, 24.166764481863947}

Out[8]= {0.965977, 5.97737, 10.6877, 14.445, 19.1554, 24.1668}

Eigensystem[%]
{{4.07693, 3.74209, 2.24183, 1.27518, 0.900416, 0.201914},
{{0.217232, -0.544886, 0.394854, -0.394854, 0.544886, -0.217232},
{-0.283218, 0.615568, -0.202149, -0.202149, 0.615568, -0.283218},
{-0.370564, 0.249471, 0.54813, -0.54813, -0.249471, 0.370564},
{0.610017, 0.179001, -0.309577, -0.309577, 0.179001, 0.610017},
{-0.561687, -0.375318, -0.208911, 0.208911, 0.375318, 0.561687},
{0.218327, 0.298387, 0.602742, 0.602742, 0.298387, 0.218327}}}

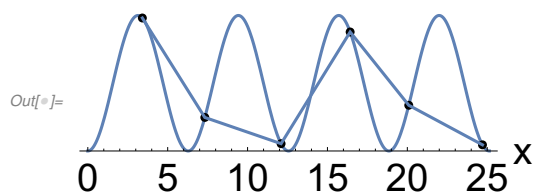
```

```
(* new
SP 59 energy=5.5325296 detHess -3.1385
3.4140560234 7.3245850433 12.0982260202
16.4206357112 20.0891872823 24.7034030558
```

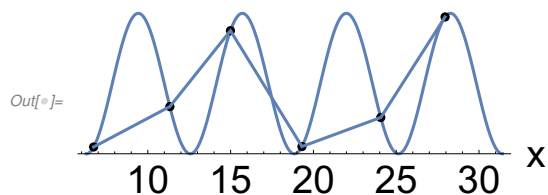
```
iMin 124 energy=5.062374 detHess 1.88
6.1904891027 10.2815003336 13.6168115153
17.8197652178 21.1655276996 25.2462591214
```

```
SP 188 energy=5.5327 detHess -3.159
6.7260021818 11.3300467548 14.9895077131
19.3071914347 24.0667033199 27.9509238771
```

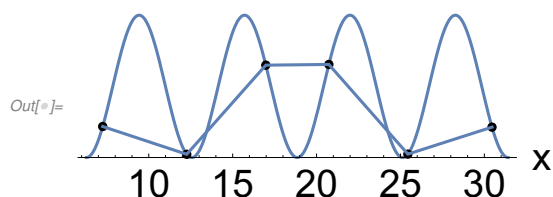
```
gMin 245 energy=4.605158 detHess 7.869
7.2546607841 12.2641725342 16.9760648871
20.7334205590 25.4421693047 30.4554319747 *)
(* SP *)
```



```
(* next SP *)
```



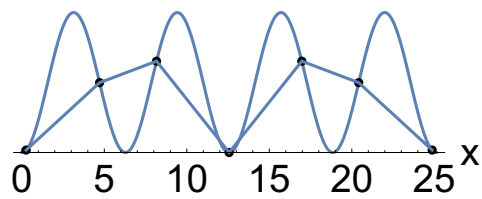
```
(* next gMin *)
```



```
(* N=7 global minimum *)
```

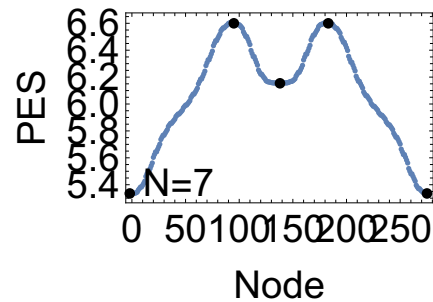
```
{0.264225, 4.71418, 8.16413, 12.5664, 16.9686, 20.4186, 24.8685}
```

(\* ENERGY 5.335880411111903` \*)



Eigensystem[%]

```
{ {4.10858, 3.33844, 2.987, 1.8316, 1.38008, 0.491834, 0.186226},
  { {-0.114611, 0.245644, -0.40291, 0.726893, -0.40291, 0.245644, -0.114611},
    {-0.373536, 0.512921, -0.312063, -1.4982 × 10-8, 0.312063, -0.512921, 0.373536},
    {0.442185, -0.45178, 0.00291307, 0.448016, 0.00291308, -0.45178, 0.442185},
    {-0.503424, -0.0673032, 0.49197, 2.44997 × 10-8, -0.49197, 0.0673032, 0.503424},
    {0.493187, 0.288621, -0.31375, -0.387364, -0.31375, 0.288621, 0.493187},
    {-0.327162, -0.482061, -0.400727, -1.91359 × 10-8, 0.400727, 0.482061, 0.327162},
    {0.219325, 0.390195, 0.489097, 0.347645, 0.489097, 0.390195, 0.219325} } }
```



(\* N=7 SP and iMin \*)

```
node95 = {0.9492379342, 5.9586638502, 10.6492345087,
14.3991836677, 19.1150023789, 24.0931612047, 28.2091284796}
```

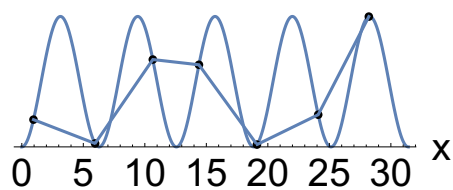
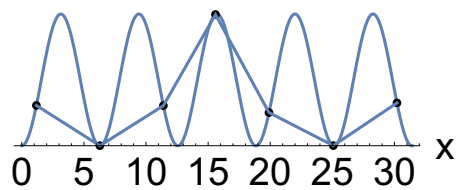
(\* ENERGY 6.606683 \*)

```
node137 = {1.1723348056, 6.2843174228, 11.3974321556,
15.5902109713, 19.9004701545, 25.0786070966,
30.2026363428}
```

(\* ENERGY 6.156568 \*)

```
{0.949238, 5.95866, 10.6492, 14.3992, 19.115, 24.0932, 28.2091}
```

```
{1.17233, 6.28432, 11.3974, 15.5902, 19.9005, 25.0786, 30.2026}
```



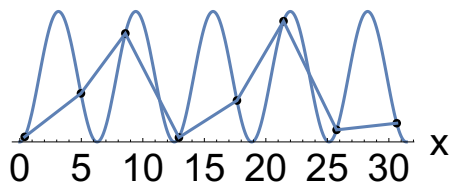
```
(* N=8 int Minimum start with a0 *)
```

```
ENERGY 6.240254144803445
```

```
SetXX = {0.3962413424987329, 4.970985194179818,  
8.57897913390411, 12.9354741534775, 17.652748664828867,  
21.439145753954012, 25.749936476707525, 30.639477350359417}
```

```
Eigenvalues[%]
```

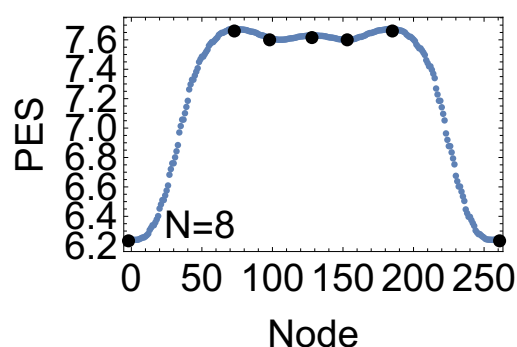
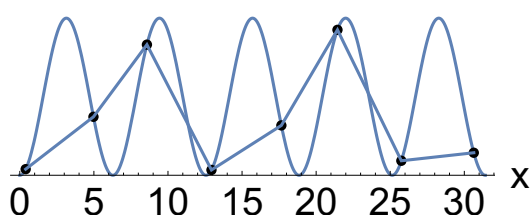
```
{4.16948, 3.70979, 3.29988, 2.28682, 1.45427, 1.21483, 0.298295, 0.0571736}
```



```
SetXX = {0.3962413424987308, 4.970985194179813,  
8.5789791339041, 12.935474153477495, 17.65274866482886,  
21.439145753953998, 25.749936476707518, 30.639477350359414}
```

## Eigensystem[%]

```
{ {4.16948, 3.70979, 3.29988, 2.28682, 1.45427, 1.21483, 0.298295, 0.0571736},
  { {-0.0931145, 0.209225, -0.30729, 0.66121, -0.510513,
    0.259831, -0.274425, 0.111733}, {0.142407, -0.254519, 0.22768,
    -0.285747, -0.00561644, 0.293298, -0.745596, 0.373474},
  { {-0.49536, 0.682287, -0.217052, -0.256212, 0.311139, -0.0345612,
    -0.236786, 0.149253}, {0.323463, -0.117839, -0.319798, 0.421633,
    0.592101, -0.375147, -0.165072, 0.287876}, {0.641544, 0.300403,
    -0.400784, -0.253351, 0.0262347, 0.277252, -0.111003, -0.428353},
  {0.339176, 0.240032, -0.0893283, -0.250933, -0.341731, -0.142229,
    0.351161, 0.704325}, {0.276557, 0.449191, 0.602702, 0.176758,
    -0.137058, -0.460061, -0.2541, -0.179562}, {0.13393, 0.249826,
    0.415325, 0.281662, 0.394589, 0.629111, 0.291992, 0.176299}} }
```



```
(* Pre-SP left at ENERGY 7.673901 detHess -1.4158 *)
node74 = {0.1240520530, 4.4383703309, 7.7899973919,
12.1395781596, 16.0752058131, 19.6517902567,
23.9472856756, 27.3161106964}
```

```
(* iMin left at ENERGY 7.60067 detHess 1.24739 *)
node107 = {-0.1407572865, 3.9089725091, 7.2644504829,
11.4511298938, 14.7397925831, 18.8523054952,
22.9675679774, 26.2543328823}
```

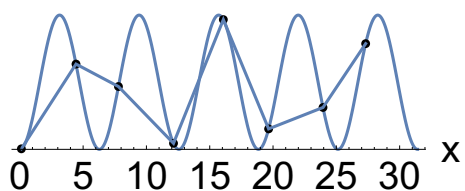
```
(* SP in between at ENERGY 7.63053 detHess -0.2425839 *)
node129 = {-0.6069949435, 3.0137528645, 6.7619925318,
10.9709530607, 14.1802166766, 18.3885537957,
22.1520450736, 25.7553331610}
```

```
(* iMin right at ENERGY 7.601356 detHess 1.37402 *)
node154 = {-1.1275817789, 2.1580593928, 6.2761608541,
10.3872379202, 13.6777150619, 17.8644879041,
21.2179510520, 25.2698414128}
```

```
(* next Post-SP ENERGY 7.67394 *)
node185 = {-2.1770620185, 1.1842023994, 5.4759953673,
9.0620747472, 12.9858236129, 17.3233542068,
20.6791433716, 24.9840087002} + 2 Pi
(* next Minimum ENERGY 6.24074 *)
node256 = {-5.5081747690, -0.6187378393, 3.6906916378,
7.4782022360, 12.1959347015, 16.5516453559,
20.1602602868, 24.7352415502} + 2 Pi
```

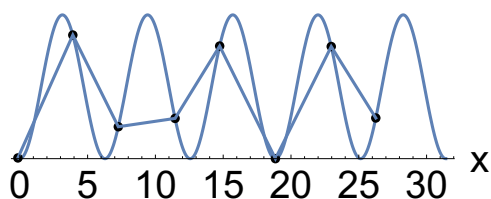
```
SetXX = node74
```

```
{0.124052, 4.43837, 7.79, 12.1396, 16.0752, 19.6518, 23.9473, 27.3161}
```



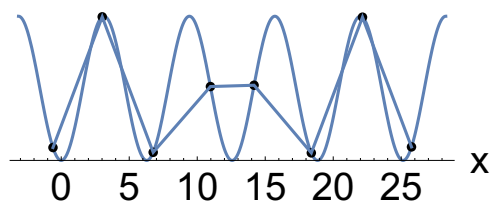
```
SetXX = node107
```

