Toric Varieties

Oberseminar Algebraische Geometrie
im Sommersemester 07

Toric varieties form a class of algebraic varieties that plays an important role in various areas both of algebraic and analytic geometry, including arithmetic and birational geometry, the study of singularities, topological mirror symmetry, and computational aspects. As the geometry of these varieties is controlled by simple combinatorial data, they form a perfect testing ground for general conjectures and an inexhaustible source of explicit examples.

We will begin with a thorough introduction to the theory of toric varieties, preparing the ground for some interesting recent applications that we will study in the second half of the seminar.

Potential participants are cordially invited to join our inaugurating meeting on

Tuesday, April 3, 2007
3.30 p.m. s.t.
Felix-Klein-Hörsaal
Mathematisches Institut
Johannisgasse 26

The covered material is also largely accessible to undergraduate students with a fair background in algebraic geometry.

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Talks

Talk 1: Cones, Fans und Polytopes

In this talk the lecturer should introduce the elementary results of convex geometry that we need to construct toric varieties. In particular, he should cover [Ful93], Chap. 1.2 and Chap. 1.5 and give the definition of a fan. Some examples and pictures are certainly helpful.

Talk 2: The Construction of Toric Varieties

Explain how one can associate a variety to a fan. The construction is contained in [Ful93], Chap. 1.3 – 1.4 and [Ewa96], Chap. VI.2 - VI.3, respectively. The necessary concepts from algebraic geometry may be assumed to be known. Again, some examples will be more than welcome.

Talk 3: The Torus Operation

This talk should cover [Bra06], Chap. 4.1 – 4.2. It contains a definition of the torus operation and the description of its orbits. One can find additional details in [Ewa96], Chap. VI.5 and [Ful93] Chap. 2.1, 3.1.

Talk 4: Torus Embeddings, Smoothness and Properness

The aim is to give complete proofs both of [Bra06], Theorem 3.2, Parts 1 – 2, i.e. of the characterisations of the smooth and the proper toric varieties, and of Theorem 4.3, which establishes the connection to the classical definition of toric varieties as torus embeddings. The details of the proofs can be found in [Bra06], Chap. 4.3 and [Ful93], Chap. 2.3 – 2.4.

Talk 5: Resolution of Singularities

For toric varieties, the process of resolving singularities is much easier than in the general case. The appropriate algorithm is described in [Ful93], Chap. 2.6, [Bra06], Chap. 6 and [Ewa96], Chap. VI.8. At the end, there is certainly enough time for some instructive examples.

Talk 6: Divisors and Projective Embeddings

The topic of this talk is the content of [Ewa96], Chap. VII.2 - VII.4, i.e. the different descriptions of the Picard group and the characterisation of projective toric varieties. Consult also the material of [Voi99], Chap. 4.2, which will be used later in Talks 10 – 12. The standard definitions from [Har77], Chap. II.6 may be assumed to be known.

Talk 7: Cohomology of Toric Varieties

The first part of the talk is devoted to the calculation of the cohomology of line bundles, as in [Ful93], Chap. 3.5. In a second part the lecturer should treat the cohomology ring of a smooth and projective toric variety by sketching the relevant material from [Ful93], Chap. 5.2. The generalisations to simplicial toric varieties can be omitted. The same topics are also contained in [Ewa96], Chap. VIII.2, VIII.4.

Talk 8: Bézout's Theorem for Toric Varieties

See [Ful93], Chap. 5.

Talk 9: Toric Resultants

See [CLO98], Chap. 7.

Talk 10: Mirror Symmetry

Give a short introduction to mirror symmetry, e.g. from the introduction of [Voi99] or from the first chapter of [CK99]. (We need the version for Calabi-Yau manifolds and mirror Hodge numbers). Present [Voi99], Chap. 4.2 on toric Fano varieties (see also [CK99], Chap. 3.5). State [Voi99], Theorem 4.30, respectively [CK99], Theorem 4.1.5 without explaining the terms in detail.

Talk 11: The results of Danilov and Khovanskii

Treat [Voi99], Chap. 4, Section 6.1; see also the original article [Bat94] and [DK87]. This is a computation of Hodge numbers of hypersurfaces in complex tori. The arguments assume familiarity with mixed Hodge structures. Present as little of the general theory as possible.

Talk 12: Proof of Batyrev's Theorem

This talk is about [Voi99] Chap. 4, Section 5 (Batyrev's desingularisation) and Section 6.2 (computation of Hodge numbers of hypersurfaces in the desingularisation). See also [Bat94],[CK99], Chap. 4.1. The classical example of [CK99], Chap. 4.2 should at least be formulated.

References

[Bat94] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau surfaces*, J. Alg. Geom. (1994), no. 3, 493–535.

[Bra06] Jean-Paul Brasselet, *Introduction to toric varieties*, online at `http://www.matcuer.unam.mx/~singularities/Notas/Brasselet.pdf`, 2006.

[CK99] David A. Cox and Sheldon Katz, *Mirror symmetry and algebraic geometry*, Math. Surveys and Monographs, no. 68, American Mathematical Society, Providence, RI, 1999.

[CLO98] David A. Cox, John Little, and Donal O'Shea, *Using algebraic geometry*, Graduate Texts in Mathematics, no. 185, Springer-Verlag, New York, 1998.

[DK87] V.I. Danilov and A.G. Khovanskii, *Newton polyhedra and an algorithm for computing Hodge-Deligne numbers*, Math. USSR (1987), no. 29, 279–298.

[Ewa96] Günter Ewald, *Combinatorial convexity and algebraic geometry*, Graduate Texts in Mathematics, no. 168, Springer-Verlag, New York, 1996.

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, no. 131, Princeton University Press, Princeton, New Jersey, 1993.

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52, Springer-Verlag, New York, 1977.

[Oda88] Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3. Folge), no. 15, Springer-Verlag, Berlin - Heidelberg, 1988.

[Voi99] Claire Voisin, *Mirror symmetry*, SMF/AMS Texts and Monographs, no. 1, American Mathematical Society, Providence, RI, 1999.