Nichols Algebras of Diagonal Type and Arithmetic Root Systems

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Introduction

The theory of Hopf algebras has its origins in the work of Hopf around 1940 on topological properties of Lie groups. He observed that for a connected Lie group G the cohomology group $H^*(G)$ with coefficients in a field has not only a graded algebra structure induced by the diagonal map $G \to G \times G$, $g \mapsto (g, g)$, but it admits also a comultiplication $\Delta : H^*(G) \to H^*(G) \otimes H^*(G)$ which comes from the multiplication map $\mathrm{m} : G \times G \to G$. Hopf algebras were studied as purely algebraic objects from the 1960s on. A good introduction to the subject can be found in [36], [1], [31].

Until the 1980s most of the known Hopf algebras were either commutative or cocommutative. This has been changed drastically with the discovery of quantized Kac–Moody algebras by Drinfel'd [13] and Jimbo [25]. After that new connections of Hopf algebra theory to several branches of mathematics like knot theory, deformation theory, Lie theory, representation theory, and combinatorics were found, and many mathematicians became interested in the subject.

One of the first main questions in any area of mathematics is the full classification of all objects of interest. In Hopf algebra theory this is a very difficult task, and it has a known solution only under strong additional hypotheses. Until now mainly semi-simple Hopf algebras, pointed Hopf algebras, and Hopf algebras of a fixed dimension n, where n is small or has only a few prime factors, have been classified. A good overview of the state of the art can be found in [2]. Nichols algebras, the central objects in this thesis, appear naturally in the classification of pointed Hopf algebras by the method of Andruskiewitsch and Schneider [5]. A Hopf algebra H over a field k is called *pointed* if all simple subcoalgebras of H are one-dimensional. Group algebras, universal enveloping algebras of Lie algebras, and quantized Kac–Moody algebras are all pointed. The coradical of a pointed Hopf algebra H is a group algebra kG. The graded coalgebra $\mathrm{gr}\, H$ associated to the coradical filtration of H is a graded Hopf algebra, and its degree zero subalgebra coincides with kG. Thus one has a canonical Hopf algebra projection $\pi : \mathrm{gr}\, H \to kG$. Call $R = \{x \in \mathrm{gr}\, H \,|\, (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\}$, where Δ denotes the coproduct of $\mathrm{gr}\, H$, the diagram of H. It is a graded braided Hopf algebra and $\mathrm{gr}\, H$ is the biproduct (bosonization) $R \# kG$ of R and kG in the sense of Radford (respectively Majid). The degree zero part of R is $k1$ and the degree one part is $P(R)$, the set of primitive elements of R. The subalgebra of R generated by $k1$ and $P(R)$ is a graded braided Hopf subalgebra of R

and is an important invariant of H — it is termed a *Nichols algebra*. The method of Andruskiewitsch and Schneider works as follows. First determine all possible Nichols algebras and the corresponding diagrams R, then find all groups G such that $R\#kG$ is a graded Hopf algebra. In the final step one has to determine all Hopf algebras H with given $\operatorname{gr} H = R\#kG$.