```
{-0.140757, 3.90897, 7.26445, 11.4511, 14.7398, 18.8523, 22.9676, 26.2543}
```



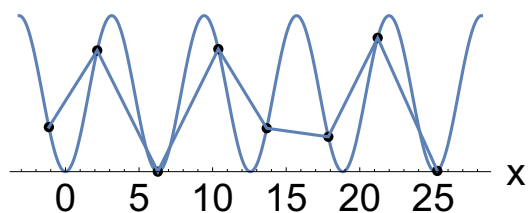
```
SetXX = node129
```

```
{-0.606995, 3.01375, 6.76199, 10.971, 14.1802, 18.3886, 22.152, 25.7553}
```



```
SetXX = node154
```

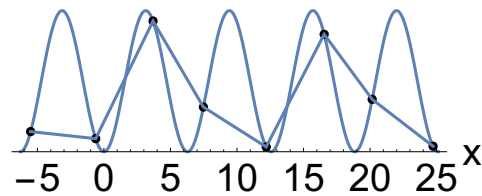
```
{-1.12758, 2.15806, 6.27616, 10.3872, 13.6777, 17.8645, 21.218, 25.2698}
```





SetXX = node256

{-5.50817, -0.618738, 3.69069, 7.4782, 12.1959, 16.5516, 20.1603, 24.7352}

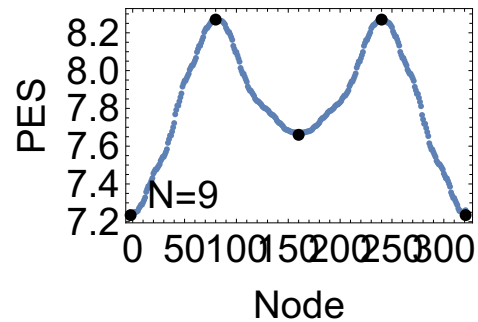
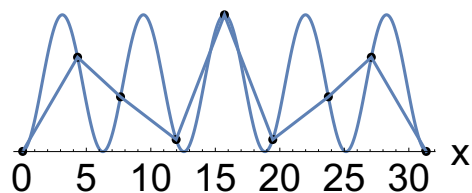


(\* N=9 local minimum start with a0 \*)

SetXX = {0.06695562702084325, 4.322651422203551, 7.653338421327639,  
11.96396414028602, 15.707962773427294, 19.45196188826518,  
23.762587562816407, 27.093274507982187, 31.348970606600645}

Eigenvalues[%]

{4.14489, 3.81555, 3.04181, 2.82583,  
1.9331, 1.66476, 0.507517, 0.334946, 0.0137671}



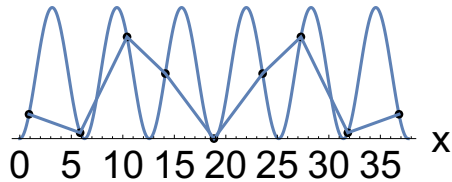
(\* N=9 global minimum \*)

Setxx = {0.3962413424987308, 4.970985194179813,  
8.5789791339041, 12.935474153477495, 17.65274866482886,  
21.439145753953998, 25.749936476707518, 30.639477350359414, 35.}

SetXX = {0.8908443549856704, 5.857237477418085, 10.410446493822011,  
14.130013998772027, 18.84955592153876, 23.569097844305492,  
27.28866534925551, 31.841874365659436, 36.80826748809185}

Eigensystem[%]

```
{ {4.22585, 3.88104, 3.70861, 2.55897,
  1.86899, 1.2411, 1.06203, 0.313137, 0.128771},
  { {0.0970978, -0.252172, 0.23456, -0.399471, 0.651745, -0.399471, 0.23456,
    -0.252172, 0.0970978}, {-0.251916, 0.567388, -0.298673, 0.159387,
    6.01646 × 10-16, -0.159387, 0.298673, -0.567388, 0.251916},
  {0.258432, -0.537502, 0.170473, 0.152076, -0.429224, 0.152076,
    0.170473, -0.537502, 0.258432}, {0.230501, -0.214415, -0.305906,
    0.554362, -1.23093 × 10-15, -0.554362, 0.305906, 0.214415, -0.230501},
  {-0.348001, 0.0836002, 0.435084, -0.266897, -0.471961, -0.266897,
    0.435084, 0.0836002, -0.348001}, {-0.579901, -0.2248, 0.204586,
    0.267066, 3.02433 × 10-15, -0.267066, -0.204586, 0.2248, 0.579901},
  {0.529368, 0.300004, 0.0252228, -0.290276, -0.299568, -0.290276, 0.0252228,
    0.300004, 0.529368}, {-0.217077, -0.285591, -0.524748, -0.309766,
    -2.13224 × 10-15, 0.309766, 0.524748, 0.285591, 0.217077}, {0.149831, 0.224744,
    0.47538, 0.402245, 0.28019, 0.402245, 0.47538, 0.224744, 0.149831} } }
```



```
(* N=9 SP energy=8.27165 detHess -8.666 *)
node81
```

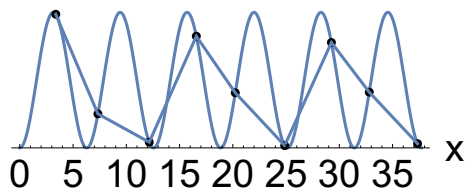
```
SetXX = {3.378723340495503, 7.33259896629844, 12.153605922357304,
  16.573469520446842, 20.231909576771407, 24.872646792608847,
  29.256212195590816, 32.80823543086986, 37.34437202781995}
```

Eigenvalues[%]

```
{4.22264, 3.96618, 3.28476, 2.52657,
  1.93172, 1.39298, 0.410781, -0.376088, 0.147783}
```

```
Sum[1 - Cos[SetXX[[i]]], {i, 9}] +
  Sum[k/2. * (SetXX[[i + 1]] - SetXX[[i]] - a0)^2, {i, 8}]
```

```
(* ENERGY 8.271596448246` *)
```

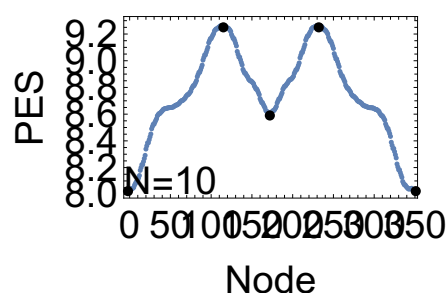
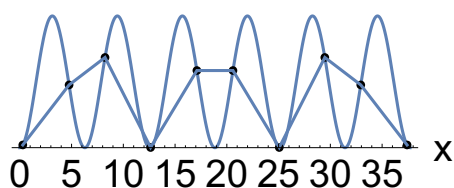


```
(* N=10 int minimum start with a0 *)
```

```
SetXX = {0.28735875090619967, 4.759569225544028, 8.23289248150686,
  12.635283811095286, 17.106533805245512, 20.592578037832013,
  25.063828031982236, 29.466219361570666, 32.9395426175335, 37.411753092171324}
```

Eigenvalues[%]

```
{4.21643, 3.94466, 3.27322, 3.12104,
  2.26455, 1.62968, 1.39867, 0.614642, 0.309624, 0.15248}
```



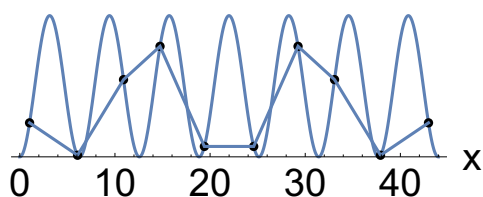
(\* N=10 NOW global minimum \*)

```
Setxx = {0.8908443549856704, 5.857237477418085, 10.410446493822011,
  14.130013998772027, 18.84955592153876, 23.569097844305492,
  27.28866534925551, 31.841874365659436, 36.80826748809185, 41.};
```

```
SetXX = {1.0257636338436487, 6.069664180337152, 10.901662355470426,
  14.738067016100848, 19.3992987368627, 24.582998413029674,
  29.24423013348191, 33.08063479419742, 37.91263296966464, 42.956533516250126}
```

Eigensystem[%]

```
{{4.315400449142172, 3.999089333373189, 3.891636026221844, 2.8752724759775616,
  2.195911928597062, 1.7791952077451836, 1.0225697188802227,
  0.9254810725953171, 0.26371862493548087, 0.11019865808552788},
  {{-0.07118389969831683, 0.1990981216326566, -0.19523121011583483,
  0.2712479118885009, -0.5861758569893921,
  0.5861758569189339, -0.2712479118266806,
```



(\* N=10 SP energy=9.2628 detHess -16.109 \*)

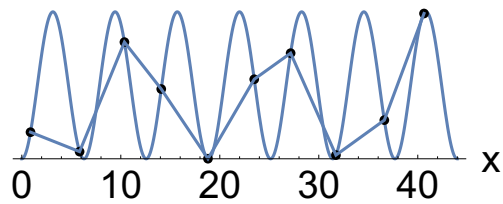
```
node114 = {0.8980549092, 5.8504229596, 10.3834109200,
  14.0979920787, 18.8118060667, 23.4878791648,
  27.1666938696, 31.7401554126, 36.6321948935,
  40.6485182833 };
```

```
SetXX = {0.8791636038878559, 5.838159507307668, 10.366674243330278,
  14.086513926815183, 18.805071020635797, 23.479157884011048,
  27.156669635789495, 31.733261728521303, 36.62188973841964, 40.62987264351739}
```

```
Eigenvalues[%]
```

```
{4.26165, 4.04819, 3.7587, 2.71197, 2.2103,
  1.63985, 1.14908, -0.381791, 0.379441, 0.149912}
```

```
(* ENERGY 9.262189985118692` *)
```

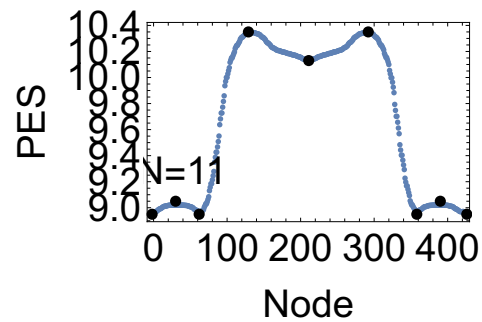
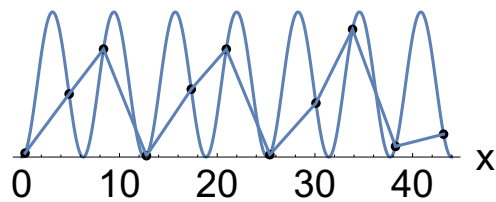


```
(* N=11 global minimum start mit a0 *)
```

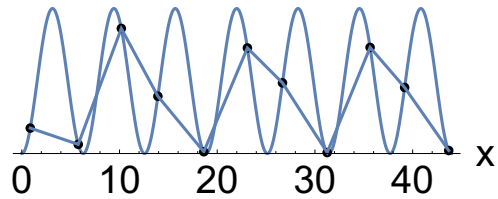
```
SetXX = {0.3300997934144975, 4.843027433778401,
  8.36447614787177, 12.758427852475593, 17.34325826752701,
  20.930168000979826, 25.389912172630254, 30.10400189958973,
  33.851412197190804, 38.247699025215695, 43.16546809904357}
```

```
Eigenvalues[%]
```

```
{4.24463, 4.0172, 3.69948, 3.25143, 2.51555,
  1.96764, 1.43746, 1.21778, 0.48661, 0.220873, 0.0870761}
```



```
(* mirror imagine globMin at node65 *)
SetXX = {0.8643567491, 5.7749334161, 10.1840392395,
13.9258055424, 18.6271482026, 23.0881471096,
26.6781883961, 31.2530661612, 35.6465341674,
39.1809022201, 43.6920991543};
```

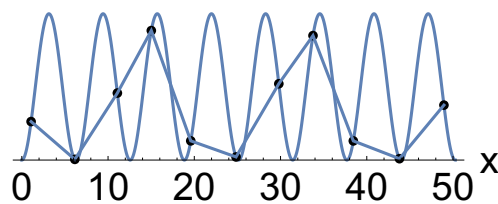


