A TREE-BASED ANALYSIS OF A FAMILY OF AUGMENTED SYSTEMS FOR THE COMPUTATION OF SINGULAR POINTS

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Abstract. The main problem of designing augmented systems or minimally extended systems for the numerical computation of singular points is to find appropriate conditions depending on the type of singular point such that the singular point becomes a regular solution of the inflated system. For the reduced problem, i.e. for the problem after having applied the Lyapunov-Schmidt reduction, this task is solved in singularity theory in form of recognition theorems. In the present paper, we show that there is a constructive transcription rule which allows to carry over these results to the original problem thus giving an explicit way how one can design augmented and minimally extended systems.

Key words. Singular points, augmented systems, minimally extended systems, Lyapunov-Schmidt reduction, contact equivalence, regularity condition, trees, elementary differentials, transcription rule.

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1. Introduction. In the last decade, great effort was laid in the development of augmented systems for the computation of singular points of a system of nonlinear equations

$$f(x,\tau,\alpha)=0,\quad f\colon D\to\mathbb{R}^n,\quad D\subseteq\mathbb{R}^{n+l}\times\mathbb{R}^p\times\mathbb{R}^q \text{ open},\quad l,p,q\ge 0, \tag{1}$$

i.e. of points $(x^*,\tau^*,\alpha^*)\in D$ with

$$\operatorname{rank} L = n-k,\quad L=f_x(x^*,\tau^*,\alpha^*),\quad k\ge 1. \tag{2}$$

Here one must distinguish between the state variables $x\in\mathbb{R}^{n+l}$ and parameter variables $\lambda=(\tau,\alpha)\in\mathbb{R}^{p+q}$. The further separation of the parameters will become necessary later on when defining classes of singular points. Since all investigations concerning singular points are of purely local nature, the objects that are treated are rather germs than functions. We use here both terms synonymously and omit formulations restricting statements to neighbourhoods of the points under consideration where we think they are obvious. Moreover, we must assume all functions to be infinitely differentiable (cp. [8]).

Trying to compute a solution with singular Jacobian say by Newton or Gauss-Newton method one may be faced with the problem that we generally have at most linear convergence or not even local convergence, i.e. the iteration may diverge even for an initial value close to the solution. Therefore a standard numerical approach to such problems is the construction of augmented systems such that the singular point becomes part of a regular solution. In singularity theory this is done for the case of fully singular problems, i.e. for the case $n=k$, in the context of deciding if a given function belongs to a certain equivalence class with respect to the so-called contact equivalence (see [8]). In order to generalize these results to (1), a lot of work was done to develop augmented system (see e.g. [4, 21, 22, 27, 26, 29, 31]) and so-called minimally extended systems (see e.g. [10, 12, 23, 24, 25]) where one wants to use as few

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additional equations as possible. Also more general concepts were developed to obtain a better understanding of the principles involved in the construction of augmented systems (see e.g. [1, 3, 5, 6, 13, 16, 20]). For a more detailed overview, see [18].

It is the aim of the present paper to develop a general principle for the construction of augmented and minimally extended systems. In particular it will be shown that many of the above mentioned approaches are in principle covered by this theory. The present paper is organized as follows. We introduce the basic concepts and results of singularity theory and the notation we use in Section 2. In Section 3 we then sketch the fundamental properties of a family of augmented systems similar to that introduced in [16]. On this basis, we develop a tree-based concept for the study of singular points in Section 4 and use this concept for the investigation of the augmented systems under consideration in Section 5. We then apply the obtained results to a number of simpler singularities in Section 6 and draw finally some conclusions in Section 7.

2. Basic concepts and notations. In the following, we sketch the two basic concepts in analyzing singular points, namely Lyapunov-Schmidt reduction and contact equivalence (see e. g. [8]).

The aim of the Lyapunov-Schmidt reduction is to eliminate the regular part of (1) in a neighbourhood of (x^*, λ^*) by means of the implicit function theorem. For this, let

$$U = (\overline{z}, z^*), \quad V = (\overline{t}, t^*) \tag{3}$$

be nonsingular matrices split according to the rank drop k of L such that

$$U^T L V = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \tag{4}$$

with Σ nonsingular. In particular, the columns of z^* and t^* form bases of corange L and kernel L, respectively. For convenience we assume that the columns $\overline{z}$ and $\overline{t}$ are orthonormalized, i.e. that $\overline{z}^T\overline{z} = I$, $\overline{t}^T\overline{t} = I$. Of course this can always be attained.

We could now use these splittings to separate the regular part of (1). This would correspond to the use of the Moore-Penrose pseudoinverse L^+ of L. For the discussion to follow this would be too restrictive and would lead to weaker results, cp. [18]. We therefore want to use a more general splitting belonging to a more general type of a generalized inverse L^- of L.

Lemma 2.1. *Choosing matrices $\tilde{z}$ and $\tilde{t}$ of the same size as z^* and t^* satisfying the transversality condition*

$$\tilde{z}^T z^*,\ \tilde{t}^T t^* \ \textit{nonsingular}, \tag{5}$$

the relation

$$L^- = (I \quad 0) \begin{pmatrix} L & \tilde{z} \\ \tilde{t}^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} = (I - t^*(\tilde{t}^T t^*)^{-1}\tilde{t}^T) L^+ (I - \tilde{z}(z^{*T}\tilde{z})^{-1} z^{*T}) \tag{6}$$

defines a generalized inverse L^- of L satisfying

$$LL^-L = L, \quad L^-LL^- = L^-, \quad L^-\tilde{z} = 0, \quad \tilde{t}^T L^- = 0. \tag{7}$$

Proof. Direct calculation shows that

$$\tilde{L} = \begin{pmatrix} L & \tilde{z} \\ \tilde{t}^T & 0 \end{pmatrix}$$

has the inverse

$$\tilde{L}^{-1} = \begin{pmatrix} (I - t^*(\tilde{t}^T t^*)^{-1}\tilde{t}^T)L^+(I - \tilde{z}(z^{*T}\tilde{z})^{-1}z^{*T}) & t^*(\tilde{t}^T t^*)^{-1} \\ (z^{*T}\tilde{z})^{-1}z^{*T} & 0 \end{pmatrix}.$$

Note that we can simplify the computations when we assume without loss of generality that the columns of z^*, t^* are orthonormalized. From this (7) follows by corresponding properties of the Moore-Penrose pseudoinverse. □

Having selected the desired generalized inverse, we need an appropriate splitting which then leads to this inverse.

LEMMA 2.2. *For*

$$\hat{z} = (I - z^*(\tilde{z}^T z^*)^{-1}\tilde{z}^T)\overline{z}, \quad \hat{t} = (I - t^*(\tilde{t}^T t^*)^{-1}\tilde{t}^T)\overline{t} \tag{8}$$

we have that $\hat{U} = (\hat{z}, z^*)$ *and* $\hat{V} = (\hat{t}, t^*)$ *are nonsingular.*

Proof. The relations (8) define elementary block column operations. Applying them on the nonsingular matrices U and V yields $\hat{U}$ and $\hat{V}$. □

Now we are ready to split (1) into

$$\hat{z}^T f(\hat{t}\hat{\xi} + t^*\xi, \lambda) = 0, \quad z^{*T} f(\hat{t}\hat{\xi} + t^*\xi, \lambda) = 0. \tag{9}$$

Because of $\hat{z}^T L\hat{t} = \overline{z}^T L\overline{t} = \Sigma$ nonsingular, the first part of (9) is uniquely solvable in a neighbourhood of x^* by $\hat{\xi} = \hat{\xi}^*(\xi, \lambda)$ due to the implicit function theorem leading to a so-called reduced function

$$h(\xi, \lambda) = z^{*T} f(x^*(\xi, \lambda), \lambda), \quad h\colon \tilde{D} \to \mathbb{R}^k, \quad \tilde{D} \subseteq \mathbb{R}^{k+l} \times \mathbb{R}^{p+q} \text{ open}, \tag{10}$$

where $x^*(\xi, \lambda) = \hat{t}\hat{\xi}^*(\xi, \lambda) + t^*\xi$. In addition, we can assume $x^* = x^*(0, 0)$ without loss of generality. In singularity theory further investigations are then based on such a reduced function and its derivatives, see also below. Although the computation of the derivatives of h is straighforward and can be found e.g. in [8, 16] at least for the first few orders, we give an outline of their determination up to order two because the methods presented below are based on the details of this process.

Since we must differentiate a vector valued function with respect to vector valued variables, we get tensors of successively higher order. Therefore it is necessary to explicitly write out the arguments on which they act. Beginning with

$$\begin{aligned} x^*(\xi, \lambda) &= \hat{t}\hat{\xi}^*(\xi, \lambda) + t^*\xi, \\ x^*_\xi(\xi, \lambda)(a_1) &= \hat{t}\hat{\xi}^*_\xi(\xi, \lambda)(a_1) + t^*a_1, \\ x^*_\lambda(\xi, \lambda)(b_1) &= \hat{t}\hat{\xi}^*_\lambda(\xi, \lambda)(b_1), \end{aligned} \tag{11}$$

we compute for $\overline{f}(\xi,\lambda) = f(x^*(\xi,\lambda),\lambda)$ by chain rule

