Series 1

1. Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{C}$ and define for a linear operator $A \in \mathrm{Hom}(V,V)$

$$\|A\|_{op} := \sup_{v \in V \setminus \{0\}} \frac{\|Av\|}{\|v\|} \in [0,\infty) \cup \{\infty\}\,.$$

Let $\mathcal{L}(V) = \{\, A \in \mathrm{Hom}(V,V) \mid \|A\|_{op} < \infty\}$, i.e. $(\mathcal{L}(V), \|\cdot\|_{op})$ is again a normed vector space. We assume $(V, \|\cdot\|)$ to be complete. One can show that then $(\mathcal{L}(V), \|\cdot\|_{op})$ is complete, too.

(a) Show $\|A\|_{op} = \sup_{\|v\|=1} \|Av\|$ and $\|A \cdot B\|_{op} \leq \|A\|_{op} \cdot \|B\|_{op}$ f.a. $A, B \in \mathcal{L}(V)$, where $A \cdot B$ denotes the composition of operators. *(1 pt)*

(b) Show that if $\|A\|_{op} < 1$, then $\mathbf{1} - A$ is invertible, and that $A + B$ is invertible if A is invertible and $\|B\| < \|A^{-1}\|^{-1}$. *(2 pts)*

(c) Show that in general $\|A^{-1}\| \neq \|A\|^{-1}$. *(1 pt)*

Background: If $\sum_{n=0}^{\infty} a_n r^n$ is a convergent series for $r > 0$ and $(a_n)_{n \in N} \in \mathbb{C}$ and if $\|A\| \leq r$, then $\sum_{n=0}^{\infty} a_n A^n$ converges in $(\mathcal{L}(V), \|\cdot\|_{op})$.
Proof: Since $(\mathcal{L}(V), \|\cdot\|)$ is a complete normed space, it suffices to show that $(\sum_{n=0}^{N} a_n A^n)_{N \in \mathbb{N}}$ is a Cauchy sequence. But this follows from the convergence from $(\sum_{n=0}^{N} a_n r^n)_{N \in N}$ as $\| \sum_{n=m}^{m+k} a_n A^n \| \leq \sum_{n=m}^{m+k} a_n r^n \to 0$ for $m \to \infty$.

2. Let X be a topological space. Then a subset $A \subset X$ is called **dense** if for all $x \in X$ and $U \subset X$ open with $x \in U$ we have $U \cap A \neq \emptyset$.

(a) Show that $GL(n, \mathbb{R}) \subset M(n \times n, \mathbb{R})$ is an open and dense subset. *(2 pts)*

(b) Let $f\colon M(n \times n, \mathbb{R}) \to \mathbb{R}$ be a continuous map. Show

$$\begin{aligned} & f(CAC^{-1}) = f(A) && \text{f.a. } A \in M(n \times n, \mathbb{R}),\ C \in GL(n, \mathbb{R}) \\ \Leftrightarrow\quad & f(AB) = f(BA) && \text{f.a. } A, B \in M(n \times n, \mathbb{R})\,, \end{aligned}$$

and that $\mathrm{tr}(CAC^{-1}) = \mathrm{tr}(A)$ for all $A \in M(n \times n, \mathbb{R})$, $C \in GL(n, \mathbb{R})$. *(1 pt)*

(c) Show that $O(n, \mathbb{R})$ is a compact space. *(1 pt)*

3. A topological space X is called **path-connected** if for any two points $x, y \in X$ there exists a continuous path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

(a) Show that if two topological spaces X and Y are homeomorphic $X \simeq Y$, then one of them is path-connected if and only if the other one is too. *(1 pt)*

(b) Show that $\mathbb{R}^2$ and $\mathbb{R}$ are not homeomorphic. *(1 pt)*

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(c) Consider $S^2 = \{ (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1 \}$ and show that $S^2 \setminus \{(0, 0, 1)\}$ and $\mathbb{R}^2$ are homeomorphic. *(1 pt)*

(d) Let $f\colon [0, 1] \to [0, 1]$ be a C^1-function, i.e. continuous, differentiable on $(0, 1)$ and f' continuous on $[0, 1]$. Assume $|f'| < 1$ on $[0, 1]$. Show that f is a contraction and has a unique fixed point in $[0, 1]$. *(1 pt)*

4. Consider the following map $\gamma\colon (0, 1) \to \mathbb{R}^2$, defined piecewise

$$\gamma(t) \;=\; \begin{cases} (0, 1-6t), & 0 < t \le \frac{1}{3}, \\ \big(\sin\frac{\pi}{2}(3t-1), -\cos\frac{\pi}{2}(3t-1)\big), & \frac{1}{3} \le t \le \frac{2}{3}, \\ (3-3t, 0), & \frac{2}{3} \le t < 1\,, \end{cases}$$

and its image $C = \gamma((0, 1))$. Let C carry the topology induced as subset of $\mathbb{R}^2$, i.e. the open subsets of C are the intersections of open subsets of $\mathbb{R}^2$ with C. *Show* that $\gamma\colon (0, 1) \to C$ is a bijection (1-1 correspondence) and continuous, but that it is not a homeomorphism. *(4 pts)*

5. Optional: Consider the vector space of twice differentiable paths $V = C^2([a, b], \mathbb{R}^n)$ and the function

$$S\colon V \to \mathbb{R}, \quad S(\gamma) = \frac{1}{2}\int_a^b |\dot\gamma(s)|^2 ds\,.$$

Show: If all directional derivatives $\partial_v S(\gamma) = 0$ where $v \in V$ with $v(a) = v(b) = 0$, then $\ddot\gamma = 0$. *(2 pts)*
Definition to be covered in class: Let V be a vector space, $f\colon V \to \mathbb{R}$ a function. Then the directional derivative of f in the direction $h \in V$ at a point $x \in V$ is defined as

$$\partial_h f(x) \;:=\; \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}\,,$$

provided that the limit exists.

Hand-In: Practice Session Wednesday Oct. 23