```
(* N=11 NOT global minimum
is not symmetric ? here first mirror intMin ? *)
```

```
SetXX = {1.077119301458117, 6.146505900649086,
11.07963825324789, 15.016301900709998, 19.590783174201118,
24.840458138808653, 29.801993868551357, 33.76445982597578,
38.43942904678951, 43.78892039027268, 48.94623792920694}
```

```
Eigensystem[%]
```

```
{{4.358832241426991, 4.196843167347291,
3.990164591844547, 3.0558577585650792, 2.5782478629207435,
2.0964124631865984, 1.5342455120642982, 0.9150290993182715,
0.7301015589742446, 0.17914555434177565, 0.06239871731127083}, {}}
```



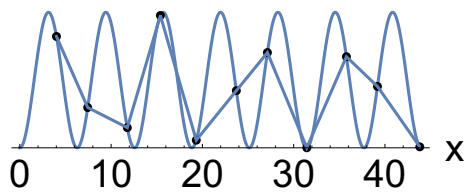
```
(* SP energy=10.3462997 detHess -4.298 *)
node134 = {3.9791227577, 7.4269135592, 11.7848884357,
15.4385309652, 19.3583577583, 23.7653157577,
27.1928824317, 31.5030900932, 35.9003509834,
39.3234834551, 43.7451811144};
```

```
SetXX = {4.01299144235139, 7.43655148202626,
11.774245328193583, 15.400091634633217, 19.328968973055037,
23.71910478717981, 27.12156481815895, 31.437916416475815,
35.776256123194514, 39.17593126992734, 43.81202782771367}
```

```
Eigenvalues[%]
```

```
{4.21092, 3.99404, 3.6794, 3.19191, 2.36496,
1.98525, 1.55144, 0.654864, 0.28946, -0.159479, 0.12037}
```

(\* ENERGY 10.3731091295288 \*)

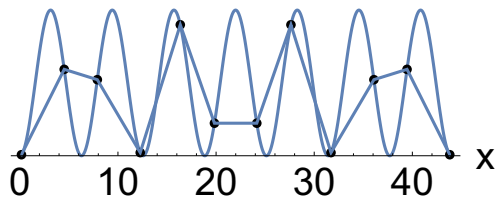


(\* N=12 global minimum start a0 \*)

```
SetXX = {0.16837484105704775, 4.524745532653849, 7.898669733386077,
  12.271595503474593, 16.35399664449332, 19.83437393092523,
  24.147922728039592, 27.62829980297333, 31.71070131127717,
  36.08362716833083, 39.45755134688683, 43.81392209461321}
```

Eigenvalues[%]

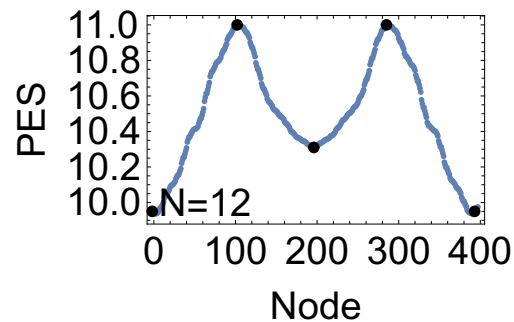
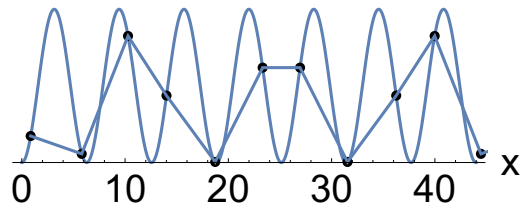
```
{4.20797, 4.03429, 3.70871, 3.09437, 2.99568, 2.24284,
  1.73889, 1.60334, 0.600866, 0.447868, 0.213926, 0.0433373}
```



(\* New Min \*)

```
node195 = {0.8457456758, 5.7945912558, 10.2740515278,
  14.0027036328, 18.7223370910, 23.3150942390,
  26.9381629503, 31.5338395216, 36.2471563044,
  39.9675249191, 44.4542734140, 49.3956781615}
```

```
SetXX = {0.8577566442172817, 5.802923897607426, 10.286080078234866,
  14.010544825125814, 18.727003700434814, 23.321216908734073,
  26.944265511245693, 31.53847871190449, 36.2549375259132,
  39.9794021855572, 44.46255849152011, 49.407725721189166}
```



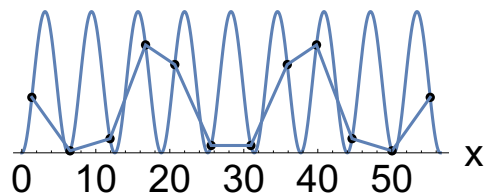
```
(* N=12 NOW global minimum *)
```

```
Setxx = {1.077119301458117, 6.146505900649086,
  11.07963825324789, 15.016301900709998, 19.590783174201118,
  24.840458138808653, 29.801993868551357, 33.76445982597578,
  38.43942904678951, 43.78892039027268, 48.94623792920694, 54.};
```

```
SetXX = {1.3551827264609373, 6.520818314447512, 11.9218568795165,
  16.72208558788227, 20.673297078878896, 25.592688188961912,
  30.95597978822508, 35.87537089998335, 39.82658246103179,
  44.626811001731554, 50.02784933198168, 55.193484861895946}
```

```
Eigensystem[%]
```

```
{{4.399732897422667, 4.265883411686537, 4.192641393945346,
  3.196927214124878, 2.7952318144623742, 2.490121990704152,
  1.8038445777621668, 1.3681099502879528, 0.7114650153797795,
  0.6813681153476272, 0.19977554283523236, 0.10028055965620365}, {}}
```



```
(* SP energy=10.99407 detHess -26.633 *)
```

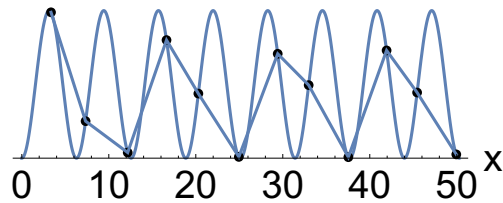
```
node106 = {3.3596971964, 7.3336728519, 12.1753142180,
  16.6357903106, 20.2959474210, 24.9483762259,
  29.4174826960, 32.9766452476, 37.5357570213,
  41.9322395194, 45.4413862168, 49.9443001444};
```

```
SetXX = {3.3565530578514564, 7.332034518606576, 12.174366031635476,
  16.634655837925706, 20.295307378282825, 24.948150990708843,
  29.417450854952904, 32.97682000846248, 37.53614012911182,
  41.93320899383533, 45.44249543271356, 49.94567213951341}
```

```
Eigenvalues[%]
```

```
{4.266, 4.10554, 3.88427, 3.24403, 2.6518, 2.21667,
  1.76587, 1.41127, 0.54536, -0.379498, 0.285685, 0.137284}
```

```
(* ENERGY 10.99407479137445 *)
```

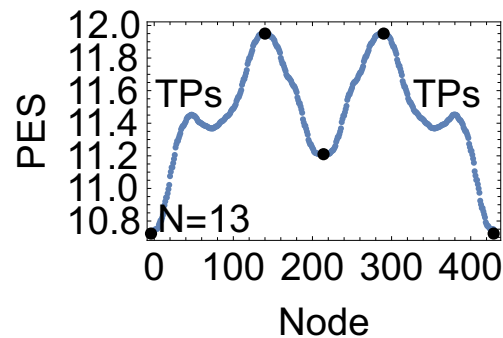
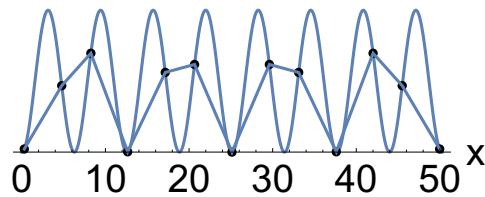


```
(* N=13 global minimum start a0 *)
```

```
SetXX = {0.2957940519783252, 4.776083744636441, 8.25840122589343,
  12.660049574927525, 17.15523994765105, 20.658049143515655,
  25.13274123920765, 29.6074333627528, 33.110242628601334, 37.60543299272491,
  42.007081458565516, 45.489398869097414, 49.96968851194649}
```

```
Eigenvalues[%]
```

```
{4.25861, 4.09522, 3.85311, 3.24441, 3.17524, 2.46608,
  2.00743, 1.54287, 1.40918, 0.680798, 0.422515, 0.234312, 0.137601}
```



```
(* new Pre-Min emerges at node85? No, it is TP
  node99 is BBP with minus to plus for < node35 is deth minus *)
```

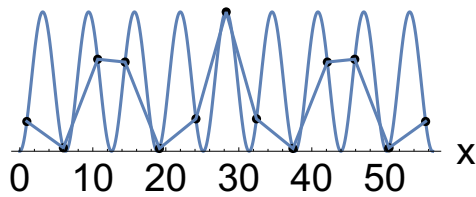
```
(* N=13 NOW global minimum *)
```



```
SetXX = {0.9607723657784886, 5.969196845054076, 10.668766707871903,
  14.421264535236208, 19.13367737910996, 24.126404507647123,
  28.274253856399927, 32.42218318863919, 37.41494744224826,
  42.12735629926849, 45.87986339824391, 50.57945462891411, 55.58788502163635}
```

```
Eigensystem[%]
```

```
{{4.314496066130787, 4.169182240655141,
  3.8925912466772017, 3.8320330846153947, 2.8850097639609076,
  2.477414018825705, 1.9538734284459227, 1.6837043123234614,
  1.0717694488730722, 1.027476968736618, 0.2997460948889715,
  0.22776219952402565, 0.00008678205261756571}, ...
```



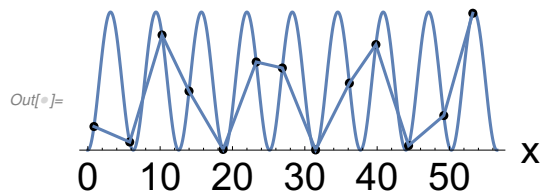
```
(* SP    energy=11.9538 detHess  -32.099 *)
node142 = {0.8319518704, 5.7805760734, 10.2474865543,
13.9814060298, 18.7032192804, 23.2792176166,
26.8949177884, 31.4923605908, 36.1661630465,
39.8406816344, 44.3566835595, 49.2383869938,
53.2642903631};
k = 1.0 ;
Setxx = node142;
SetAA = Table[A[i], {i, 13}];
SetXX = Setxx + SetAA;
as = 6.283185307180;
a0 = 4.188790204787;
(* The formula for the FK model for N=13 *)
VVV = Sum[1 - Cos[SetXX[[i]]], {i, 13}] +
  Sum[k/2. * (SetXX[[i+1]] - SetXX[[i]] - a0)^2, {i, 12}];
GGG = Table[D[VVV, A[i]], {i, 13}];
Set00 = Table[-0.172, {i, 13}];
Set22 = Table[0.172, {i, 13}];
SetAA00 = Riffle[SetAA, Set00];
SetA0 = Partition[SetAA00, 2];
SetAA22 = Riffle[SetA0, Set22];
SetFl = Flatten[SetAA22];
SetFin = Partition[SetFl, 3];
G2G2G2 = Sum[GGG[[i]]^2, {i, 13}];
NMinimize[G2G2G2, SetFin]
HHH = Table[D[GGG, A[j]], {j, 13}];

SetXX = {0.8521064899896589, 5.793565679951261, 10.264734631174466,
  13.991289384019682, 18.707222861378554, 23.281303372476522,
  26.89450593330393, 31.489529382077798, 36.15808923927823,
  39.827092301026916, 44.344842863487855, 49.21724904009374, 53.2231123445653}
```

```
In[ ]:= Eigenvalues[%]
```

```
Out[ ]:= {4.28539, 4.1492, 3.94906, 3.72276, 2.75699, 2.41166, 1.99139,
1.57178, 1.18469, 0.487213, -0.383713, 0.246304, 0.118387}
```