$$(12)\qquad \begin{aligned}
\overline{f}(\xi,\lambda) &= f(x^*(\xi,\lambda),\lambda),\\
\overline{f}_\xi(\xi,\lambda)(a_1) &= f_x(x^*(\xi,\lambda),\lambda)(x^*_\xi(\xi,\lambda)(a_1)),\\
\overline{f}_\lambda(\xi,\lambda)(b_1) &= f_x(x^*(\xi,\lambda),\lambda)(x^*_\lambda(\xi,\lambda)(b_1)) +\\
&\quad + f_\lambda(x^*(\xi,\lambda),\lambda)(b_1)),\\
\overline{f}_{\xi\xi}(\xi,\lambda)(a_1,a_2) &= f_{xx}(x^*(\xi,\lambda),\lambda)(x^*_\xi(\xi,\lambda)(a_1),x^*_\xi(\xi,\lambda)(a_2)) +\\
&\quad + f_x(x^*(\xi,\lambda),\lambda)(x^*_{\xi\xi}(\xi,\lambda)(a_1,a_2)),\\
\overline{f}_{\xi\lambda}(\xi,\lambda)(a_1)(b_1) &= f_{xx}(x^*(\xi,\lambda),\lambda)(x^*_\xi(\xi,\lambda)(a_1),x^*_\lambda(\xi,\lambda)(b_1)) +\\
&\quad + f_{x\lambda}(x^*(\xi,\lambda),\lambda)(x^*_\xi(\xi,\lambda)(a_1))(b_1) +\\
&\quad + f_x(x^*(\xi,\lambda),\lambda)(x^*_{\xi\lambda}(\xi,\lambda)(a_1)(b_1)),\\
\overline{f}_{\lambda\lambda}(\xi,\lambda)(b_1,b_2) &= f_{xx}(x^*(\xi,\lambda),\lambda)(x^*_\lambda(\xi,\lambda)(b_1),x^*_\lambda(\xi,\lambda)(b_2)) +\\
&\quad + f_{x\lambda}(x^*(\xi,\lambda),\lambda)(x^*_\lambda(\xi,\lambda)(b_1))(b_2) +\\
&\quad + f_x(x^*(\xi,\lambda),\lambda)(x^*_{\lambda\lambda}(\xi,\lambda)(b_1,b_2)) +\\
&\quad + f_{x\lambda}(x^*(\xi,\lambda),\lambda)(x^*_\lambda(\xi,\lambda)(b_2))(b_1) +\\
&\quad + f_{\lambda\lambda}(x^*(\xi,\lambda),\lambda)(b_1,b_2)).
\end{aligned}$$

Evaluating at the origin, we observe that in every derivative of $\overline{f}$ there is one term containing $L = f_x(x^*,\lambda^*)$. We want to use this fact to eliminate the derivatives of x^* from (12) evaluated at the origin. Let for the moment v denote some derivative of x^*. Then because of $\tilde{t}^T\hat{t} = 0$ by construction (11) yields a relation of the form $\tilde{t}^T v = r_d$ with some residual r_d. Furthermore, from (12) we obtain $\hat{z}^T Lv = \hat{z}^T r_v$ with a second residual r_v. Setting $d = (z^{*T}\tilde{z})^{-1}z^{*T}r_v$, a short computation shows that

$$\begin{pmatrix} L & \tilde{z}\\ \tilde{t}^T & 0\end{pmatrix}\begin{pmatrix} v\\ d\end{pmatrix} = \begin{pmatrix} r_v\\ r_d\end{pmatrix}.$$

Using the results of Lemma 2.1 we then find $v = L^- r_v + t^*(\tilde{t}^T t^*)^{-1} r_d$. In particular, we have $r_d = \tilde{t}^T t^* a_1$, $r_v = 0$ for x^*_ξ and $r_d = 0$, $r_v = -f_\lambda(x^*,\lambda^*)(b_1)$ for x^*_λ or

$$(13)\qquad \begin{aligned}
x^*_\xi(0,0)(a_1) &= t^* a_1,\\
x^*_\lambda(0,0)(b_1) &= -L^- f_\lambda(x^*,\lambda^*)(b_1).
\end{aligned}$$

Summarizing we end up with

$$(14)\qquad \begin{aligned}
h(0,0) &= 0,\\
h_\xi(0,0)(a_1) &= z^{*T} f_x(x^*,\lambda^*)(t^* a_1) = 0,\\
h_\lambda(0,0)(b_1) &= z^{*T} f_\lambda(x^*,\lambda^*)(b_1),\\
h_{\xi\xi}(0,0)(a_1,a_2) &= z^{*T} f_{xx}(x^*,\lambda^*)(t^* a_1, t^* a_2)\\
h_{\xi\lambda}(0,0)(a_1)(b_1) &= z^{*T} f_{xx}(x^*,\lambda^*)(t^* a_1, -L^- f_\lambda(x^*,\lambda^*)(b_1)) +\\
&\quad + z^{*T} f_{x\lambda}(x^*,\lambda^*)(t^* a_1)(b_1),\\
h_{\lambda\lambda}(0,0)(b_1,b_2) &= z^{*T} f_{xx}(x^*,\lambda^*)(-L^- f_\lambda(x^*,\lambda^*)(b_1), -L^- f_\lambda(x^*,\lambda^*)(b_2)) +\\
&\quad + z^{*T} f_{x\lambda}(x^*,\lambda^*)(-L^- f_\lambda(x^*,\lambda^*)(b_1))(b_2) +\\
&\quad + z^{*T} f_{x\lambda}(x^*,\lambda^*)(-L^- f_\lambda(x^*,\lambda^*)(b_2))(b_1) +\\
&\quad + z^{*T} f_{\lambda\lambda}(x^*,\lambda^*)(b_1,b_2).
\end{aligned}$$

Because there is no unique reduced function h fixed by f due to the freedom in the choice of the used bases, we consider classes of singular points which we then call singularities. Therefore the second concept here is the so-called contact equivalence, see again [8]. Two functions g and h are contact equivalent if there is a neighbourhood of the origin such that

$$h(\xi,\tau,\alpha)=T(\xi,\tau,\alpha)g(\varrho(\xi,\tau,\alpha),\sigma(\tau,\alpha),\alpha) \tag{15}$$

with

$$\begin{array}{lll} T\colon \mathbb{R}^{k+l}\times\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^{k,k}, & T(0,0,0) \text{ nonsingular}, & \\ \varrho\colon \mathbb{R}^{k+l}\times\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^{k+l}, & \varrho_\xi(0,0,0) \text{ nonsingular}, & \varrho(0,0,0)=0, \\ \sigma\colon \mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^p, & \sigma_\tau(0,0) \text{ nonsingular}, & \sigma(0,0)=0. \end{array} \tag{16}$$

We then have the important property that two reduced functions of the same original function are contact equivalent, see [2, 8, 15]. Note that we distinguish here for the first time between so-called bifurcation parameters τ and unfolding parameters α. In singularity theory one topic then is to describe the different equivalence classes in terms of conditions on h and its derivatives with respect to ξ and τ and to decide for a given h to what class it belongs. For isolated singularities it is clear that this must lead to a system of equations which has the desired singular point as isolated solution. To obtain a square system of equations (i.e. a system consisting of the same number of equations as the number of unknowns) which determines a singular point it may be necessary to introduce additional parameters α. The number q of parameters α is the so-called codimension of the particular singularity defined in singularity theory. The most popular case is that of a scalar equation h, i.e. $k=1$, with a scalar state variable ξ and a scalar bifurcation parameter τ, i.e. $l=0$ and $p=1$. Figure 1 shows all corresponding equivalence classes for $0\le q\le 3$ arranged in a hierarchical way, cp. [8, 13, 14]. Writing the system of determining equations in the form

$$\begin{aligned} h(\xi,\tau,\alpha)&=0,\\ \tilde b(\xi,\tau,\alpha)&=0, \end{aligned} \tag{17}$$

one obtains the function $\tilde b$ if one starts at the top of the hierarchy and gathers all equations one meets when following the arrows to reach the desired type of singular points. The results of singularity theory then guarantees full column rank of the first two block columns of the Jacobian

$$\tilde W=\begin{pmatrix} 0 & h_\tau(\xi^*,\tau^*,\alpha^*) & h_\alpha(\xi^*,\tau^*,\alpha^*) \\ \tilde b_\xi(\xi^*,\tau^*,\alpha^*) & \tilde b_\tau(\xi^*,\tau^*,\alpha^*) & \tilde b_\alpha(\xi^*,\tau^*,\alpha^*) \end{pmatrix} \tag{18}$$

at the desired singular point (ξ^*,τ^*,α^*). To obtain a nonsingular Jacobian, the derivatives with respect to α must complete these two block columns to a nonsingular matrix. This is an additional condition on the parameters α to be general enough. One therefore speaks of so-called generic unfolding parameters α requiring that they should satisfy the above requirement for a specific problem.

In the context of reduced functions the system (17) therefore solves the problem of constructing appropriate augmented systems when $\tilde b$ is chosen as indicated by results of singularity theory. Note that in general $\tilde b$ maps into $\mathbb{R}^{l+p+q}$.

It remains the question how one can utilize such results for the construction of augmented systems for the original equation (1). Observe that a reduced function is only defined via the application of the implicit function theorem on the first equation of (9) or, in an alternative approach, on a system of the form

$$f(x,\tau,\alpha)-\tilde z h=0,\quad \tilde t^T x=\xi \tag{19}$$

FIG. 1. *Hierarchy of singularities for $k = 1$, $l = 0$, and $p = 1$*

that also implicitly defines a reduced function h in the unknowns (ξ, τ, α). In both cases h is numerically available only by means of an iterative solver for nonlinear equations, say Newton's method. Since we also need derivatives of h (cp. Figure 1), we would then be forced to apply numerical differentiation of an iteratively defined function or symbolic differentiation of Newton's method. Moreover, the results so obtained would have to be used as residuals within a second nonlinear equation solver for (17). Such nested iterations are known to behave extremely sensible in particular with respect to errors in the residuals (which are caused by the stopping criterion of the inner iteration) and should therefore be avoided.

Assuming that we have a finite algorithm to evaluate f at a given point, we can derive (e.g. by a symbolic process) finite algorithms for its derivatives. Our aim therefore is the construction of defining equations for augmented and minimally extended systems that use finite compositions of directional derivatives of f. Moreover, we want to have the same defining equations for all singular points belonging to the same class with respect to contact equivalence.

3. A family of augmented systems. On the basis of the analytical results of the previous section, it seems to be necessary in designing augmented systems to combine the determination of the nullspaces z^* and t^* with the original problem (1) and a certain number of additional conditions describing the type of singularity one wants to compute.