The symmetric algebra and the exterior algebra of a vector space, which appear in many areas of mathematics, are the most prominent examples of Nichols algebras, and they have been known for a long time. The systematic study of Nichols algebras started with the paper [32] where Nichols considered bialgebras of type one. In his article skew-differential operators (see Section 1.3) are introduced and graded pointed Hopf algebras with very small coradical are analyzed. Independently, in Woronowicz' work [38] about differential calculi on quantum groups the external algebra associated to a Hopf bimodule is defined explicitly by generators and relations. Lusztig [29] defined the quantized enveloping algebra $U_q(\mathfrak{n}_+)$, where $\mathfrak{n}_+$ is the nilpotent part of a semi-simple Lie algebra with respect to a fixed set of positive roots, as the quotient of a tensor algebra by the kernel of a bilinear form. In this approach the Cartan matrix is used to define the bilinear form, and the positive roots in the corresponding root system can be identified with the Poincaré–Birkhoff–Witt generators of $U_q(\mathfrak{n}_+)$. The relationship of Nichols algebras, Lusztig's construction, and Woronowicz' external algebras was observed by Schauenburg [34]. If R is the diagram of a pointed Hopf algebra H then $P(R)$ is a braided vector space (see Section 1.1). Schauenburg proved that the Nichols algebra corresponding to R is uniquely determined by this structure of $P(R)$ in a way described by Woronowicz (see also Definition 1.2.2). In what follows the Nichols algebra corresponding to a braided vector space (V,σ) will be denoted by $\mathcal{B}(V)$. Andruskiewitsch and Schneider [5] introduced the above mentioned method to classify pointed Hopf algebras, and performed the classification in the case where the base field has characteristic zero and is algebraically closed, and the diagram of the pointed Hopf algebra H is a quantum linear space. As a consequence they obtained the classification of pointed Hopf algebras of order p^3, where p is an odd prime, and the characterization of the pointed Hopf algebras whose coradical is abelian and has index p or p^2. A d-dimensional braided vector space (V,σ) is termed *of diagonal type* if V admits a basis $\{x_i \,|\, 1 \le i \le d\}$ such that for all $i,j \in \{1,2,\dots,d\}$ one has $\sigma(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for certain $q_{ij} \in k^* = k \setminus \{0\}$. Rosso [33] proved the first remarkable general classification result: if V is a connected braided vector space of diagonal type over the complex numbers, and the structure constants q_{ii} of V are positive, then $\mathcal{B}(V)$ has finite Gel'fand–Kirillov dimension if and only if V is of finite Cartan type. Note that these Nichols algebras are all infinite dimensional. Rosso gave also a construction method of $U_q(\mathfrak{n}_+)$ by induction over the rank of the Cartan matrix. Andruskiewitsch and Graña [3] showed the equivalence of several different definitions of a Nichols algebra. Graña [17] determined all Nichols algebras of dimension less than 32. He [16] as well as Milinski and Schneider [30] proved that Nichols algebras are free modules over their Nichols subalgebras. In the papers [3], [17], [30], and [4] one can find a few finite dimensional Nichols algebras of nonabelian group type. On the homepage of Graña `http://mate.dm.uba.ar/~matiasg/zoo.html` all currently known finite dimensional Nichols algebras of nonabelian group type are described.

Kharchenko [27] showed that any Hopf algebra, which is generated by a group algebra of an abelian group and a braided vector space of diagonal type, has a restricted Poincaré–Birkhoff–Witt basis. In his paper he also related his set of Poincaré–Birkhoff–Witt generators to Lyndon words and found restrictions on the heights of the Poincaré–Birkhoff–Witt generators. Andruskiewitsch and Schneider [6] showed under a weak condition on the structure constants q_{ij} that finite dimensional Nichols algebras of diagonal type are in fact of finite Cartan type (see Section 2.10). Moreover they classified coradically graded Hopf algebras of order p^4, where p is an odd prime. In the survey article [7] Andruskiewitsch and Schneider classified pointed Hopf algebras H such that the diagram of H is of Cartan type A_n. They listed the known rank 2 Nichols algebras of diagonal but not Cartan type — these were found previously by Nichols [32] and Graña [17]. At that time it was not yet clear, how to deal with these examples. Inspired by the work of Kharchenko [27], Ufer [37] showed that braided Hopf algebras generated by a braided vector space with triangular braiding (which is more general than a diagonal one) admit a restricted Poincaré–Birkhoff–Witt basis. This is an important information for the classification of pointed Hopf algebras of finite Gel'fand–Kirillov dimension with abelian coradical. In [4] Andruskiewitsch and Graña related Nichols algebras of nonabelian group type to the topological notion of racks. In a series of papers [18, 19, 22, 21, 20, 23] a classification of finite dimensional rank 2 and rank 3 Nichols algebras of diagonal type and finite dimensional Nichols algebras of Cartan type of arbitrary rank over fields of characteristic zero was carried out, and a sufficient set of relations in the rank 2 case was given. The first proofs were very technical. For them