```
(* ENERGY 11.95710650261535 *)
```

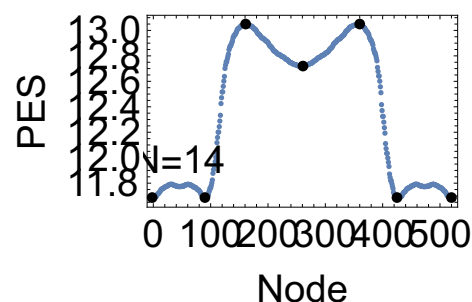
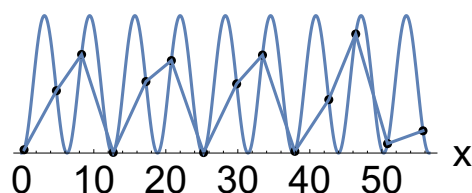


```
(* N=14 global minimum start a0 *)
```

```
SetXX = {0.31052366665005143, 4.804871171360356,
8.303492106719586, 12.702772964016688, 17.238033589666536,
20.774123404443383, 25.248286134278842, 29.837736841787503,
33.427214880111805, 37.921234902516616, 42.635555942915346,
46.37487250256273, 50.7951084635092, 55.72055504570824}
```

```
Eigenvalues[%]
```

```
{4.27508, 4.13549, 3.91745, 3.6954, 3.23064, 2.6271, 2.22691,
1.79906, 1.43538, 1.21508, 0.594005, 0.348732, 0.181493, 0.0958194}
```



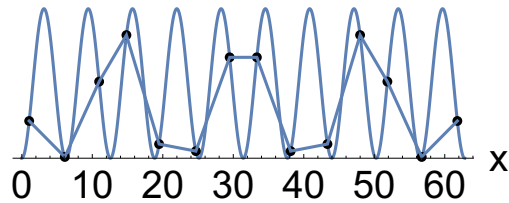
```
(* N=14 int minimum *)
```

```
ENERGY 12.638884121625996
```

```
SetXX = {1.0453908385398527, 6.099301677830099, 10.97036341917065,
14.841742937621973, 19.47500870799889, 24.693738884446866,
29.487432438331172, 33.34442063346469, 38.138114187349, 43.356844363796974,
47.99011013417388, 51.86148965262521, 56.732551393965764, 61.78646223325601}
```

```
Eigensystem[%]
```

```
{ {4.371215715433188, 4.245312502184315,
  3.9795543251689387, 3.941870846432025, 3.075252601850487,
  2.7051581141040275, 2.1771280795376353, 1.9349280611084927,
  1.4873676734428236, 0.9605062183502496, 0.9306416348505805,
  0.30293148735768965, 0.1563813728134431, 0.08683417356029356}, ...
```



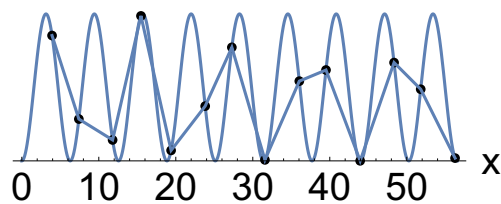
```
(* SP detHess -10.908 energy=13.043208 *)
node162 = {3.9381266191, 7.4168957113, 11.8016538439,
15.4940793226, 19.3987617379, 23.8254541563,
27.2866647062, 31.5826200655, 36.0444980482,
39.5098866588, 43.9466183573, 48.3476788321,
51.8083446114, 56.2686202571 };

SetXX = {3.9279515890547763, 7.4089559706747075, 11.792559970423916,
15.4772980905137, 19.390661344570766, 23.81910840689218, 27.28044017199012,
31.57992801218312, 36.04268313952589, 39.50910247891238,
43.94705099272133, 48.34976064620892, 51.81136964878551, 56.27266843488813}
```

Eigenvalues[%]

```
{4.25086, 4.10881, 3.90885, 3.65687, 3.18121, 2.49627, 2.17188,
1.87628, 1.50466, 0.692972, 0.394234, 0.205497, -0.197688, 0.0936039}
```

```
(* ENERGY 13.0434627405587 *)
```



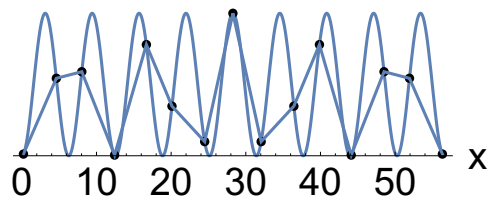
```
(* N=15 local minimum start a0 *)
```

ENERGY 12.992321413634606

```
SetXX = {0.21895691624366576, 4.624958682799388, 8.034780043793038,
12.428301845041144, 16.6841931261935, 20.11169298816606,
24.49193456821756, 28.274333882308976, 32.056733196399556,
36.43697477645088, 39.864474638423474, 44.12036591957546,
48.51388772082354, 51.9237090818171, 56.32971084837272}
```

Eigenvalues[%]

```
{4.23783, 4.13182, 3.92846, 3.66248, 3.12888, 3.08368, 2.38824, 2.06789,
 1.63282, 1.56423, 0.648015, 0.542706, 0.278316, 0.191923, 0.00215093}
```

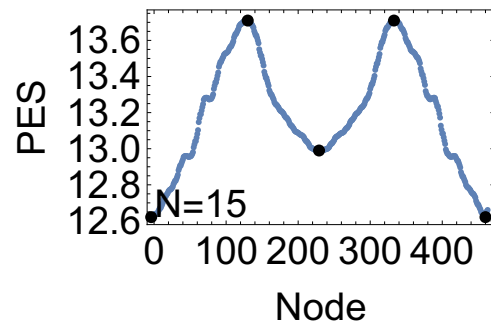


(\* NEw Min \*)

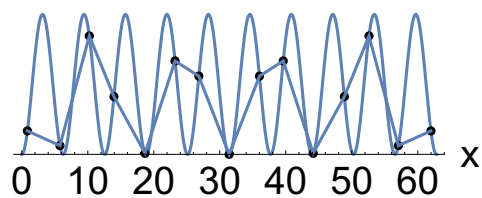
```
node230 = {0.8496522990, 5.7855693988, 10.2294078511,
13.9677208983, 18.6804152715, 23.2220707863,
26.8226144708, 31.4034337322, 36.0100709008,
39.6150939910, 44.1597299715, 48.8760445721,
52.6055765449, 57.0669649885, 62.0035362806};
```

ENERGY 12.639030890836644

```
SetXX = {0.843255522463437, 5.778857837586836, 10.231241403199965,
13.961780769673899, 18.67697936603188, 23.220456763843792,
26.821676814675886, 31.415926535897935, 36.01017625711999,
39.61139630795208, 44.15487370576399, 48.87007230212197,
52.6006116685959, 57.05299523420903, 61.988597549332425}
```



(\* glob Min \*)

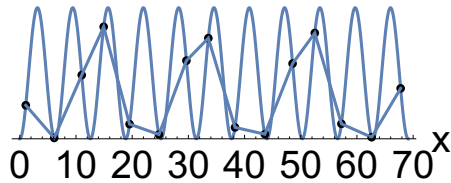


(\* N=15 ? minimum \*)

```
SetXX = {1.0606392095525503, 6.122097037827037, 11.023162385046916,
14.924608449090119, 19.531714827448955, 24.769291480639076,
29.651367472585374, 33.55215696135409, 38.29730245464852,
43.60559603363205, 48.546034718733665, 52.49750170953767,
57.23821056541024, 62.61510399473767, 67.7769415947914}
```

Eigensystem[%]

```
{ {4.396684694456103, 4.293324773508658, 4.19060060076055,
  3.9764073843602614, 3.1805798784741928, 2.863810144484446,
  2.532959022263621, 2.1117660970483945, 1.7183803819817243,
  1.3716467438582602, 0.9273755920303295, 0.7144781917051511,
  0.2591240279665751, 0.13703557669982036, 0.07557935187129156}, ... }
```



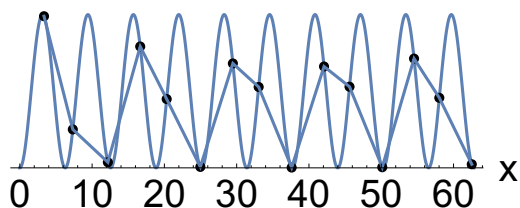
```
(* SP detHess -80.473 energy=13.70971 *)
node128 = {3.2817261482, 7.2967571041, 12.1605141973,
16.6294655397, 20.3019060537, 24.9673400144,
29.4681258923, 33.0391401430, 37.6087809226,
42.0882135767, 45.6194499975, 50.1484856321,
54.5607911726, 58.0588211141, 62.5550128336 };

SetXX = {3.348947132474862, 7.33186556531588, 12.181549803195997,
16.655840982290194, 20.317952985197262, 24.974826763124568,
29.474441575884814, 33.041978284480145, 37.60798880062566,
42.08300232728757, 45.611488025189736, 50.13826924589308,
54.53818009702061, 58.03320785789412, 62.52451785577087}
```

Eigenvalues[%]

```
{4.28584, 4.17837, 4.01737, 3.83011, 3.22738, 2.71303, 2.38684, 2.02557,
1.66802, 1.42208, 0.627724, 0.397173, -0.380592, 0.225766, 0.131645}
```

```
(* ENERGY 13.7094888353605 *)
```



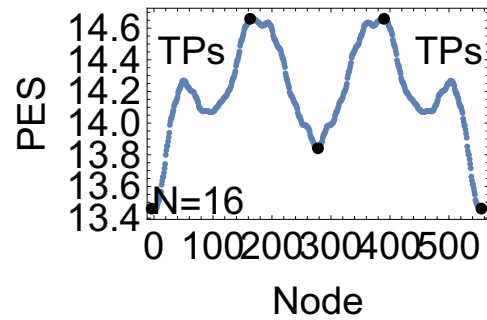
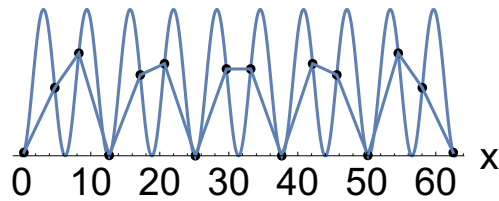
```
(* N=16 global minimum start a0 *)
Setxx = Table[a0 * (i - 1), {i, 16}]
```

ENERGY 13.458660887977139

```
SetXX = {0.2988583026438764, 4.782077816584061, 8.267724614882008,
12.668993827995324, 17.172706219767246, 20.682037001763117,
25.157323273259454, 29.65718911364469, 33.17466395815119,
37.674529798536426, 42.14981607003276, 45.65914685202863,
50.16285924380055, 54.56412845691386, 58.049775255211806, 62.53299476915199}
```

Eigenvalues[%]

```
{4.27951, 4.17077, 4.00222, 3.80096, 3.23059, 3.19863, 2.5776, 2.22445,
 1.85674, 1.49994, 1.41625, 0.718366, 0.507489, 0.323241, 0.195898, 0.129947}
```



```
(* N=16 int minimum *)
```

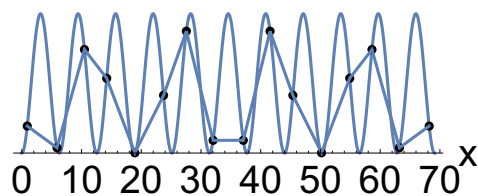
```
Setxx = {1.0606392095525503, 6.122097037827037, 11.023162385046916,
 14.924608449090119, 19.531714827448955, 24.769291480639076,
 29.651367472585374, 33.55215696135409, 38.29730245464852,
 43.60559603363205, 48.546034718733665, 52.49750170953767,
 57.23821056541024, 62.61510399473767, 67.7769415947914, 72.3};
```