These considerations lead to augmented systems of the general form

$$
\begin{aligned}
f_x(x,\lambda)t_i + \tilde{z}\beta_i &= 0, && i = 1,\ldots,k+l,\\
\tilde{t}^T t &= I, &&\\
f_x(x,\lambda)^T z_j + \tilde{t}\gamma_j &= 0, && j = 1,\ldots,k,\\
\tilde{z}^T z &= I, &&\\
f(x,\lambda) &= 0, &&\\
b(x,\lambda,t,z) &= 0 &&
\end{aligned}
\tag{20}
$$

for the vector of unknowns $\mathrm{col}(x, \lambda, \mathrm{col}(t_i), \mathrm{col}(z_j), \mathrm{col}(\beta_i), \mathrm{col}(\gamma_j))$. Here we use the shorthand notation

$$\begin{array}{rr} \mathrm{col}(t_i)^T = (t_1^T, \ldots, t_{k+l}^T), & \mathrm{col}(z_j)^T = (z_1^T, \ldots, z_k^T), \\ \mathrm{col}(\beta_i)^T = (\beta_1^T, \ldots, \beta_{k+l}^T), & \mathrm{col}(\gamma_j)^T = (\gamma_1^T, \ldots, \gamma_k^T), \\ t = (t_1, \ldots, t_{k+l}), & z = (z_1, \ldots, z_k). \end{array} \tag{21}$$

The quantities β_i and γ_j are vectors of length k and $k+l$, respectively. Finally b is a smooth $\mathbb{R}^{l+p+q}$-valued function depending on the type of singular point to be computed.

For the actual computation of a singular point, one must solve the nonlinear equations (20) by some iteration method, say Newton's method. In order to show local uniqueness of the desired solution (and quadratic convergence of Newton's method), we must show that the desired solution is regular which means that the corresponding Jacobian at the solution is nonsingular.

THEOREM 3.1. *Let* $\mathrm{col}(x^*, \lambda^*, \mathrm{col}(t_i^*), \mathrm{col}(z_j^*), \mathrm{col}(\beta_i^*), \mathrm{col}(\gamma_j^*))$ *be a perfect solution of (20), i.e. let*

$$\mathrm{col}(\beta_i^*) = 0, \quad \mathrm{col}(\gamma_j^*) = 0, \tag{22}$$

and define

$$\begin{array}{ll} T = \mathrm{col}(f_{xx}(x^*, \lambda^*)(t_i^*)), & b^T = \dfrac{\partial b}{\partial x}(x^*, \lambda^*, t^*, z^*), \\ Z = \mathrm{col}(z_j^{*T} f_{xx}(x^*, \lambda^*)), & r^T = \dfrac{\partial b}{\partial \lambda}(x^*, \lambda^*, t^*, z^*), \\ C = \mathrm{col}(f_{x\lambda}(x^*, \lambda^*)(t_i^*)), & p^T = \dfrac{\partial b}{\partial\, \mathrm{col}(t_i)}(x^*, \lambda^*, t^*, z^*), \\ E = \mathrm{col}(z_j^{*T} f_{x\lambda}(x^*, \lambda^*)), & \\ P = f_\lambda(x^*, \lambda^*), & q^T = \dfrac{\partial b}{\partial\, \mathrm{col}(z_j)}(x^*, \lambda^*, t^*, z^*). \end{array} \tag{23}$$

Further, let $\tilde{z}$, $\tilde{t}$ *be chosen such that* $\tilde{L}$ *is nonsingular. Then the Jacobian belonging to (20) at the above solution is nonsingular if and only if the matrix*

$$W = \begin{pmatrix} 0 & z^{*T} P \\ B & R \end{pmatrix} \tag{24}$$

of size $k+l+p+q$ *is nonsingular where*

$$\begin{array}{ll} B = \overline{B} t^*, & \overline{B} = b^T - p^T \,\mathrm{diag}(L^-) T - q^T \,\mathrm{diag}(L^{-T}) Z, \\ R = \overline{R} - \overline{B} L^- P, & \overline{R} = r^T - p^T \,\mathrm{diag}(L^-) C - q^T \,\mathrm{diag}(L^{-T}) E \end{array} \tag{25}$$

and $\mathrm{diag}(*)$ *denotes a block diagonal matrix of matching size with entries* $*$ *in the diagonal.*

Proof. Note first that some of the letters in the right hand part of (23) are already used for different quantities. To avoid confusion we use those of (23) only as transposed matrices such that the transposition can be seen as superscript distinguishing between the different meanings.

Arranging equations and unknowns appropriately, in particular looking at pairs (t_i, β_i) and (z_j, γ_j), the Jacobian M under consideration has the block structure

$$M = \begin{pmatrix} \mathrm{diag}(\tilde{L}) & 0 & \mathrm{diag}\begin{pmatrix} I \\ 0 \end{pmatrix} T & \mathrm{diag}\begin{pmatrix} I \\ 0 \end{pmatrix} C \\ 0 & \mathrm{diag}(\tilde{L}^T) & \mathrm{diag}\begin{pmatrix} I \\ 0 \end{pmatrix} Z & \mathrm{diag}\begin{pmatrix} I \\ 0 \end{pmatrix} E \\ 0 & 0 & L & P \\ p^T \,\mathrm{diag}\,(\,I \quad 0\,) & q^T \,\mathrm{diag}\,(\,I \quad 0\,) & b^T & r^T \end{pmatrix}.$$

Because of $\tilde{L}$ nonsingular, we can build the corresponding Schur complement. Thus nonsingularity of M is equivalent to nonsingularity of

$$D = \begin{pmatrix} L & P \\ \overline{B} & \overline{R} \end{pmatrix}.$$

Using the matrices U and V of the previous section, we get

$$\begin{pmatrix} U^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} L & P \\ \overline{B} & \overline{R} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \Sigma & 0 & \overline{z}^T P \\ 0 & 0 & z^{*T} P \\ \overline{B}\overline{t} & \overline{B}t^* & \overline{R} \end{pmatrix}$$

and nonsingularity of D is equivalent to nonsingularity of

$$\begin{pmatrix} 0 & z^{*T} P \\ \overline{B}t^* & \overline{R} - \overline{B}L^+ P \end{pmatrix}$$

because of $L^+ = \overline{t}\Sigma^{-1}\overline{z}^T$. Recalling the relation (6) for L^-, we see that by appropriate row and column operations we can replace L^+ by L^-. □

One may argue that with respect to efficiency such augmented systems are too large compared with the original problem. Instead, one should try to construct so-called minimally extended systems

$$\begin{aligned} f(x,\lambda) &= 0, \\ \hat{b}(x,\lambda) &= 0, \end{aligned} \tag{26}$$

where $\hat{b}$ maps into $\mathbb{R}^{l+p+q}$ as b above. In this context, $\hat{b}$ is also called test function for the singularity under consideration. Note that (17) is a minimally extended system for the reduced function h in this sense. Of course, the function $\hat{b}$ as b before must be numerically accessible in the sense of the considerations at the end of Section 2. Observing that for

$$\tilde{L}(x,\lambda) = \begin{pmatrix} f_x(x,\lambda) & \tilde{z} \\ \tilde{t}^T & 0 \end{pmatrix} \tag{27}$$

we have

$$\begin{aligned} \frac{\partial}{\partial x}(\tilde{L}(x,\lambda)^{-1}v)(a_1) &= -\tilde{L}(x,\lambda)^{-1} \begin{pmatrix} f_{xx}(x,\lambda)(a_1) & 0 \\ 0 & 0 \end{pmatrix} \tilde{L}(x,\lambda)^{-1}v, \\ \frac{\partial}{\partial \lambda}(\tilde{L}(x,\lambda)^{-1}v)(b_1) &= -\tilde{L}(x,\lambda)^{-1} \begin{pmatrix} f_{x\lambda}(x,\lambda)(b_1) & 0 \\ 0 & 0 \end{pmatrix} \tilde{L}(x,\lambda)^{-1}v \end{aligned} \tag{28}$$

we immediately arrive at the following connection between the above augmented systems and minimally extended systems.

Corollary 3.2. *For given b in (20) let $\hat{b}$ be chosen by*

$$\hat{b}(x,\lambda) = b(x,\lambda,t(x,\lambda),z(x,\lambda)) \tag{29}$$

with

$$\begin{aligned} t(x,\lambda) &= (\,I \quad 0\,)\,\tilde{L}(x,\lambda)^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}, \\ z(x,\lambda) &= (\,I \quad 0\,)\,\tilde{L}(x,\lambda)^{-T} \begin{pmatrix} 0 \\ I \end{pmatrix}. \end{aligned} \tag{30}$$

If $(x^*, \lambda^*, \mathrm{col}(t_i^*), \mathrm{col}(z_j^*), 0, 0)$ *is a regular solution of (20) then* (x^*, λ^*) *is a regular solution of (26).*

Proof. Because of $t^* = t(x^*, \lambda^*)$ and $z^* = z(x^*, \lambda^*)$ according to the proof of Lemma 2.1 the vector (x^*, λ^*) solves (26). Using (28) we then obtain exactly the matrix D of the proof of Theorem 3.1 as Jacobian of (26) at (x^*, λ^*). □

Consequently, if we have found an appropriate augmented system of the form (20) we also have succeeded in constructing a minimally extended system. We must therefore no longer distinguish between these two tasks. Moreover, using block techniques like those presented in [17] for (20) should yield the same efficient numerical procedure in both cases.

Now the remaining problem is to specify the function b in (20). Comparing W of (24) with $\tilde{W}$ of (18) we find that they are of the same size and of the same block structure. The question then is if we can choose b for given $\tilde{b}$ in such a way that B coincides with $\tilde{b}_\xi(\xi^*, \lambda^*)$ and R with $\tilde{b}_\lambda(\xi^*, \lambda^*)$ for some reduced function h of f. In view of Figure 1, we must at least be able to treat finite compositions of sums and products of derivatives of h where the derivatives are of arbitrary order.