Kharchenko's Poincaré–Birkhoff–Witt basis and skew-differential operators were used. Later the notion of Weyl groupoid and arithmetic root system were introduced which allowed to simplify the proofs enormously and made the classification in rank 3 possible. Recently Andruskiewitsch and Schneider [8] classified finite dimensional pointed Hopf algebras under the following assumptions: the base field has characteristic zero and is algebraically closed, the coradical kG is abelian, and all prime factors of $|G|$ are bigger than 7. The proof used partially results of [18, 19, 22]. The knowledge of Nichols algebras of diagonal type is also helpful in the nonabelian case, see for example [17] and [9].

The main motivation for the subject of this thesis was the following question of Andruskiewitsch [2], [5].

Question 5.9. Given a braided vector space V of diagonal type, decide when $\mathcal{B}(V)$ is finite dimensional. If so, compute $\dim_k \mathcal{B}(V)$, and give a "nice" presentation by generators and relations.

This question consists of two in their nature different parts, and the present thesis, where the results of [18], [19], [22], [21], [20], [23] are summarized, contributes to both of them.

As mentioned above, to the first part of the question there exist many partial answers. If the dimension of V is one then $\mathcal{B}(V)$ is finite dimensional if and only if q_{11} is a primitive mth root of 1, where $2 \le m < \infty$ [32]. In this case one has $\mathcal{B}(V) = k[x]/(x^m)$. If V is of Cartan type, then one has the classification results of Rosso, Andruskiewitsch and Schneider, and there exist a few simple examples of rank 2 which are not of Cartan type. It is also known that Nichols algebras of diagonal type admit many obvious Nichols subalgebras, and therefore an induction on the rank (like for semi-simple Lie algebras) could be a good way for a complete classification. However, even for rank 2 Nichols algebras of diagonal type the answer to the above question was missed, see [2, Question 5.40]. One of the most important problems was to find a method which allows to detect infinite dimensional Nichols algebras. The main idea in [19] was to find new (infinite dimensional) Nichols subalgebras $\mathcal{B}(W)$ of a Nichols algebra $\mathcal{B}(V)$, more precisely, to consider also braided subspaces $W \subset \mathcal{B}(V)$, $W \not\subset V$. Luckily, for all infinite dimensional Nichols algebras it was possible to find infinite dimensional Nichols subalgebras. The finite dimensionality of the remaining rank 2 Nichols algebras was proven in [18] by determining a sufficiently large set of relations using