```
ENERGY 13.841078454303204
```

```
SetXX = {0.9116634870314859, 5.890977358616636, 10.48806144933132,
 14.211189543198087, 18.931579171603076, 23.73390012209519,
 27.550969035368283, 32.02994864965849, 37.085087813955575,
 41.564065949012296, 45.38113615095428, 50.18345812468619, 54.90384771815711,
 58.626975886583196, 63.224060543674184, 68.20337449554611}
```

Eigensystem[%]

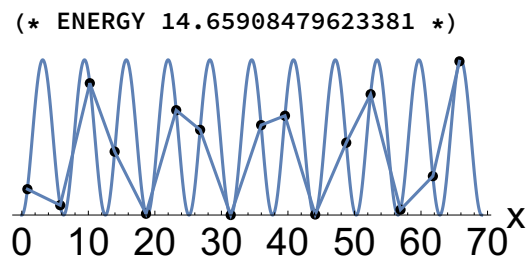
```
{{4.327433514442583, 4.209613042596356, 4.080437990694952, 3.8181863460630514,
 3.785260765810064, 2.914284350638373, 2.568556957145767,
 2.2512739991478288, 1.8065623136662707, 1.6051125105216872,
 1.1207366245733201, 1.0955037192922958, 0.36698750256548385,
 0.2685124156633272, 0.15606386516497128, 0.049553150136532054}, ...
```



```
(* SP detHess -87.52 energy=14.65891 *)
node166 = {0.8595375228, 5.7872973752, 10.2392444109,
13.9638318920, 18.6734344410, 23.2078246098,
26.8042628673, 31.3956326372, 35.9667099014,
39.5508169548, 44.0957279186, 48.7538265638,
52.4136734948, 56.9114080016, 61.7639799954,
65.7403746555 };

SetXX = {0.8406680996417462, 5.7745471898528775, 10.22143806115392,
13.953303768724526, 18.66831420720913, 23.203073557789146,
26.801539084498526, 31.395206303576455, 35.9681547729305,
39.553901581005576, 44.099592565475874, 48.76231018754752,
52.427313444936146, 56.92268129186927, 61.78340356941231, 65.7774751464371}

Eigenvalues[%]
{4.29687, 4.20457, 4.0588, 3.88246, 3.70561, 2.77576, 2.52755, 2.19767, 1.85447,
1.53817, 1.20048, 0.566821, -0.382158, 0.347394, 0.189966, 0.106749}
```



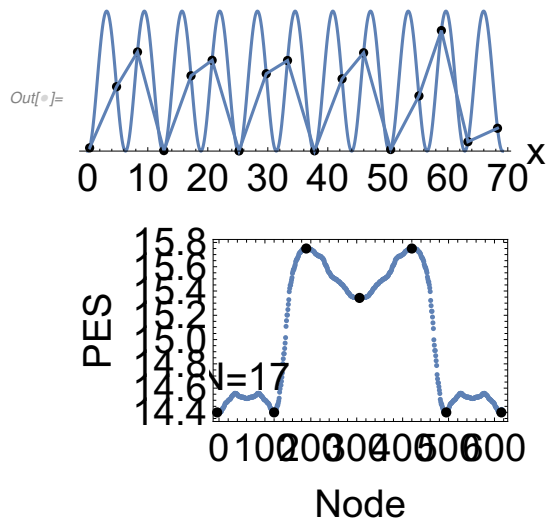
```
(* N=17 global minimum start a0 *)

ENERGY 14.400271988365342

SetXX = {0.3040402378547049, 4.7922080134875085, 8.283559637225704,
12.684052847821967, 17.201956847584047, 20.72280872885792,
25.1982682779649, 29.73920799310776, 33.285752223235086, 37.78791933598534,
42.37877725361058, 45.970170539471056, 50.47584072178482,
55.19032118416705, 58.92728433989826, 63.35532305538308, 68.28325022956797}
```

Eigenvalues[%]

```
{4.2902, 4.19593, 4.04485, 3.85324, 3.69286, 3.22239, 2.68717, 2.38443, 2.04186,
 1.70065, 1.43582, 1.21305, 0.658361, 0.44437, 0.274203, 0.159381, 0.0985685}
```



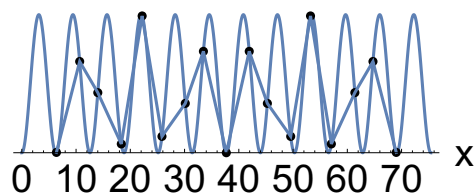
```
(* iMin below is at 15.197 ? → it is a parallel iMin *)
```

```
(* Find here iMin2 with the profile *)
```

```
node300 = {6.3175960767, 10.5536772449, 13.8858164012,
18.1865329673, 21.8717473760, 25.6760794759,
29.9974079409, 33.3303082726, 37.6047615249,
41.7850043810, 45.1551605893, 49.4471807951,
53.0092148667, 56.9586955456, 61.3068110311,
64.6559730612, 68.9732198756};
```

```
ENERGY 15.344977449704743
```

```
SetXX = {6.381106557259297, 10.667661599653439, 14.007499980971602,
18.338943374345156, 22.18167501525583, 25.83503082094946,
30.134353800512063, 33.47521117763972, 37.69911184307762, 41.92301250851561,
45.263869885643224, 49.563192865205856, 53.216548670899726,
57.05928031181013, 61.39072370518363, 64.73056208650176, 69.01711712889582}
```



```
(* N=17 iMin2 is compressed min, iMin1 is stretched min *)
```

```
(* N=17 NOT global minimum but iMin1 *)
```

```
Setxx = {0.9116634870314859, 5.890977358616636,
10.48806144933132, 14.211189543198087, 18.931579171603076,
23.73390012209519, 27.550969035368283, 32.02994864965849,
37.085087813955575, 41.564065949012296, 45.38113615095428,
50.18345812468619, 54.90384771815711, 58.626975886583196,
63.224060543674184, 68.20337449554611, 74.};
```

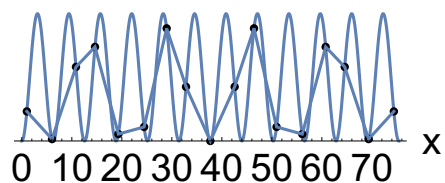


ENERGY 15.19719249238205

```
SetXX = {1.005793240475058, 6.039170347755231, 10.830946836061406,
  14.63624393925799, 19.31956556015155, 24.455781898866412,
  28.965572419004225, 32.837871133935096, 37.69911186714765,
  42.56035267515917, 46.43265146399108, 50.942442122716024, 56.07865860864467,
  60.7619807026667, 64.56727766863675, 69.3590538067699, 74.39243081475276}
```

Eigensystem[%]

```
{ {4.351848385144607, 4.280334091624835,
  4.117439196418111, 3.9235422243287315, 3.897714917447867,
  3.0180073759050634, 2.7804855207161325, 2.403458521484392,
  2.0119958431159946, 1.8324583836520916, 1.476540485943689,
  1.000803558408512, 0.9874268584238932, 0.3200785795191945,
  0.24447637436337755, 0.13350203636446328, 0.04425718971008478}, ... }
```



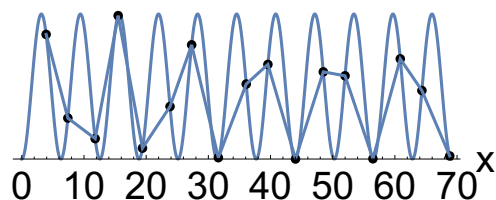
```
(* SP detHess -28.8 energy=15.7496 *)
node190 = {3.9148713557, 7.4074851856, 11.8020645124,
  15.5046075732, 19.4091076443, 23.8444140548,
  27.3193503202, 31.6105908313, 36.0952685243,
  39.5804922194, 44.0178711527, 48.4908165856,
  51.9844715361, 56.4671660214, 60.8684489633,
  64.3458174375, 68.8215714105 };
```

```
SetXX = {3.9104548203748197, 7.403927099594186, 11.79782275925142,
  15.496626424205559, 19.40519727685091, 23.841256409448825,
  27.316070062747293, 31.609078316012234, 36.094039582515116,
  39.57958821132773, 44.0175679673949, 48.49081122805912, 51.984765091035555,
  56.4677151025769, 60.86980084141023, 64.347455729683, 68.82358783404753}
```

Eigenvalues[%]

```
{4.27276, 4.17111, 4.02775, 3.84941, 3.64568, 3.20349, 2.57969, 2.31066, 2.0374,
  1.79462, 1.47154, 0.732645, 0.484553, 0.295547, -0.205488, 0.169054, 0.0871038}
```

```
(* ENERGY 15.749625528218 *)
```



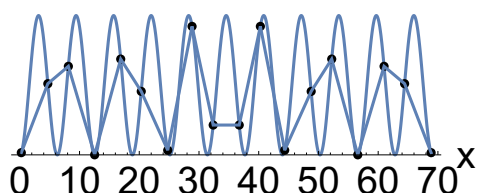
```
(* N=18 int minimum start a0 *)
```

ENERGY 15.67944532011809

```
SetXX = {0.25211446088309447, 4.690356937539036, 8.128842149323019,
  12.529790475941185, 16.89416693568626, 20.33159205654154,
  24.76508070892394, 28.83913613442725, 32.37794276876456, 36.73709553271063,
  40.27590194983638, 44.34995744450903, 48.783445982806, 52.22087096859984,
  56.58524749146999, 60.98619561117685, 64.42468090194703, 68.86292356335206}
```

Eigenvalues[%]

```
{4.25648, 4.18716, 4.03938, 3.86748, 3.6292,
  3.15924, 3.13792, 2.48033, 2.22328, 1.94672, 1.5661, 1.52918,
  0.689138, 0.608008, 0.351175, 0.248109, 0.155414, 0.0218526}
```

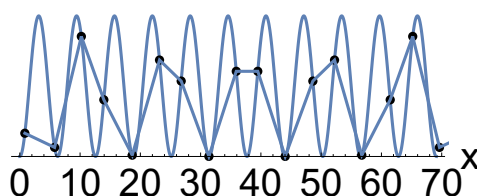
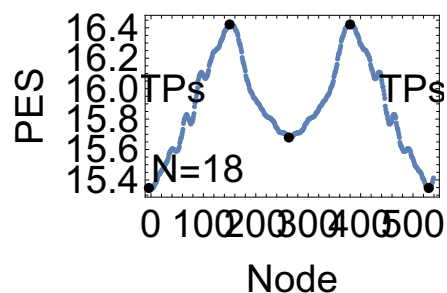


(\* new Min \*)

```
node265 = {0.7943597384, 5.7379107086, 10.1628087909,
  13.9148744368, 18.6423346761, 23.1640535248,
  26.7638922913, 31.3619102635, 35.9059382272,
  39.4745902936, 44.0223680046, 48.6102058473,
  52.2016100270, 56.7270196807, 61.4298372084,
  65.1468643775, 69.5995114456, 74.5179006437};
```

ENERGY 15.34679759329071

```
SetXX = {0.8373247876041916, 5.76896984505151, 10.208762982570493,
  13.942449286611, 18.65723793019716, 23.18089191090796,
  26.776272346738843, 31.369008773595183, 35.91484464941496,
  39.48337903674009, 44.029214912559866, 48.621951339416206,
  52.21733177524709, 56.74098575595788, 61.45577439954405,
  65.18946070358456, 69.62925384110353, 74.56089889855085}
```



GlobMin

(\* N=18 NOT global minimum \*)

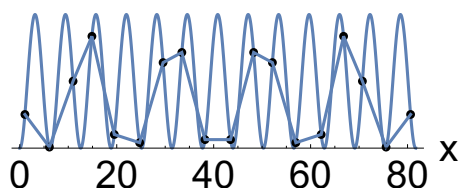
```
Setxx = {1.005793240475058, 6.039170347755231,
  10.830946836061406, 14.63624393925799, 19.31956556015155,
  24.455781898866412, 28.965572419004225, 32.837871133935096,
  37.69911186714765, 42.56035267515917, 46.43265146399108,
  50.942442122716024, 56.07865860864467, 60.7619807026667,
  64.56727766863675, 69.3590538067699, 74.39243081475276, 79.};
```

ENERGY 16.67637428914199