4. Representation of derivatives by trees. Treating higher order derivatives of a reduced function, one must deal with a large number of terms due to successive application of the chain rule. Inspired by the techniques used for the determination of order conditions for the construction of high order Runge-Kutta schemes (see e.g. [11]), we introduce a graph based method to handle the special structure of these terms. In particular, the basic mathematical objects will be so-called labelled rooted trees.

DEFINITION 4.1. *A monotonically indexed labelled rooted tree T is a pair of functions $T = (T', T'')$ where*

$$T' \colon \{2, \ldots, N\} \to \{1, \ldots, N\}, \quad T'' \colon \{1, \ldots, N\} \to \mathcal{L}$$

for some $N \in \mathbb{N}$ and given set of symbols $\mathcal{L}$ such that

$$T'(i) < i \text{ for all } i \in \{2, \ldots, N\}. \tag{31}$$

We call $i \in \{1, \ldots, N\}$ a vertex and a pair $(i, T'(i))$ for $i \in \{2, \ldots, N\}$ an edge of the tree. Further, we call $T''(i)$ the label of the vertex i. The vertex 1 is called root of the tree. The function T' is often referred to as son-father mapping. In this context, the vertex i is called son of the vertex $T'(i)$. Then ${T'}^{-1}(\{i\})$ is the set of all vertices that have the vertex i as father.

Two monotonically indexed labelled rooted trees $T = (T', T'')$ and $\tilde{T} = (\tilde{T}', \tilde{T}'')$ with the same N are called equivalent if there is a permutation π on $\{1, \ldots, N\}$ such that

$$\begin{aligned} \tilde{T}'(\pi(i)) &= \pi(T'(i)) && \text{for all } i \in \{2, \ldots, N\}, \\ \tilde{T}''(\pi(i)) &= T''(i) && \text{for all } i \in \{1, \ldots, N\}. \end{aligned} \tag{32}$$

The corresponding equivalence classes are called labelled rooted trees.

As in classical graph theory, we can visualize (monotonically indexed) labelled rooted trees by choosing N points in the plain and connecting point i with point $T'(i)$. Finally we attach the label $T''(i)$ to each point i.

Note that we need the function T' for a simple mathematical description of a tree as a graph without cycles. In particular, we can formulate additional requirements on the structure of the trees more easily. The equivalence classes can therefore be seen as the actual trees allowing several equivalent descriptions by choosing different indexations.

Of course, we must now choose trees in such a way that we can mimic the process of Section 2 leading by differentiation and insertion to the derivatives of the reduced function at the singular point.

DEFINITION 4.2. *Let $\mathcal{L}$ be a disjoint union of the form $\mathcal{L} = \{z, x, \lambda\} \cup I_1 \cup I_2$ where $|I_1| = m_1$, $|I_2| = m_2$ with $m_1, m_2 \in \mathbb{N}_0$. A labelled rooted tree is called an R-tree of order (m_1, m_2) if and only if for a representation $T = (T', T'')$ and therefore for all representations the following properties hold.*

(i) The root has label z, i.e.

$$T''(1) = z.$$

(ii) There is exactly one vertex with label z and for each $j \in I_1 \cup I_2$ there is exactly one vertex with label j, i.e.

$$|T''^{-1}(\{z\})| = 1, \quad |T''^{-1}(\{j\})| = 1 \text{ for all } j \in I_1 \cup I_2.$$

(iii) A vertex with label x has at least one son. A vertex with label λ has exactly one son. Vertices with labels in $I_1 \cup I_2$ have no sons.

$$\begin{array}{ll} |T'^{-1}(\{i\})| \geq 1 & \text{for all } i \in \{1, \ldots, N\} \text{ with } T''(i) = x \\ |T'^{-1}(\{i\})| = 1 & \text{for all } i \in \{1, \ldots, N\} \text{ with } T''(i) = \lambda \\ |T'^{-1}(\{i\})| = 0 & \text{for all } i \in \{1, \ldots, N\} \text{ with } T''(i) \in I_1 \cup I_2 \end{array}$$

(iv) The father of the vertices with labels x or λ is the root. Vertices with labels in I_1 have a vertex with label x as father and vertices with labels in I_2 have a vertex with label x or λ as father. More precisely, we have

$$\begin{array}{lll} T'(i) = j, \quad T''(i) \in \{x, \lambda\} & \Longrightarrow & T''(j) = z, \\ T'(i) = j, \quad T''(i) \in I_1 & \Longrightarrow & T''(j) = x, \\ T'(i) = j, \quad T''(i) \in I_2 & \Longrightarrow & T''(j) \in \{x, \lambda\}. \end{array}$$

The set of all R-trees of order (m_1, m_2) is denoted by RT_{m_1,m_2} and the set of all R-trees of arbitrary order by RT.

We identify the sets I_1 and I_2 with the sets $\{1, \ldots, m_1\}$ and $\{1, \ldots, m_2\}$. To distinguish between elements in I_1 and I_2 we call vertices with labels in I_1 fat vertices and vertices with labels in I_2 meagre vertices. Table 1 gives a list of all R-trees with $m_1 + m_2 \leq 2$. Note that by definition R-trees have only three generations of vertices. The old generation consists of the root, the middle generation of the vertices with labels x or λ, and the young generation of the vertices with labels in $I_1 \cup I_2$.

As next step, we relate R-trees to terms arising when we differentiate the function $\overline{f}(\xi, \lambda)$ defined in (12). For this we use the fact that each labelled rooted tree T with $N > 0$ can be written in the form

$$T = {}_u[T_1, \ldots, T_m] \tag{33}$$

where the right hand side denotes the tree we get if we take the labelled rooted trees $T_1, \ldots, T_m$ and connect their roots to a new vertex with label u which then becomes the

TABLE 1
List of R-trees with $m_1 + m_2 \le 2$

new root. Note that the trees $T_1, \dots, T_m$ are uniquely determined by T up to order. Let $\mathcal{P}(T)$ denote the set of all labelled rooted trees which we get from T when we successively go over by (33) to corresponding subtrees. Accordingly let $\mathcal{P}(\mathrm{RT})$ denote the set of all subtrees which can arise for some R-tree. Finally let $\varrho\colon \mathcal{P}(\mathrm{RT}) \to \mathcal{L}$ be the function which yields the label of the root.

DEFINITION 4.3. *Let $F\colon \mathcal{P}(RT) \to C^\infty(\tilde{D})$ be defined by*

(i) $$F(T)(\xi,\lambda) = f_{\underbrace{x\cdots x}_{l_1}\underbrace{\lambda\cdots\lambda}_{l_2}}(x^*(\xi,\lambda),\lambda)(F(T_1)(\xi,\lambda),\dots,F(T_{l_1})(\xi,\lambda))\cdot (F(\tilde{T}_1)(\xi,\lambda),\dots,F(\tilde{T}_{l_2})(\xi,\lambda))$$
for $T = {}_z[T_1,\dots,T_{l_1},\tilde{T}_1,\dots,\tilde{T}_{l_2}]$, $\varrho(T_1) = \dots = \varrho(T_{l_1}) = x$, $\varrho(\tilde{T}_1) = \dots = \varrho(\tilde{T}_{l_2}) = \lambda$,

(ii) $$F(T)(\xi,\lambda) = x^*_{\underbrace{\xi\cdots\xi}_{l_1}\underbrace{\lambda\cdots\lambda}_{l_2}}(\xi,\lambda)(F(T_1)(\xi,\lambda),\dots,F(T_{l_1})(\xi,\lambda))\cdot (F(\tilde{T}_1)(\xi,\lambda),\dots,F(\tilde{T}_{l_2})(\xi,\lambda))$$
for $T = {}_x[T_1,\dots,T_{l_1},\tilde{T}_1,\dots,\tilde{T}_{l_2}]$, $\varrho(T_1),\dots,\varrho(T_{l_1}) \in I_1$, $\varrho(\tilde{T}_1),\dots,\varrho(\tilde{T}_{l_2}) \in I_2$,

(iii) $F(T)(\xi,\lambda) = F(T_0)(\xi,\lambda)$ *for* $T = {}_\lambda[T_0]$,

(iv) $F(T)(\xi,\lambda) = a_{\varrho(T)}$ *for* $\varrho(T) \in I_1$,

(v) $F(T)(\xi,\lambda) = b_{\varrho(T)}$ *for* $\varrho(T) \in I_2$

with the trivial extension $F(\cdot_z)(\xi,\lambda) = f(x^*(\xi,\lambda),\lambda)$. The function $F(T)$ with $T \in RT$ is called elementary differential of T.

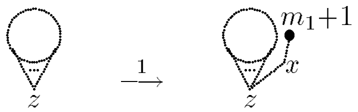
Note that the elementary differential of an R-tree is well-defined because of the symmetry of higher order derivative tensors with respect to their arguments.

LEMMA 4.4. *For all $(m_1,m_2) \in \mathbb{N}_0 \times \mathbb{N}_0$ we have*

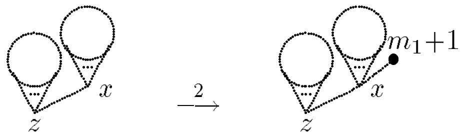
$$\overline{f}_{\underbrace{\xi\cdots\xi}_{m_1}\underbrace{\lambda\cdots\lambda}_{m_2}}(\xi,\lambda)(a_1,\dots,a_{m_1})(b_1,\dots,b_{m_2}) = \sum_{T\in RT_{m_1,m_2}} F(T)(\xi,\lambda) \tag{34}$$

Proof. The claim holds trivially for order $(0,0)$. Let now the claim hold for order (m_1, m_2). Let $T \in \mathrm{RT}_{m_1,m_2}$ and $F(T)$ its elementary differential.

Differentiating $F(T)$ with respect to ξ we must differentiate the first argument of the derivative of f and each of the first m_1 arguments which arose by chain rule. In the first case we obtain the elementary differential belonging to the R-tree we get when we add a vertex with label x and an edge to the root as well as a fat vertex with label m_1+1 and an edge to the new vertex with label x.