Kharchenko's results [27]. It turned out that many of these finite dimensional algebras have the same dimension. Therefore it was likely that there exists an additional structure which could explain this phenomenon. On the other hand, there was a mysterious connection between Lie theory and Nichols algebras of diagonal type, and the Weyl group plays a crucial role in the theory of semi-simple Lie algebras. Therefore it was natural to ask whether there exists a structure in our setting which could play the role of the Weyl group. Since any Nichols algebra $\mathcal{B}(V)$ of diagonal type with $\dim_k V = d$ is $\mathbb{Z}^d$-graded and has a graded restricted Poincaré–Birkhoff–Witt basis [27], it is natural to consider the $\mathbb{Z}^d$-degrees of the restricted Poincaré–Birkhoff–Witt generators as a set of positive roots. In [22] it was shown that the vector space $\mathcal{B}(V) \otimes k[y_i]$, where $y_i \in V^*$ is homogeneous with respect to the $\mathbb{Z}^d$-grading, has an algebra structure which contains a Nichols algebra $\mathcal{B}(W)$ of diagonal type and rank d different from $\mathcal{B}(V)$. Moreover, the root system of $\mathcal{B}(W)$ was described with the help of that of $\mathcal{B}(V)$. It is also possible to recover $\mathcal{B}(V)$ from $\mathcal{B}(W)$. This fact leaded to the definitions of a Weyl groupoid, Weyl equivalence, and arithmetic root system. In contrast to the Weyl group of a semi-simple Lie algebra the elements of a Weyl groupoid map one single set of simple roots to another one instead of permuting all roots. This is the reason why composition of maps is only partially possible. Nevertheless, since sets of simple roots correspond to braided vector spaces of diagonal type, the action of the Weyl groupoid can be used to introduce an equivalence relation, the so called Weyl equivalence, for Nichols algebras of diagonal type. It turned out [20] that this equivalence relation can be used to simplify enormously the classification of finite dimensional rank 2 Nichols algebras of diagonal type. In order to deal with the algebras of higher rank the investigation of subsystems of arithmetic root systems [23] is very useful. The methods in [23], where the rank 3 case was considered, seem to be sufficient to detect all Nichols algebras of diagonal type with an infinite set of restricted Poincaré–Birkhoff–Witt generators.

By Lusztig's construction [29] of quantized enveloping algebras of semi-simple Lie algebras it was known that the quantum Serre relations are sufficient to define Nichols algebras of Cartan type as a quotient of the tensor algebra. However in general one needs more complicated relations. In order to manage the large variety of finite dimensional rank 2 Nichols algebras of diagonal type the set of degrees of their restricted Poincaré–Birkhoff–Witt generators were encoded in a full binary tree [18]. This is possible by an old result of Stern [35], see also [12], about pairs of relatively prime positive integers, and since there is at most one generator of a given degree, and there is no generator of degree (m_1, m_2) if m_1 and m_2 are not coprime. In [18] the results of Kharchenko [27] were interpreted in terms of such binary trees, and a sufficient set of relations was determined using skew-differential operators. Together with the results in [18] this gave a complete answer to the above question of Andruskiewitsch under the assumption that $\dim_k V = 2$. It is still an open problem how to obtain all relations if $\dim_k V > 2$.

The structure of the present thesis is the following. In Sections 1.1–1.3 Nichols algebras of diagonal type and related structures and tools are introduced. In Sections 1.4 and 1.5 the fundamental results of Kharchenko are formulated for Nichols algebras of diagonal type, and they are interpreted in terms of full binary trees. Section 1.6 contains Theorem 1.6.1 which is the main result of Chapter 1. It describes all rank two Nichols algebras of diagonal type having a finite Poincaré–Birkhoff–Witt basis with help of generators and explicit relations. The associated tables in Appendix A allow additionally to obtain easily the restricted Poincaré–Birkhoff–Witt generators and their degrees and heights. The proof of Theorem 1.6.1 is given in Section 1.7.