```
SetXX = {1.051922101071843, 6.109090527751039, 10.993042252390733,
  14.876997267371694, 19.499535150783473, 24.727242964100377,
  29.560474149613537, 33.433947200249804, 38.20907117179658,
  43.47233705401451, 48.24746045434879, 52.120933583489546,
  56.95416535841302, 62.181873206172995, 66.80441071357276,
  70.6883658950222, 75.57231809594131, 80.62948656477586}
```

Eigensystem[%]

```
{4.396024019217051, 4.315745470027977, 4.210788263997386,
  3.973588557645227, 3.960364856011498, 3.169950591395251,
  2.9116744262937395, 2.6183103804239902, 2.1646884271525817,
  2.0108721550020214, 1.6540595812373537, 1.3678054891836913, ...}
```



(\* SP detHess -234.2 energy=16.42248 \*)

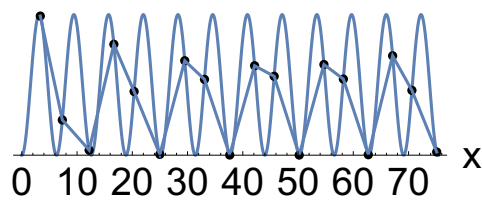
```
node153 = {3.3052174963, 7.3102476772, 12.1710608032,
  16.6467798042, 20.3156392629, 24.9790213246,
  29.4892881626, 33.0622015314, 37.6322677375,
  42.1355396052, 45.6766477356, 50.2101327437,
  54.6882962952, 58.2080945688, 62.7239677320,
  67.1321647178, 70.6240709038, 75.1140703125 };
```

```
SetXX = {3.3463211779975697, 7.33181002153847, 12.184036968244893,
  16.663177282885297, 20.325880258873987, 24.984124074431453,
  29.49429721780834, 33.06538360509676, 37.63337783107967, 42.13568537384801,
  45.67578947179474, 50.20837585444798, 54.68388666805237, 58.20230066458557,
  62.7172856736442, 67.11795374945268, 70.60811504218435, 75.09525768179688}
```

Eigenvalues[%]

```
{4.29641, 4.22015, 4.10053, 3.95229, 3.79357,
  3.22105, 2.74558, 2.49564, 2.20038, 1.89858, 1.60822, 1.42846,
  0.678933, 0.480379, -0.380961, 0.313018, 0.192364, 0.127467}
```

(\* ENERGY 16.42234711841684 \*)



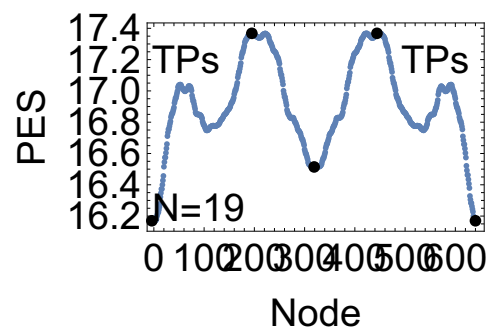
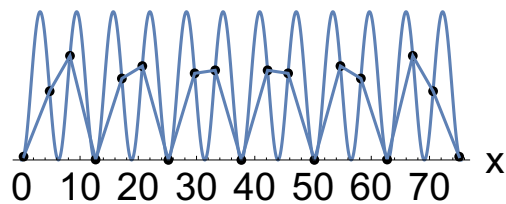
(\* N=19 global minimum start a0 \*)

ENERGY 16.1697565918655

SetXX = {0.2999629464267178, 4.784237959041587, 8.271092999335744,  
12.672211039772842, 17.178972008750442, 20.690707628881302,  
25.16611929335612, 29.67490282509036, 33.198140084755195,  
37.699111843077524, 42.200083601399854, 45.723320861064686,  
50.23210439279892, 54.707516057273736, 58.2192516774046,  
62.726012646382195, 67.1271306868193, 70.61398572711346, 75.09826073972832}

Eigenvalues[%]

{4.29138, 4.21383, 4.09177, 3.93431, 3.76975, 3.22382,  
3.20895, 2.64466, 2.36571, 2.06155, 1.76006, 1.47668, 1.42137,  
0.741155, 0.569859, 0.400299, 0.265566, 0.17361, 0.125491}



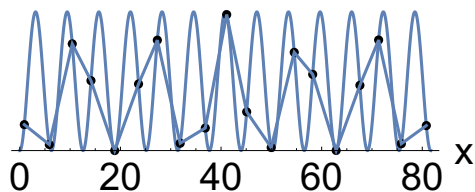
```
(* asymm iMin by node316 energy 16.51374 detHess 13.070664
? SP by node322 16.5139331 detHess -9.4701394354085178
the other asymm iMin is by node328 *)
node316 = {0.8849669681, 5.8487428816, 10.3916141062,
14.1113922675, 18.8308382807, 23.5315677460,
27.2327585602, 31.7971499914, 36.7335977941,
40.8477040808, 44.9548108405, 49.8882216939,
54.4532572984, 58.1527756273, 62.8517391784,
67.5705875255, 71.2897828945, 75.8320781364,
80.7947446388};
Setxx = node322;

ENERGY iMin 16.5137381196538

SetXX = {0.8949504720700667, 5.863918833039066, 10.425796649492808,
14.145653509459018, 18.865474363306372, 23.601213154008164,
27.337722863239264, 31.879787282922937, 36.86925579970363,
41.12089015516819, 45.095990443704025, 49.96842568349522,
54.54815370481437, 58.21879836235233, 62.88451334521037,
67.60286449845873, 71.32293344315661, 75.84682729312082, 80.80442874031928}
```

Eigensystem[%]

```
{{4.326517417753234, 4.2475863173167365, 4.130862284567683,
4.005381646561338, 3.7828632367473425, 3.7533198723664323,
2.910876393332114, 2.649210680737644, 2.369078545794972,
2.09206021057105, 1.716779515248826, 1.5692997967766136, ...
```

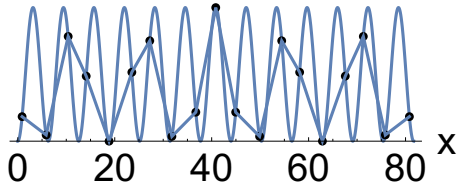


```
Setxx = node316;
```

```
SetXX = {0.8858152531327883, 5.849036479056216, 10.391619358450106,
14.111106192790265, 18.830253465046592, 23.53009947942216,
27.23045251641375, 31.795168105704597, 36.730099730497095,
40.84070449666728, 44.95130926283749, 49.88624088763, 54.45095647692085,
58.15130951391245, 62.851155528288025, 67.57030280054437,
71.28978963488453, 75.83237251427842, 80.79559374020184}
```

Eigensystem[%]

```
{ {4.325513346934483, 4.248036501458918, 4.12920854672572,
  4.007308496591662, 3.7774295839673466, 3.75845047727504,
  2.90868279663722, 2.650994848500223, 2.366877997292911, 2.09574108167885,
  1.7139906200202066, 1.572192846095176, 1.1470501939749824,
  1.1345468160309697, 0.4175332281220663, 0.3486192673342825,
  0.17655675031624973, 0.14231639380120864, -0.004636335224854626}, ...
```



```
(* N=19 NOT global minimum *)
```

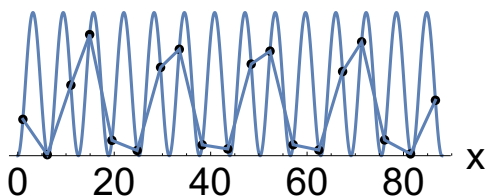
```
Setxx = {1.051922101071843, 6.109090527751039, 10.993042252390733,
  14.876997267371694, 19.499535150783473, 24.727242964100377,
  29.560474149613537, 33.433947200249804, 38.20907117179658,
  43.47233705401451, 48.24746045434879, 52.120933583489546,
  56.95416535841302, 62.181873206172995, 66.80441071357276,
  70.6883658950222, 75.57231809594131, 80.62948656477586, 85.};
```

```
ENERGY 18.3114264816786
```

```
SetXX = {1.0564689226206105, 6.11588306449547, 11.008774362156286,
  14.901752724048647, 19.516400261000122, 24.749557208402813,
  29.608838914241954, 33.49590790584618, 38.25611905491653,
  43.54497847626549, 48.41032587092346, 52.31583178896182,
  57.108538934223695, 62.432323267948625, 67.36712232515785,
  71.31756624012695, 76.07501622659268, 81.45876190681707, 86.62169560292172}
```

```
Eigenvalues[%]
```

```
{4.41245, 4.34575, 4.25323, 4.18571, 3.97204, 3.2368,
  3.01839, 2.74987, 2.50792, 2.11816, 1.82456, 1.53745, 1.29774,
  0.93155, 0.709675, 0.296132, 0.195058, 0.115339, 0.0782792}
```



```
highMin
```

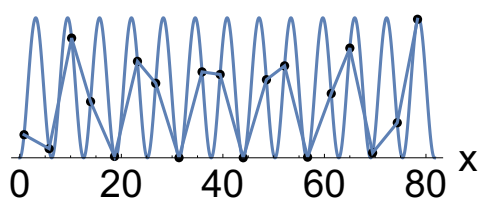
```
(* SP detHess -223.5 energy=17.3678877 *)
node195 = {0.8410132318, 5.7705477800, 10.2095694269,
13.9418445428, 18.6552296338, 23.1754926066,
26.7694435721, 31.3613333825, 35.8986527333,
39.4621509437, 44.0073453971, 48.5775894088,
52.1545731242, 56.6814569836, 61.3407580116,
65.0030731107, 69.4905683202, 74.3448533804,
78.3299642217 };
Setxx = node195;

SetXX = {0.8362837031775643, 5.767231401590475, 10.204814298088763,
13.939091943158669, 18.653816788810822, 23.174050028055344,
26.768575912858363, 31.36098754912644, 35.898487831438985,
39.46228244921904, 44.00763013088686, 48.57830808365616,
52.155750321624154, 56.682594081783904, 61.342964161830025,
65.00668677941411, 69.49345861474771, 74.349683386089, 78.33921216012361}
```

Eigenvalues[%]

```
{4.30337, 4.23755, 4.12884, 3.99042, 3.83657, 3.69751,
2.78517, 2.60097, 2.33531, 2.04898, 1.76355, 1.5206, 1.20695,
0.624684, 0.429727, -0.381531, 0.27272, 0.161832, 0.102397}
```

```
(* ENERGY 17.3678983996635 *)
```



```
(* N=20 int minimum *)
```

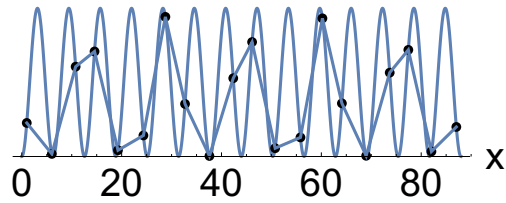
```
Setxx = {1.0564689226206105,
6.11588306449547, 11.008774362156286, 14.901752724048647,
19.516400261000122, 24.749557208402813, 29.608838914241954,
33.49590790584618, 38.25611905491653, 43.54497847626549,
48.41032587092346, 52.31583178896182, 57.108538934223695,
62.432323267948625, 67.36712232515785, 71.31756624012695,
76.07501622659268, 81.45876190681707, 86.62169560292172, 91.};
```

ENERGY 17.813147479307876

```
SetXX = {0.9919627720988959, 6.017854297568608, 10.78151711434355,
14.568002821377465, 19.263105506727968, 24.360070400635585,
28.75898512473716, 32.69200005374494, 37.58189767962685, 42.35484936171315,
46.129405307702626, 50.74243941641908, 55.814551405226894,
60.216732010624604, 64.11642674647877, 68.97543866214188,
73.69530384104249, 77.42388464150437, 82.05078595298225, 87.03872174875049}
```

Eigenvalues[%]