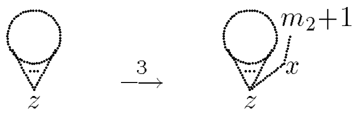


In the second case we obtain the elementary differential belonging to the R-tree we get when we add a fat vertex with label m_1+1 and an edge to the vertex with label x that represents the argument which we differentiate.

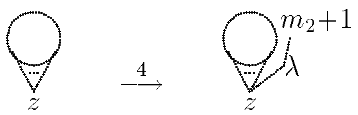


In both cases we obtain R-trees of order (m_1+1, m_2).

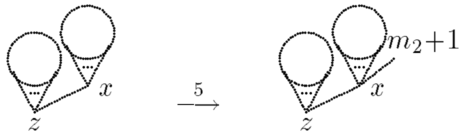
Differentiating $F(T)$ with respect to λ we must differentiate both arguments of the derivative of f and each of the first m_1 arguments which arose by chain rule. In the first case this corresponds to



and to



whereas in the second case this corresponds to



In all cases we obtain R-trees of order (m_1, m_2+1).

This shows that the sum on the right hand side of (34) runs over a subset of RT_{m_1,m_2}. To complete the proof we must show that every R-tree actually occurs and that it only occurs once.

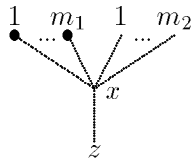
Let $T = (T', T'') \in \mathrm{RT}_{m_1+1,m_2}$. The father i of the fat vertex with the label m_1+1 must have the label x. Now we have the two possibilities $|T'^{-1}(\{i\})| = 1$ and $|T'^{-1}(\{i\})| > 1$. We therefore obtain T in the first case by the above process 1 and in the second case by the above process 2 from R-trees of order (m_1, m_2).

Let $T = (T', T'') \in \mathrm{RT}_{m_1, m_2+1}$. The father i of the meagre vertex with label m_2+1 must have the label x or λ. Now we have the three possibilities $T''(i) = x$ and $|T'^{-1}(\{i\})| = 1$ or $|T'^{-1}(\{i\})| > 1$ as well as $T''(i) = \lambda$. We therefore obtain T by the above processes 3, 5 and 4 from R-trees of order (m_1, m_2).

Hence, for all possibilities there is exactly one way of differentiation to obtain a selected elementary differential. With this the proof is finished. □

Following further the lines of Section 2 we must now compute the derivatives of x^* at the origin in terms of derivatives of x^* of lower order. Again this starts with the observation that there is exactly one term containing L for each order (m_1, m_2) with $m_1 + m_2 \geq 1$. This term is the elementary differential belonging to the tree

(35)



with one vertex with label x and none with label λ. If we denote the set RT_{m_1,m_2} without this special R-tree by $\mathrm{RT}^*_{m_1,m_2}$ we conclude the following.

LEMMA 4.5. *For $m_1 + m_2 \geq 1$ we have*

$$x^*_{\underbrace{\xi\cdots\xi}_{m_1}\underbrace{\lambda\cdots\lambda}_{m_2}}(0,0)(a_1,\ldots,a_{m_1})(b_1,\ldots,b_{m_2}) = \tag{36}$$
$$= \begin{cases} t^* a_1 & \text{for } m_1 = 1,\ m_2 = 0 \\ -\sum\limits_{T \in RT^*_{m_1,m_2}} L^- F(T)(0,0) & \text{else} \end{cases}.$$

Proof. For

$$v = x^*_{\underbrace{\xi\cdots\xi}_{m_1}\underbrace{\lambda\cdots\lambda}_{m_2}}(0,0)(a_1,\ldots,a_{m_1})(b_1,\ldots,b_{m_2})$$

we have

$$Lv = -\sum_{T \in RT^*_{m_1,m_2}} F(T)(0,0).$$

In the notation of Section 2 we therefore get

$$r_d = \begin{cases} (\tilde{t}^T t^*)(a_1) & \text{for } m_1 = 1,\ m_2 = 0 \\ 0 & \text{else} \end{cases}.$$

and

$$r_v = -\sum_{T \in RT^*_{m_1,m_2}} F(T)(0,0)$$

which then leads to (36). □

Now we must successively replace the derivatives of x^* by the expressions (36) until no explicit derivative is present any longer and the process stops.

For the construction of a tree representation for such terms we introduce a new label l substituting the label z to indicate the presence of the factor $-L^-$ in (36).

Because of the above iterative replacement this new label will play the role of a root concerning vertices with labels x or λ and the role of a vertex with label x concerning the root. In addition we must include that terms are replaced for $m_1 + m_2 \geq 1$.

DEFINITION 4.6. *Let $\mathcal{L}$ be a disjoint union of the form $\mathcal{L} = \{z, l, x, \lambda\} \cup I_1 \cup I_2$ where $|I_1| = m_1$, $|I_2| = m_2$ with $m_1, m_2 \in \mathbb{N}_0$. A labelled rooted tree is called an X-tree of order (m_1, m_2) if and only if for a representation $T = (T', T'')$ and therefore for all representations the following properties hold.*

(i) The root has label z, i.e.

$$T''(1) = z.$$

(ii) There is exactly one vertex with label z and for each $j \in I_1 \cup I_2$ there is exactly one vertex with label j, i.e.

$$|T''^{-1}(\{z\})| = 1, \quad |T''^{-1}(\{j\})| = 1 \text{ for all } j \in I_1 \cup I_2.$$

(iii) The root has at least one son. If it has exactly one son, this must be a vertex with label λ. A vertex with label l has at least one son. If it has exactly one son, this must be a vertex with label λ. A vertex with label x or λ has exactly one son. Vertices with labels in $I_1 \cup I_2$ have no sons.

$$\begin{array}{ll}
|T'^{-1}(\{1\})| \geq 1 & \\
|T'^{-1}(\{1\})| = 1, & T'(j) = 1 \implies T''(j) = \lambda \\
|T'^{-1}(\{i\})| \geq 1 & \text{for all } i \in \{1, \ldots, N\} \text{ with } T''(i) = l \\
|T'^{-1}(\{i\})| = 1, & T''(i) = l,\ T'(j) = i \implies T''(j) = \lambda \\
|T'^{-1}(\{i\})| = 1 & \text{for all } i \in \{1, \ldots, N\} \text{ with } T''(i) \in \{x, \lambda\} \\
|T'^{-1}(\{i\})| = 0 & \text{for all } i \in \{1, \ldots, N\} \text{ with } T''(i) \in I_1 \cup I_2
\end{array}$$

(iv) There are only edges of the kinds

$$\begin{array}{llll}
T'(i) = j, & T''(i) \in \{l, x, \lambda\} & \implies & T''(j) \in \{z, l\}, \\
T'(i) = j, & T''(i) \in I_1 & \implies & T''(j) = x, \\
T'(i) = j, & T''(i) \in I_2 & \implies & T''(j) = \lambda.
\end{array}$$

The set of all X-trees of order (m_1, m_2) is denoted by XT_{m_1,m_2} and the set of all X-trees of arbitrary order by XT.

Table 2 gives a list of all X-trees with $m_1 + m_2 \leq 2$ again with the distinction of fat and meagre vertices as before for R-trees.

DEFINITION 4.7. *Let $F^*: \mathcal{P}(XT) \to C^\infty(\tilde{D})$ be defined by*

(i) $F^*(T)(\xi, \lambda) = f_{\underbrace{x \cdots x}_{l_1} \underbrace{\lambda \cdots \lambda}_{l_2}}(x^*(\xi, \lambda), \lambda)(F^*(T_1)(\xi, \lambda), \ldots, F^*(T_{l_1})(\xi, \lambda)) \cdot$
$\cdot (F^*(\tilde{T}_1)(\xi, \lambda), \ldots, F^*(\tilde{T}_{l_2})(\xi, \lambda))$
for $T = {}_z[T_1, \ldots, T_{l_1}, \tilde{T}_1, \ldots, \tilde{T}_{l_2}]$, $\varrho(T_1), \ldots, \varrho(T_{l_1}) \in \{l, x\}$,
$\varrho(\tilde{T}_1) = \cdots = \varrho(\tilde{T}_{l_2}) = \lambda$,

(ii) $F^*(T)(\xi, \lambda) = -L^- F^*({}_z[T_1, \ldots, T_m])(\xi, \lambda)$ *for* $T = {}_l[T_1, \ldots, T_m]$,

(iii) $F^*(T)(\xi, \lambda) = t^* F^*(T_0)(\xi, \lambda)$ *for* $T = {}_x[T_0]$,

(iv) $F^*(T)(\xi, \lambda) = F^*(T_0)(\xi, \lambda)$ *for* $T = {}_\lambda[T_0]$,

(v) $F^*(T)(\xi, \lambda) = a_{\varrho(T)}$ *for* $\varrho(T) \in I_1$,

(vi) $F^*(T)(\xi, \lambda) = b_{\varrho(T)}$ *for* $\varrho(T) \in I_2$.

TABLE 2
List of X-trees with $m_1 + m_2 \leq 2$

The function $F^*(T)$ with $T \in XT$ is called elementary differential of T.

Again the elementary differential of an X-tree is well-defined by symmetry. Now we finally arrive at the following representation of the derivatives of the reduced function h at the origin or (ξ^*, λ^*) respectively using X-trees.

THEOREM 4.8. *Regarding empty sums to be zero we have*

$$h_{\underbrace{\xi\cdots\xi}_{m_1}\underbrace{\lambda\cdots\lambda}_{m_2}}(\xi^*, \lambda^*)(a_1, \ldots, a_{m_1})(b_1, \ldots, b_{m_2}) = \sum_{T \in XT_{m_1,m_2}} z^{*T} F^*(T)(\xi^*, \lambda^*) \tag{37}$$

for all $(m_1, m_2) \in \mathbb{N}_0 \times \mathbb{N}_0$.