```
{4.34938, 4.28747, 4.18098, 4.05811, 3.89305, 3.81752, 3.00878,
 2.80277, 2.51936, 2.24152, 1.90177, 1.73283, 1.46668, 1.08965,
 1.01101, 0.361881, 0.27313, 0.222584, 0.0698858, 0.0240661}
```



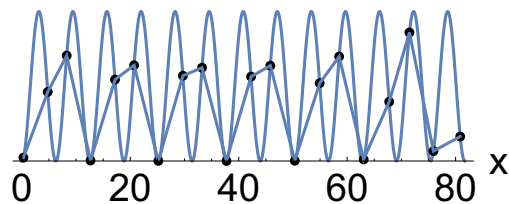
(\* N=20 global minimum \*)

ENERGY 17.11244725240905

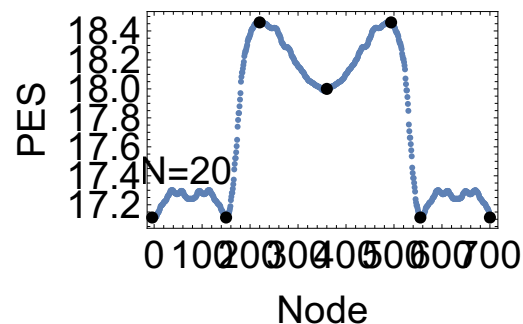
```
SetXX = {0.3018007102659365, 4.787830922325957, 8.276705410102506,
 12.677554992818353, 17.189360046606, 20.705158644015437, 25.18067326399918,
 29.704101573435114, 33.23745798746348, 37.739544808276584,
 42.2820537044092, 45.832929263556856, 50.34490548569288,
 54.936221451551965, 58.52840465712425, 63.038130491674885,
 67.75267423160793, 71.48886148991701, 75.91958094353838, 80.8483581333425}
```

Eigensystem[%]

```
{{4.2987568931719435, 4.230975070209038,
 4.120647534312439, 3.9749821968347563, 3.8110334893605518,
 3.6911327699997187, 3.2192377399657826,
 2.7220498868375484, 2.487374209689931, 2.2070889814894645,
 1.9163900469391906, 1.6394781998939822, 1.4364885006692671,
```



(\* stat points by NT - SP node74 - 78, node220 SP nach TP  
node357 is iMin and 636 ... \*)





```
(* iMin 18.0 ist 2. iMin the true one! energy=17.98625 *)
```

```
node358 = {6.4447525293, 10.7931241953, 14.1619189828,  
18.5304074542, 22.5851377839, 26.0801967294,  
30.3871884562, 33.8375315364, 37.9472500061,  
42.3025680505, 45.6638134046, 50.0189355668,  
54.1300009985, 57.5794848845, 61.8866881425,  
65.3832978713, 69.4363914753, 73.8053357587,  
77.1745240513, 81.5226705987};
```

```
Setxx = node358;
```

```
ENERGY 17.98566441423823
```

```
(* SP energy 18.459144744734658 detHess -78.1 *)
```

```
node221 = {3.9068904665, 7.4045365494, 11.8028709652,  
15.5097514880, 19.4135484502, 23.8519100581,  
27.3320178274, 31.6210475114, 36.1137627961,  
39.6065839699, 44.0432631760, 48.5408706480,  
52.0502844165, 56.5368862419, 61.0117068172,  
64.5174543450, 69.0166190201, 73.4175231470,  
76.9012682734, 81.3827191286};
```

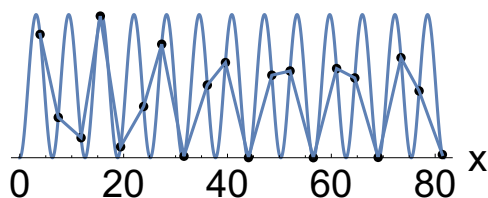
```
Setxx = node221;
```

```
SetXX = {3.9039120027373158, 7.402101469121865, 11.79991863988285,  
15.504152113691276, 19.410788651339516, 23.849655419667037,  
27.329626141934508, 31.619922574382677, 36.112803124759594,  
39.60580398986921, 44.04292026771263, 48.54062253653452, 52.05016914534049,  
56.53692828543379, 61.01194821598952, 64.51783555641897,  
69.01709630295552, 73.41857148573168, 76.90247038661369, 81.384155678273}
```

```
Eigenvalues[%]
```

```
{4.28619, 4.20986, 4.10162, 3.96441, 3.80893, 3.64134, 3.21222,  
2.63766, 2.40957, 2.17167, 1.9444, 1.72927, 1.45506, 0.755254,  
0.554197, 0.375344, 0.241148, -0.208391, 0.149779, 0.0846739}
```

```
(* ENERGY 18.4591459822515 *)
```



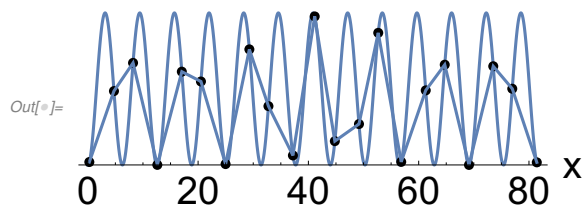
```
(* N=21 global minimum *)
```

```
ENERGY 18.379301293682634
```

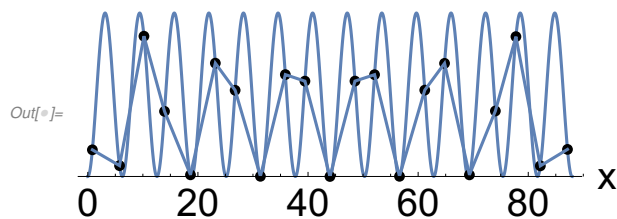
```
SetXX = {0.2776481439393259, 4.740532931702765, 8.20381383018079,
  12.606525111801544, 17.049380114000893, 20.518427468468115,
  24.98266943336167, 29.297402098979905, 32.7584249033583, 37.193500724096495,
  41.14423422543093, 44.79607713420319, 49.17480851235125, 52.66660148327359,
  56.833031983026245, 61.28000987214449, 64.72716741296388,
  69.1221294953277, 73.52418270584565, 76.97186330926134, 81.41953988870132}
```

```
In[*]:= Eigenvalues[%]
```

```
Out[*]:= {4.26879, 4.21974, 4.10927, 3.97745, 3.81948, 3.61541, 3.18461,
  3.16063, 2.54366, 2.34654, 2.08048, 1.86738, 1.53373, 1.49891,
  0.716041, 0.649553, 0.414613, 0.312778, 0.198887, 0.145717, 0.00712197}
```



```
(* gMin by 299 energy=18.057687 detHess 439.2 *)
SetXX = {0.8315389509, 5.7626785291, 10.1964982413,
  13.9329487361, 18.6486190646, 23.1647019675,
  26.7586537265, 31.3510869738, 35.8787260823,
  39.4373512751, 43.9819905941, 48.5263233570,
  52.0847956841, 56.6125462924, 61.2041319950,
  64.7973374735, 69.3136595036, 74.0272992864,
  77.7608470543, 82.1969411204, 87.1260330724};
```

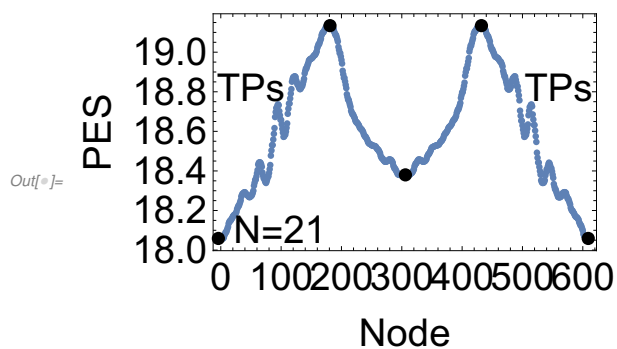


```
GlobMin
```

```

(* SP 181 energy=19.134379 detHess -642.65
   3.3401418542 7.3290227422 12.1832481750
  16.6636552524 20.3273490885 24.9867212471
  29.5005917736 33.0732308482 37.6421304399
  42.1540794587 45.6989787960 50.2332557296
  54.7353115134 58.2666410050 62.7871595107
  67.2629993331 70.7781278916 75.2890004564
  79.6908668289 83.1795406532 87.6655755621
   iMin 306 energy=18.3800554 detHess -14.43090
   6.5518285832 11.0083763029 14.4650059654
  18.8683760119 23.2905650377 26.7493521375
  31.2070899330 35.4575058077 38.9246030910
  43.3326726065 47.1358546536 50.9270721364
  55.3326615299 58.8005314934 63.0453025578
  67.5019059792 70.9594054382 75.3797172250
  79.7815236069 83.2369928526 87.6923463934
   SP 432 energy=19.13422618 detHess -629.33
   6.5765057381 11.0633924102 14.5525778499
  18.9567138586 23.4678028427 26.9833199187
  31.4599525553 35.9805969902 39.5121319816
  44.0144739406 48.5489871380 52.0940956703
  56.6061531136 61.1756642503 64.7488191112
  69.2626531880 73.9235665667 77.5890977605
  82.0684622345 86.9252879929 90.9200608927
   gMin 609 energy=18.05884066 detHess 439.72
   7.1250513005 12.0531670768 16.4903122461
  20.2225100176 24.9352007676 29.4516332972
  33.0444919047 37.6356823451 42.1634858116
  45.7218876716 50.2660775830 54.8108626199
  58.3695612445 62.8971482502 67.4899840470
  71.0842914482 75.6002597192 80.3168923558
  84.0547253122 88.4874550360 93.4195454622 *)

```



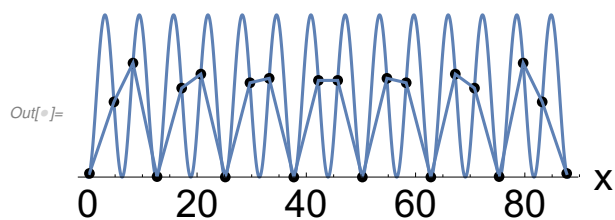
```
(* N=22 global minimum *)
k = 1.0 ;
as = 6.283185307180;
a0 = 4.188790204787;
Setxx = Table[a0 * (i - 1), {i, 22}];
```

```
ENERGY 18.8810840829232
```

```
SetXX = {0.3003590304519469, 4.785012417737669, 8.27230172802019,
12.673363689099304, 17.18121470753595, 20.693819457374026,
25.16926449878362, 29.68122469076857, 33.20658734708861,
37.70787701729747, 42.217931749491164, 45.74666255102305,
50.256717283216744, 54.75800695342561, 58.28336960974564,
62.79532980173059, 67.27077484314017, 70.78337959297825,
75.2912306114149, 79.69229257249401, 83.17958188277653, 87.66423527006226}
```

```
In[*]:= Eigenvalues[%]
```

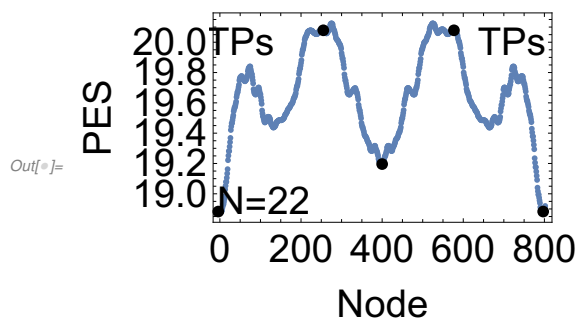
```
Out[*]:= {4.29876, 4.24065, 4.14853, 4.02658, 3.88463, 3.75004, 3.22049,
3.21357, 2.68735, 2.46284, 2.20776, 1.94619, 1.69441, 1.46318, 1.42522,
0.755857, 0.615924, 0.463494, 0.331515, 0.229025, 0.159457, 0.122629}
```