Proof. The successive elimination of the derivatives of x^* from elementary differentials of R-trees by means of (36) can be interpreted as replacing vertices with label x which have more than one son successively by trees lying in $\mathrm{RT}^*_{m_1,m_2}$. The order (m_1, m_2) depends on the number of fat and meagre sons. Changing the label z into l then includes the factor $-L^-$ in (36). This replacing stops when all vertices with label x have exactly one son. In this case (36) occurs as part of the inductive definition of F^*. Because $z^{*T}L = 0$ there is no X-tree with a root having exactly one son with label x. Because there is no R-trees in $\mathrm{RT}^*_{m_1,m_2}$ with a root having exactly one son with label x there are no X-trees with vertices with label l having exactly one son with label x.

These considerations explain that the sum on the right hand side of (37) runs over a subset of XT_{m_1,m_2}. To complete the proof we must show that every X-tree actually occurs and that it only occurs once.

Take some X-tree. We say that a vertex with label l is higher than another one if we must go along a larger number of edges to reach the root. This gives us an order for the vertices with label l of this tree. Now select a vertex with label l which is at least as high as the other ones. The tree consisting of this vertex as root with label z and of all its descendants is an R-tree in some $\mathrm{RT}^*_{m_1,m_2}$. Therefore this part of the

X-tree came from inserting (36). In this way we can successively follow back the above elimination process and decide from which R-tree the selected X-tree arose. Moreover there is only one way possible. □

By this result we have now knowledge of the exact structure of the derivatives of the reduced function at the singular point. For example applying (37) to the trees of Table 2 at once yields the formulas given in (14). In the next section we want to use this result for the examination of the regularity condition (cp. (24)) of the augmented systems under consideration.

5. The regularity condition. In this section we are going to construct a constructive rule how we can get a suitable function b for (20) when a suitable function $\tilde{b}$ for (17) is known. In particular we want to achieve $B = \tilde{B}$, $R = \tilde{R}$ such that the regularity condition for (20) (i.e. W nonsingular) coincides with that for (17) (i.e. $\tilde{W}$ nonsingular). Because we want to use (37) we must choose h according to the values t^*, z^* obtained as part of a perfect solution of (20).

We start with the simplest possible function $\tilde{b}$ one can think of, namely a scalar function of the form

$$\tilde{b}(\xi,\lambda) = e^T h_{\underbrace{\xi\cdots\xi}_{m_1}\underbrace{\lambda\cdots\lambda}_{m_2}}(\xi,\lambda)(a_1,\ldots,a_{m_1})(b_1,\ldots,b_{m_2}) \tag{38}$$

where $e, a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2}$ are basis vectors of the canonical bases in the corresponding spaces. In this case we have

$$\begin{aligned} \tilde{B} = \tilde{b}_\xi(\xi^*,\lambda^*) &= \sum_{T\in \mathrm{XT}_{m_1+1,m_2}} e^T z^{*T} F^*(T)(\xi^*,\lambda^*) \text{ with } a_{m_1+1} = I, \\ \tilde{R} = \tilde{b}_\lambda(\xi^*,\lambda^*) &= \sum_{T\in \mathrm{XT}_{m_1,m_2+1}} e^T z^{*T} F^*(T)(\xi^*,\lambda^*) \text{ with } b_{m_2+1} = I. \end{aligned} \tag{39}$$

To all possible functions $\tilde{b}$ of the form (38) with $m_1 + m_2 \geq 1$ we assign a function $b = \mathcal{U}(\tilde{b})$ in the variables (x,λ,t,z) with the help of an operator $\mathcal{U}$ defined by

$$\mathcal{U}(\tilde{b})(x,\lambda,t,z) = \begin{cases} e^T z^T f_x(x,\lambda)(ta_1) & \text{for } m_1 = 1,\ m_2 = 0 \\ \displaystyle\sum_{T\in \mathrm{XT}_{m_1,m_2}} e^T z^T F(T)(x,\lambda) & \text{else} \end{cases}. \tag{40}$$

The function $F: \mathcal{P}(\mathrm{XT}) \to C^\infty(D)$ is given by the following replacement in the definition of $F^*(T)(\xi^*,\lambda^*)$ (note that the elementary differentials of R-trees were also denoted by F but they are no longer needed)

$$x^* = x^*(\xi^*,\lambda^*) \leftarrow x, \quad \lambda^* \leftarrow \lambda, \quad t^* \leftarrow t, \quad L^- \leftarrow (\, I \quad 0\,)\begin{pmatrix} f_x(x,\lambda) & \tilde{z} \\ \tilde{t}^T & 0 \end{pmatrix}^{-1}\begin{pmatrix} I \\ 0 \end{pmatrix} \tag{41}$$

for given $\tilde{t}$, $\tilde{z}$ satisfying (5).

Because the operator $\mathcal{U}$ transcribes a function $\tilde{b}$ involving derivatives of a reduced function h into a function b involving the original function f, we refer to it as transcription rule.

Defining an operator $\mathcal{D}$ for all differentiable functions $\tilde{b}$ in the variables (ξ,λ) by

$$\mathcal{D}(\tilde{b}) = (\, \tilde{b}_\xi(\xi^*,\lambda^*) \quad \tilde{b}_\lambda(\xi^*,\lambda^*)\,) \tag{42}$$

and an operator $\mathcal{B}$ for all differentiable functions b in the variables (x, λ, t, z) by

$$\mathcal{B}(b) = (\, B \quad R\,) \tag{43}$$

where B, R are obtained by (25) via

$$\begin{aligned}
\overline{B} &= b_x(x^*, \lambda^*, t^*, z^*)(\cdot) - \sum_{i=1}^{k+l} b_{t_i}(x^*, \lambda^*, t^*, z^*)(L^- f_{xx}(x^*, \lambda^*)(t_i^*, \cdot)) - \\
&\quad - \sum_{j=1}^{k} z_j^{*T} f_{xx}(x^*, \lambda^*)(L^- b_{z_j}^T(x^*, \lambda^*, t^*, z^*), \cdot) \\
\overline{R} &= b_\lambda(x^*, \lambda^*, t^*, z^*)(\cdot) - \sum_{i=1}^{k+l} b_{t_i}(x^*, \lambda^*, t^*, z^*)(L^- f_{x\lambda}(x^*, \lambda^*)(t_i^*)(\cdot)) - \\
&\quad - \sum_{j=1}^{k} z_j^{*T} f_{x\lambda}(x^*, \lambda^*)(L^- b_{z_j}^T(x^*, \lambda^*, t^*, z^*))(\cdot)
\end{aligned} \tag{44}$$

we are now able to show $\tilde{B} = B$, $\tilde{R} = R$ for the above defined class of functions $\tilde{b}$. With the help of the operators $\mathcal{U}$, $\mathcal{D}$, and $\mathcal{B}$ we can state this as follows.

THEOREM 5.1. *For $m_1 + m_2 \geq 1$ and $\tilde{b}$ of the form (38), we have*

$$\mathcal{D}(\tilde{b}) = \mathcal{B}(\mathcal{U}(\tilde{b})). \tag{45}$$

Proof. For $m_1 = 1$, $m_2 = 0$ the claim follows from

$$\begin{aligned}
\tilde{B} &= e^T h_{\xi\xi}(\xi^*, \lambda^*)(a_1, a_2) = \\
&= e^T z^{*T} f_{xx}(x^*, \lambda^*)(t^* a_1, t^* a_2) = \overline{B} t^* = B
\end{aligned}$$

and

$$\begin{aligned}
\tilde{R} &= e^T h_{\xi\lambda}(\xi^*, \lambda^*)(a_1)(b_1) = \\
&= e^T z^{*T} f_{xx}(x^*, \lambda^*)(t^* a_1, -L^- f_\lambda(x^*, \lambda^*)(b_1)) + e^T z^{*T} f_{x\lambda}(x^*, \lambda^*)(t^* a_1)(b_1) = \\
&= \overline{R} - \overline{B} L^- P = R
\end{aligned}$$

because of $Lt^* = 0$, $z^{*T} L = 0$.

In the general case we must apply $\mathcal{B}$ on the sum in (40). Comparing with (39) the process of building the matrices B and R for $\mathcal{U}(\tilde{b})$ must yield an iterative construction scheme where starting from the trees in XT_{m_1,m_2} the process for B must give all trees in XT_{m_1+1,m_2} and that for R all trees in XT_{m_1,m_2+1}. This is similar to the iterative construction of R-trees by differentiation shown in the proof of Lemma 4.4.

Let $T \in XT_{m_1,m_2}$ be given. To determine its contribution to B and R we must compute the terms on the right hand side of (44) for $b = e^T z^T F(T)(x, \lambda)$. Recalling the definition of F we have derivatives with respect to x and λ for the root (argument of derivative of f) and for vertices with label l (argument of derivative of f as well as argument in $\tilde{L}(x, \lambda)$). Derivatives with respect to t_i occur for vertices with label x and derivatives with respect to z_j occur for the root (because the factor $e^T z^T$ in front $F(T)$). Because e and $a_1, \ldots, a_{m_1}$ are canonical basis vectors there is exactly one term in the sums of (44) for a selected vertex with label x and for the root in the last two cases. Taking into account the direction of differentiation and the formulas (28) we

TABLE 3

Processes involved in the application of operator $\mathcal{B}$

get terms of the form $e^T z^* F^*(\tilde{T})(\xi^*, \lambda^*)$ with $\tilde{T} \in \mathrm{XT}_{m_1+1,m_2}$, $a_{m_1+1} = I$ for B and $\tilde{T} \in \mathrm{XT}_{m_1,m_2+1}$, $b_{m_2+1} = I$ for R. See Table 3 for a list of all cases.

Again we must show that all X-trees actually occur. This follows similar as in the proof to Lemma 4.4 by examining an extended list of cases.