```

(* SP 255 energy=20.069532619 detHess -33.77
   0.2751706592 5.0574573746 8.8986917703
  13.2420787814 18.2109158234 22.5836487476
  26.3979448948 31.1659098551 35.6864546941
  39.3030420334 43.9190804964 48.4719444032
  52.0495128540 56.6044329321 61.2150892795
  64.8268019448 69.3499024387 74.1057137121
  77.8999973824 82.2913312707 87.2554689123
  91.5684363043
   iMin 400 energy=19.19606105 detHess 122.03
   0.8655845461 5.8162665726 10.3168116179
  14.0390065152 18.7563875450 23.3807349289
  27.0214559610 31.6120651542 36.3975577954
  40.2190777090 44.6229560211 49.6245581224
  54.0282236324 57.8499196085 62.6355078817
  67.2260100999 70.8667222154 75.4911187457
  80.2082767869 83.9302001788 88.4309555443
  93.3813500045
   SP 577 energy=20.0786900 detHess -628.7
   3.3557830383 7.3380084229 12.1900467773
  16.6745810836 20.3361463381 24.9941683693
  29.5140606049 33.0882576359 37.6573044387
  42.1845560148 45.7374492263 50.2733969922
  54.8172592101 58.3739918686 62.8985068310
  67.4896262096 71.0822366626 75.5973032886
  80.3101371105 84.0428099870 88.4788985234
  93.4069162599
   gMin 797 energy=18.88278299 detHess 1468.50
   6.5778044825 11.0633038797 14.5510960491
  18.9544357870 23.4624632201 26.9754348762
  31.4516696191 35.9636398350 39.4891384774
  43.9907022497 48.5006710224 52.0294017624
  56.5395421475 61.0405570421 64.5657836592
  69.0777333666 73.5523867138 77.0646259014
  81.5722983261 85.9710764509 89.4578787449
  93.9416784967 *)

```



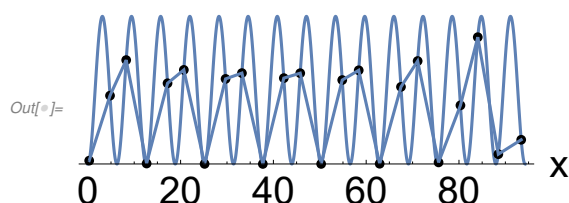
(\* N=23 high Min ? \*)

ENERGY 19.824158448638084

```
SetXX = {0.3010130229515318, 4.786291060390571, 8.27429861392992,
12.675265810021605, 17.1849131131682, 20.698960766394503,
25.174447471293057, 29.69162832910392, 33.220567485040895,
37.72228935472911, 42.24718666097617, 45.78555319346769,
50.297022416033904, 54.8400263682953, 58.392515918011284,
62.90795785915637, 67.49943114322446, 71.09190826807892, 75.60306425684115,
80.3176313119007, 84.053550306866, 88.48521054687437, 93.41428562075589}
```

```
In[ ]:= Eigenvalues[%]
```

```
Out[ ]:= {4.30405, 4.25307, 4.16916, 4.05635, 3.92169, 3.78251, 3.68995, 3.21806,
2.74363, 2.55805, 2.32496, 2.07618, 1.82748, 1.59965, 1.43699, 1.21152,
0.725098, 0.569524, 0.420835, 0.296105, 0.200489, 0.137273, 0.0997207}
```



```
(* SP node95 energy=19.9654207 detHess -72.6
0.3118294095 4.8213633706 8.3368291668
12.7379713601 17.3098733556 20.8822593452
25.3498495513 30.0328464913 33.7334104570
38.1679171374 43.0542446187 47.1401179412
51.2097638387 56.0894855348 60.5259920374
64.2207111803 68.8989251697 73.3627042840
76.9325376877 81.5017056852 85.8921360155
89.4057817807 93.9110400707
iMin node185 energy=19.82540989 detHess 1426.6
0.8406990175 5.7676721275 10.2016638369
13.9345934062 18.6470750260 23.1584564861
26.7501410065 31.3407397825 35.8562226242
39.4084949198 43.9511794323 48.4627512484
52.0010996012 56.5258957250 61.0279217771
64.5570009197 69.0741914978 73.5505465525
77.0649888388 81.5748296507 85.9782927808
89.4668433957 93.9530455691
SP node254 energy=21.17012183 detHess -227.9
3.8929896114 7.3945038008 11.7923021805
15.4910503188 19.4050144001 23.8463113875
27.3277687975 31.6206388856 36.1167944980
39.6130164766 44.0509519885 48.5574884174
52.0734216987 56.5613681269 61.0620145758
64.5824046048 69.0866821235 73.5626071866
77.0733926660 81.5787362648 85.9815874306
89.4682011567 93.9525585220
iMin node413 energy=20.65604589 detHess 19.66
6.4675105045 10.8436844127 14.2313714279
18.6146244810 22.7651012242 26.2166105249
```

30.5518947350 34.1267120509 38.1191336948  
 42.5193357443 45.9253463469 50.2628672573  
 54.5977729706 58.0040505763 62.4036754319  
 66.3880866392 69.9696365477 74.3054936224  
 77.7534644737 81.9092161541 86.2908097179  
 89.6777018862 94.0544848860

SP node573 energy=21.16990132 detHess -267.13

6.5827480145 11.0664391795 14.5526402102  
 18.9537666456 23.4589152883 26.9693667866  
 31.4446932737 35.9487825312 39.4689445938  
 43.9693642123 48.4568512534 51.9724879405  
 56.4788624938 60.9154884533 64.4112313044  
 68.9069373315 73.1960410676 76.6779347975  
 81.1177615190 85.0233152416 88.7298922995  
 93.1292212966 96.6290789044

gMin node642 energy=19.8247609 detHess 1288.3

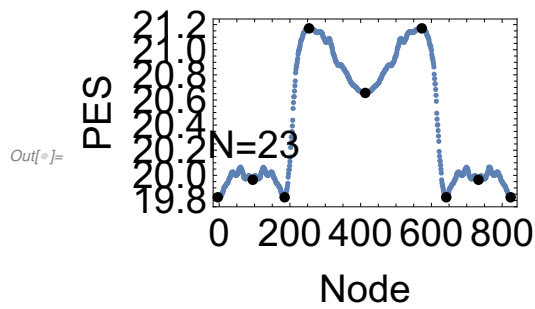
6.5757755419 11.0622912401 14.5510316888  
 18.9553455906 23.4652519484 26.9798294287  
 31.4564804994 35.9736744185 39.5028014632  
 44.0049312225 48.5296931216 52.0680355171  
 56.5796423098 61.1222186949 64.6744176102  
 69.1899142779 73.7802169000 77.3716337289  
 81.8830905779 86.5948645408 90.3267843775  
 94.7615587915 99.6878052204

SP node733 energy=19.964862 detHess -72.70

6.6174082999 11.1231030729 14.6369186264  
 19.0284357632 23.5978803007 27.1679704451  
 31.6321363166 36.3108313872 40.0061362934  
 44.4424478300 49.3228424854 53.3941248324  
 57.4783570961 62.3640235337 66.7987398663  
 70.4987088570 75.1812208351 79.6484290759  
 83.2205563555 87.7921828485 92.1922507962  
 95.7075191086 100.2166269474

gMin node824 energy=19.825269609 detHess 1466.9

7.1119802085 12.0424049898 16.4725121044  
 20.2104077522 24.9263458837 29.4373509260  
 33.0303522091 37.6224018833 42.1378168079  
 45.6904500168 50.2336646383 54.7450668091  
 58.2834458058 62.8084099021 67.3099329760  
 70.8387810640 75.3559556405 79.8308747561  
 83.3446660365 87.8541858692 92.2535214474  
 95.7411851527 100.2258556857 \*)



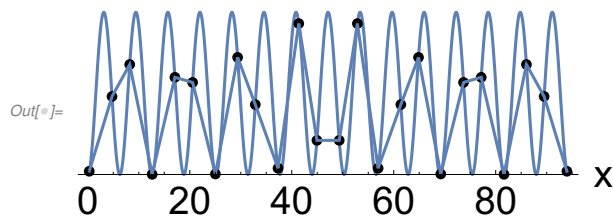
(\* N=24 global minimum \*)

ENERGY 21.081332957673204

```
SetXX = {0.2826275153555484, 4.750297544858847, 8.218685910842531,
  12.621303685875642, 17.078826811601534, 20.556270216465897,
  25.02449116736468, 29.38467319243669, 32.84900574719804, 37.30387001438159,
  41.373703008447755, 44.935417842068304, 49.31235948689404,
  52.874073254654135, 56.943908011670544, 61.39877266566901,
  64.86310510415498, 69.22328768534719, 73.69150842258675, 77.16895198914246,
  81.62647552701674, 86.02909325624334, 89.4974817792414, 93.96515195303873}
```

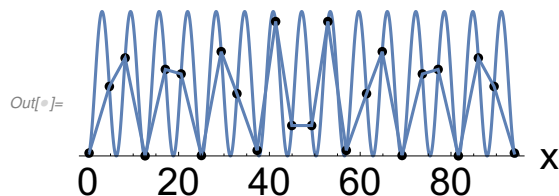
In[ ]:= Eigenvalues [%]

```
Out[ ]:= {4.27688, 4.24327, 4.15346, 4.0547, 3.92285, 3.78926, 3.60464, 3.19295, 3.18828,
  2.58423, 2.43727, 2.20634, 1.97715, 1.8145, 1.49754, 1.48598, 0.733674,
  0.686974, 0.460326, 0.379618, 0.248658, 0.182229, 0.133049, 0.0137329}
```



(\* gMin node354 energy=20.76879041 detHess 1220. \*)

```
SetXX = {0.8329566722, 5.7628777893, 10.1956519219,
  13.9316636549, 18.6466337966, 23.1600715825,
  26.7531794927, 31.3450554952, 35.8661197694,
  39.4213608765, 43.9651549286, 48.4918075985,
  52.0389696256, 56.5656600955, 61.1093420785,
  64.6645106735, 69.1855890763, 73.7771596644,
  77.3699934064, 81.8835086505, 86.5977505898,
  90.3327188675, 94.7663079708, 99.6954995598};
```





```

(* SP node216 energy=21.8458415detHess -1857.3
   3.3451927542 7.3318462898 12.1852560173
16.6667104191 20.3296924292 24.9885676519
29.5037682448 33.0766692439 37.6455277938
42.1608279330 45.7073823455 50.2420578235
54.7533108099 58.2896717565 62.8115823155
67.3132235065 70.8414310130 75.3575578167
79.8330299583 83.3467814378 87.8560639289
92.2570289834 95.7448864628 100.2300301015
   iMin node 354 energy=21.082405272 detHess 129.69
   6.5696485276 11.0368206463 14.5048432698
18.9060309995 23.3636637915 26.8408899629
31.3086981128 35.6694832058 39.1336980185
43.5886505631 47.6600444660 51.2206043484
55.5975483716 59.1604262669 63.2286944026
67.6834696161 71.1479213402 75.5074972551
79.9761294013 83.4537908011 87.9112025606
92.3152479436 95.7840092264 100.2521731400
   SP node493 energy=21.8462390962 detHess -1747.2385
   6.5826672662 11.0680278573 14.5560120604
18.9575555094 23.4668887191 26.9807363669
31.4564098058 35.9725554576 39.5008072019
44.0025199887 48.5244542355 52.0608429744
56.5721229598 61.1068559898 64.6534544172
69.1687657875 73.7377787215 77.3108074041
81.8259931747 86.4852599075 90.1487063681
94.6298968643 99.4839732184 103.4721271145
   gMin node705 energy=20.7694519457 detHess 1200.75
   7.1234591118 12.0512393200 16.4863704190
20.2193553670 24.9322083592 29.4458698012
33.0381824895 37.6291714301 42.1502769646
45.7053078749 50.2487765257 54.7755400219
58.3227027840 62.8492835570 67.3932939324
70.9486753596 75.4697125905 80.0621778488
83.6558158434 88.1691028278 92.8854757658
96.6235046439 101.0546961438 105.9860023475 *)

```