Let $T \in \mathrm{XT}_{m_1+1,m_2}$, $(m_1, m_2) \neq (1, 0)$. The aim is to show that it is obtainable by exactly one of the processes in the first column of Table 3 from an X-tree of order (m_1, m_2). We start with the observation that the father i of the fat vertex with label $m_1 + 1$ must have the label x. The father j of i can be a vertex with label l or the root.

If it is the root, i.e. $j = 1$, then we must have $|T'^{-1}(\{1\})| > 1$. If $|T'^{-1}(\{1\})| = 2$ and the second son is also a vertex with label x we have $(m_1, m_2) = (2, 0)$ where we already proved the result. Let therefore the second son be a vertex with label l or λ. In the first case we get T by process 5, in the second case we get T by process 1. If $|T'^{-1}(\{1\})| > 2$ we get T by process 1, too.

If it is a vertex with label l, i.e. $j \neq 1$, then we must have $|T'^{-1}(\{j\})| > 1$. If $|T'^{-1}(\{j\})| = 2$ the second son can be a vertex with label l or x. In the first case we get T by process 3, in the second case we get T by process 4. If $|T'^{-1}(\{j\})| > 2$ we get T by process 2.

For $T \in \mathrm{XT}_{m_1,m_2+1}$ we must proceed accordingly with a further case if the grandfather of the meagre vertex with label $m_2 + 1$ has more that one son or not when he has the label l. We think it is reasonable to omit the details here. □

To extend the definition of the operator $\mathcal{U}$ to more complicated functions $\tilde{b}$ let $\mathcal{E}$ be the operator which for a function in the variables (x, λ, t, z) gives its value at $(x^*, \lambda^*, t^*, z^*)$ and let $\tilde{\mathcal{E}}$ be the operator which for a function in the variables (ξ, λ) gives its value at (ξ^*, λ^*). We then have

$$\mathcal{E}(\mathcal{U}(\tilde{b})) = \tilde{\mathcal{E}}(\tilde{b}) \tag{46}$$

for all $\tilde{b}$ such that the left hand side is defined. Assuming now $\mathcal{U}$ to be componentwise, additive, and multiplicative we conserve the central property (45) at least for special compositions of functions of the form (38).

COROLLARY 5.2. *Let $\tilde{b}_i$, $i = 1, \ldots, 4$ be differentiable functions in the variables (ξ, λ) such that $\tilde{b} = \tilde{b}_1\tilde{b}_2 + \tilde{b}_3\tilde{b}_4$ is defined and such that $\mathcal{D}(\tilde{b}_i) = \mathcal{B}(\mathcal{U}(\tilde{b}_i))$ holds for $i = 1, \ldots, 4$. Then for this $\tilde{b}$ again (45) holds.*

Proof. By definition we find

$$\begin{aligned}
\mathcal{D}(\tilde{b}) &= \mathcal{D}(\tilde{b}_1\tilde{b}_2 + \tilde{b}_3\tilde{b}_4) = \\
&= \mathcal{D}(\tilde{b}_1)\tilde{\mathcal{E}}(\tilde{b}_2) + \tilde{\mathcal{E}}(\tilde{b}_1)\mathcal{D}(\tilde{b}_2) + \\
&\qquad + \mathcal{D}(\tilde{b}_3)\tilde{\mathcal{E}}(\tilde{b}_4) + \tilde{\mathcal{E}}(\tilde{b}_3)\mathcal{D}(\tilde{b}_4) = \\
&= \mathcal{B}(\mathcal{U}(\tilde{b}_1))\mathcal{E}(\mathcal{U}(\tilde{b}_2)) + \mathcal{E}(\mathcal{U}(\tilde{b}_1))\mathcal{B}(\mathcal{U}(\tilde{b}_2)) + \\
&\qquad + \mathcal{B}(\mathcal{U}(\tilde{b}_3))\mathcal{E}(\mathcal{U}(\tilde{b}_4)) + \mathcal{E}(\mathcal{U}(\tilde{b}_3))\mathcal{B}(\mathcal{U}(\tilde{b}_4)) = \\
&= \mathcal{B}(\mathcal{U}(\tilde{b}_1)\mathcal{U}(\tilde{b}_2) + \mathcal{U}(\tilde{b}_3)\mathcal{U}(\tilde{b}_4)) = \\
&= \mathcal{B}(\mathcal{U}(\tilde{b}_1\tilde{b}_2 + \tilde{b}_3\tilde{b}_4)) = \\
&= \mathcal{B}(\mathcal{U}(\tilde{b}))
\end{aligned}$$

utilizing that both $\mathcal{D}$ and $\mathcal{B}$ are operators combining differentiation and evaluation. □

By induction relation (45) is at least valid for all vector function $\tilde{b}$ in the variables (ξ, λ) whose components consist of a finite composition of sums and products of functions of the form (38). Because of the linearity of derivative tensors with respect to their arguments we find among these compositions terms of the form

$$w^T h_{\underbrace{\xi\cdots\xi}_{m_1}\underbrace{\lambda\cdots\lambda}_{m_2}}(\xi, \lambda)(u_1, \ldots, u_{m_1})(v_1, \ldots, v_{m_2})$$

with arbitrary vectors w, $u_1, \ldots, u_{m_1}$, $v_1, \ldots, v_{m_2}$ of matching sizes and determinants of matrices with entries of this form. Comparing with Figure 1 these are exactly the kind of functions we should be able to handle.

Concluding this section we want to summerize the obtained results not on the level of the technical statement (45) but on the level of constructing augmented systems and to give a concise outline of the whole logic construction.

THEOREM 5.3. *Let (x^*, λ^*) be a singular solution of (1). Let $\tilde{t}$, $\tilde{z}$ and t^*, z^* be given such that $\tilde{t}^T t^* = I$, $\tilde{z}^T z^* = I$ and such that the columns of t^* and z^* span kernel and corange of $L = f_x(x^*, \lambda^*)$. Choose h to be a reduced function of f obtained by using t^*, z^* and $\hat{t}$, $\hat{z}$ as described in Section 2 with the singular point at (ξ^*, λ^*). Finally let a function $\tilde{b}$ be given such that we can apply the transcription rule $\mathcal{U}$.*

Then (ξ^*, λ^*) is a regular solution of (17) if and only if $(x^*, \lambda^*, \mathrm{col}(t_i^*), \mathrm{col}(z_j^*), 0, 0)$ is a regular solution of (20) with $b = \mathcal{U}(\tilde{b})$ if and only if (x^*, λ^*) is a regular solution of (26) with $\hat{b}$ according to (29).

6. Applications. The main result of the previous section can be used in the following manner. Assume that we have a recognition type result from singularity theory for determining if a singular point of a reduced problem is of some type. Assume furthermore that the transcription rule $\mathcal{U}$ can be applied to the determining function $\tilde{b}$. Then we at once have also a suitable augmented and a minimally extended system for the computation of corresponding singular points of a non-reduced problem. This procedure is in particular practicable for all singularities of Figure 1. In this section we want to carry out this for some selected classes of singular points. Moreover we want to show that many of the well-known augmented and minimally extended systems developed so far are generalized by this approach.

6.1. Turning points. Let $k = 1$, $l = 0$, $p = 1$. Turning points are defined to be singular solutions of $f(x, \tau, \alpha) = 0$ such that a reduced function h is contact equivalent to g with $g(\xi, \tau, \alpha) = \xi^2 + \tau$. Figure 1 then yields $q = 0$ (i.e. no additional unfolding parameters α are needed so that we can omit them) and

$$\tilde{b}(\xi, \tau) = h_\xi(\xi, \tau). \tag{47}$$

Using (40) we get

$$b(x, \tau, t, z) = z^T f_x(x, \tau) t \tag{48}$$

for (20). Because $b(x, \tau, t, z) = 0$ implies $\beta = \gamma = 0$ the augmented system can be reduced to

$$\begin{aligned} f_x(x, \tau) t &= 0, \\ \tilde{t}^T t &= 1, \\ f(x, \tau) &= 0 \end{aligned} \tag{49}$$

which is due to Seydel [26, 27].

For a minimally extended system we obtain

$$\hat{b}(x, \tau) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} f_x(x, \tau) & \tilde{z} \\ \tilde{t}^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} f_x(x, \tau) \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} f_x(x, \tau) & \tilde{z} \\ \tilde{t}^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{50}$$

which for $f_x(x, \tau)$ nonsingular can be rewritten in the form

$$\hat{b}(x, \tau) = (\tilde{t}^T f_x(x, \tau)^{-1} \tilde{z})^{-1}. \tag{51}$$

This test function can be found explicitly or implicitly e.g. in Seydel [26, 27, 28], Griewank/Reddien [10], Werner [30], or Pönisch/Schwetlick [25].

6.2. Bogdanov-Takens points. If we take $k = 1$, $l = 0$, $p = 1$, $q = 1$, we can expect to find paths of turning points $(x(\alpha), \tau(\alpha))$. Let the nullvectors be continuously parametrized by α along such a path as $t(\alpha)$, $z(\alpha)$. We then speak of a Bogdanov-Takens point (x^*, τ^*, α^*) if $z(\alpha)^T t(\alpha)$ has a simple root α^*. This condition requires $f_x(x^*, \tau^*, \alpha^*)$ to have a zero eigenvalue with a nontrivial Jordan block of size two. By Schur complement we at once get (20) with

$$b(x, \tau, \alpha, t, z) = \begin{pmatrix} z^T f_x(x, \tau, \alpha) t \\ z^T t \end{pmatrix} \tag{52}$$

for a suitable augmented system. As possible test function we may take $\hat{b} = (\hat{b}_1, \hat{b}_2)$ with $\hat{b}_1$ as in (50) with additional parameter α and

$$\hat{b}_2(x, \tau, \alpha) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} f_x(x, \tau, \alpha) & \tilde{z} \\ \tilde{t}^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} f_x(x, \tau, \alpha) & \tilde{z} \\ \tilde{t}^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{53}$$

There are several ways to generalize Bogdanov-Takens points. One possibility (see [18, 19]) is to look for singular points with rank deficiency k and a nontrivial Jordan block of size two. A suitable augmented system is given by

$$b(x, \tau, \alpha, t, z) = \begin{pmatrix} z^T f_x(x, \tau, \alpha) t \\ \det(z^T t) \end{pmatrix} \tag{54}$$

where the first component is a short hand notation for k^2 scalar components requiring $q = k^2$. A second possibility (see [9, 18]) is to look for singular points with rank deficieny one and a nontrivial Jordan block of size $m+1$, $m \in \mathbb{N}$. Here a suitable augmented system is given by

$$b(x,\tau,\alpha,t,z) = \begin{pmatrix} z^T f_x(x,\tau,\alpha)t \\ z^T t \\ z^T L^-(x,\tau,\alpha)t \\ \vdots \\ z^T L^-(x,\tau,\alpha)^{m-1}t \end{pmatrix} \tag{55}$$

with $q = m$ and

$$L^-(x,\tau,\alpha) = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} f_x(x,\tau,\alpha) & \tilde{z} \\ \tilde{t}^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix}. \tag{56}$$

6.3. Simple bifurcation points. Simple bifurcation points are defined both in the case $k=1$, $l=0$, $p=1$ by h being contact equivalent to g with $g(\xi,\tau,\alpha) = \xi^2 - \tau^2$ and in the case $k=1$, $l=1$, $p=0$ by h being contact equivalent to g with $g(\xi,\alpha) = \xi_1^2 - \xi_2^2$. In either case we have $q=1$. Figure 1 yields

$$\tilde{b}(\xi,\tau,\alpha) = \begin{pmatrix} h_\xi(\xi,\tau,\alpha) \\ h_\tau(\xi,\tau,\alpha) \end{pmatrix} \tag{57}$$

and therefore

$$b(x,\tau,\alpha,t,z) = \begin{pmatrix} z^T f_x(x,\tau,\alpha)t \\ z^T f_\tau(x,\tau,\alpha) \end{pmatrix}. \tag{58}$$

A corresponding minimally extended system gets the form of that proposed by Pönisch [23]. In the second case we find

$$\tilde{b}(\xi,\alpha) = h_\xi(\xi,\alpha) \tag{59}$$

and

$$b(x,\alpha,t,z) = z^T f_x(x,\tau,\alpha)t \tag{60}$$

both consisting of two scalar components. With the special choice of the unfolding parameter α by replacing $f(x,\alpha) = 0$ by $f(x) - z\alpha = 0$ together with a reduction similar to that for turning points we get

$$\begin{aligned} f_x(x)^T z &= 0, \\ \tilde{z}^T z &= 1, \\ f(x) - z\alpha &= 0 \end{aligned} \tag{61}$$

which is due to Moore [21].

6.4. Hysteresis points. Another example with $k=1$, $l=0$, $p=1$, $q=1$ are the hysteresis points defined by h being contact equivalent to g with $g(\xi,\tau,\alpha) = \xi^3 + \tau$. Here Figure 1 yields

$$\tilde{b}(\xi,\tau,\alpha) = \begin{pmatrix} h_\xi(\xi,\tau,\alpha) \\ h_{\xi\xi}(\xi,\tau,\alpha) \end{pmatrix} \tag{62}$$

and therefore

$$b(x,\tau,\alpha,t,z) = \begin{pmatrix} z^T f_x(x,\tau,\alpha)t \\ z^T f_{xx}(x,\tau,\alpha)(t,t) \end{pmatrix}. \tag{63}$$

The corresponding minimal extended system is similar to those given by Pönisch [24] or Janovsky [12].

6.5. Regular corank two singularities. The simplest singularity for $k = 2$, $l = 0$, $p = 1$ is that belonging to

$$\tilde{b}(\xi,\tau,\alpha) = h_\xi(\xi,\tau,\alpha). \tag{64}$$

Note that this consists of four scalar components. Thus we have $q = 3$. For a confirmation of (64) and more details on this type of singular points including a normal form see e.g. [7, 16, 19]. As for turning points this leads to the formally same function

$$b(x,\tau,\alpha,t,z) = z^T f_x(x,\tau,\alpha)t \tag{65}$$

but the dimensions of the nullspaces are different. In particular this prevents the application of a reduction as done before for turning points. Finally we get a function $\hat{b}$ similar to (50) as test function.

7. Conclusions. The main effort in developing suitable augmented and minimally extended systems is to find appropriate functions b and $\hat{b}$ that match with the results offered by singularity theory. Treating this problem many papers deal only with one kind of singularities and with either augmented or minimally extended systems. The constructive transcription rule developed in the present paper now covers all such approaches. Most of the augmented and minimally extended systems one can find in the literature or slight modifications of them are now brought together as special cases of a general principle. Moreover, because of the generality of this transcription rule appropriate augmented and minimally extended systems are at hand as soon as a recognition type result as for example those of Figure 1 is available. In particular, the results of this paper include suitable systems for all singularities with corank $k = 1$ and codimension $q \leq 3$. The numerical task of developing such systems is now replaced by the analytical task of getting recognition theorems.

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$h_\xi = 0$

$q = 0$ $\quad \boxed{\xi^2 + \tau}$

$h_{\xi\xi} = 0 \qquad h_\tau = 0$

$q = 1$ $\quad \boxed{\xi^3 + \tau} \quad \boxed{\xi^2 \pm \tau^2}$

$h_{\xi\xi\xi} = 0 \qquad h_\tau = 0 \qquad h_{\xi\xi} = 0 \qquad h_{\xi\tau}^2 - h_{\xi\xi} h_{\tau\tau} = 0$

$q = 2$ $\quad \boxed{\xi^4 + \tau} \quad \boxed{\xi^3 + \tau\xi} \quad \boxed{\xi^2 + \tau^3}$

$h_{\xi\xi\xi\xi} = 0 \qquad h_\tau = 0 \qquad h_{\xi\xi\xi} = 0 \qquad h_{\xi\tau} = 0 \qquad h_{\xi\xi} = 0 \qquad D_3 = 0$

$q = 3$ $\quad \boxed{\xi^5 + \tau} \quad \boxed{\xi^4 + \tau\xi} \quad \boxed{\xi^3 + \tau^2} \quad \boxed{\xi^2 \pm \tau^4}$

Figure 1

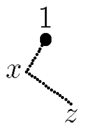
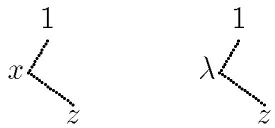
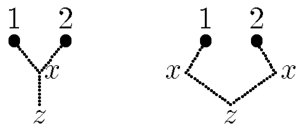
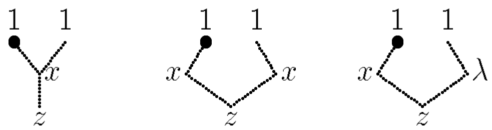
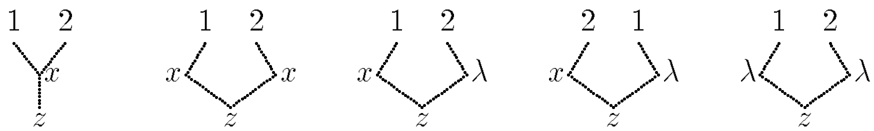
$m_1 = 0$ $m_2 = 0$	
$m_1 = 1$ $m_2 = 0$	
$m_1 = 0$ $m_2 = 1$	
$m_1 = 2$ $m_2 = 0$	
$m_1 = 1$ $m_2 = 1$	
$m_1 = 0$ $m_2 = 2$	

Table 1

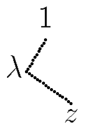
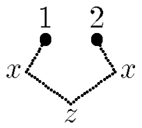
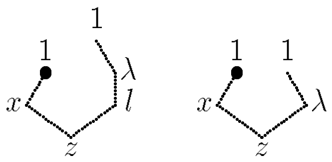
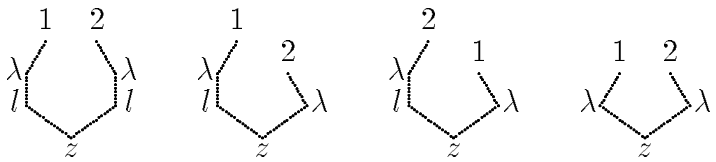
$m_1 = 0$ $m_2 = 1$	
$m_1 = 2$ $m_2 = 0$	
$m_1 = 1$ $m_2 = 1$	
$m_1 = 0$ $m_2 = 2$	

Table 2

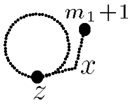
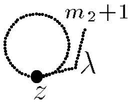
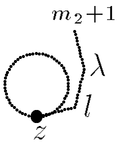
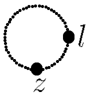
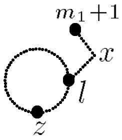
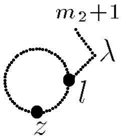
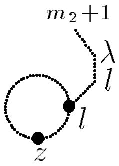
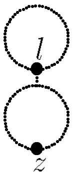
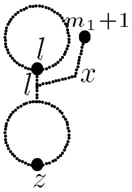
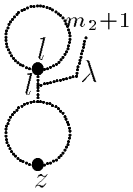
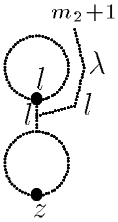
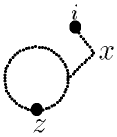
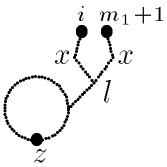
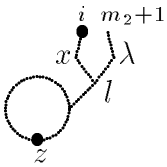
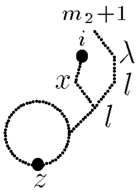
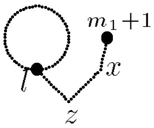
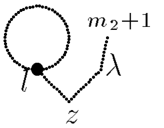
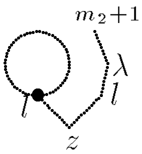
Process	$\bar{B}t^*$	$\bar{R}$	$-\bar{B}L^-P$
1 			
2 			
3 			
4 			
5 			

Table 3