Chapter 2 is devoted to the general theory of arithmetic root systems. In Section 2.3, after a short recollection of the necessary structures in Sections 2.1 and 2.2, the most crucial result of this chapter, Proposition 2.3.1, is stated. It tells about certain relationships between Nichols algebras of diagonal type, and the description of these relationships in terms of the associated root systems. Proposition 2.3.1 is then used to define in Sections 2.4 and 2.5 the Weyl groupoid and the arithmetic root system associated to a pair (χ, E), where E is a basis of $\mathbb{Z}^d$ and $\chi : \mathbb{Z}^d \times \mathbb{Z}^d \to k^* = k \setminus \{0\}$ is a bicharacter on $\mathbb{Z}^d$ with values in a field k of characteristic zero. Arithmetic root systems are in their nature closely related to ordinary root systems as it can be seen in Propositions 2.5.1 and 2.5.2. Moreover, Theorem 2.5.3 gives a one-to-one correspondence between arithmetic root systems and Nichols algebras of diagonal type with a finite set of restricted Poincaré–Birkhoff–Witt generators. This theorem allows to reduce the problem of classification of finite dimensional Nichols algebras of diagonal type to that of arithmetic root systems, which is mainly a combinatorial problem. In Section 2.6 Weyl equivalence is defined and with Example 2.6.1 the typical way of application is demonstrated. The results on subsystems of arithmetic root systems given in Section 2.7 are the main tools for the classification of arithmetic root systems of rank 3 and higher. In particular, Proposition 2.7.1 tells that intersections of arithmetic root systems with subspaces of $\mathbb{R}^d$ are always arithmetic root subsystems. In Section 2.8 a necessary and sufficient criterion for the finiteness of a Weyl groupoid is given. In Section 2.9 it is recalled why it is sufficient to consider connected arithmetic root systems. In the last section of Chapter 2 arithmetic root systems of Cartan type are classified. Theorem 2.10.2 generalizes results of Rosso, Andruskiewitsch and Schneider in the sense that all technical assumptions on the structure constants of the braiding are removed. Finally, in Chapter 3 a classification of general rank 2 and rank 3 arithmetic root systems is given. The main results are Theorem 3.1.1 and Theorem 3.2.1.

In the thesis the following notation will be used. If R is a ring, M an R-module, and $S \subset R$, $N \subset M$, then write $S \cdot N := \{rm \,|\, r \in S,\, m \in N\}$ and $SN := \{\sum_j r_j m_j \,|\, r_j \in S, m_j \in N, r_j = 0 \text{ for all but finitely many } j\}$. Further, if not stated otherwise, k is a field of characteristic zero, $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ denotes the set of natural numbers strictly greater than 0, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Unadorned tensor product $\otimes$ means tensor product over k. The symbol R_m, $m \geq 2$, denotes the set of primitive mth roots of 1 in k. For an algebra B let B^{op} denote B with the opposite product. Similarly, if C is a coalgebra then C^{cop} means C with the opposite coproduct. For the coproduct and the antipode of a Hopf algebra the symbols $\varDelta$ and κ are used. The coproduct of elements a of a coalgebra is written in the Sweedler notation $\varDelta(a) = a_{(1)} \otimes a_{(2)}$.

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Chapter 1

Generators and Relations for Nichols Algebras of Diagonal Type

1.1 Yetter–Drinfel'd Modules

Let k be a field, G a group, and V a finite dimensional vector space over k. Then V is called a *Yetter–Drinfel'd module* over the group algebra kG if for all $g \in G$ there exist subspaces $V_g \subset V$ such that $V = \bigoplus_{g\in G} V_g$, and V is a left G-module with left action $. : kG \otimes V \to V$ which satisfies the condition $g.v \in V_{ghg^{-1}}$ for all $g, h \in G$ and $v \in V_h$. In Hopf algebraic terminology one says that V is a left kG-module and a left kG-comodule with left coaction $\delta : V \to kG \otimes V$, and the condition

$$\delta(a.v) = a_{(1)}v_{(-1)}\kappa(a_{(3)}) \otimes a_{(2)}.v_{(0)}$$

holds for all $a \in kG$ and $v \in V$, where $\delta(v) = v_{(-1)} \otimes v_{(0)}$.

Any Yetter–Drinfel'd module has a braiding $\sigma \in \mathrm{Aut}_k(V \otimes V)$ defined by the formula $\sigma(v \otimes w) = g.w \otimes v \in V_{ghg^{-1}} \otimes V_g$ for all $v \in V_g$ and $w \in V_h$, and hence σ satisfies the braid relation

$$(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)$$

on $V \otimes V \otimes V$. More generally, the braiding σ and its inverse are defined by the rules

$$\sigma(v \otimes w) =(v_{(-1)}.w) \otimes v_{(0)}, \qquad \sigma^{-1}(v \otimes w) =w_{(0)} \otimes (\kappa^{-1}(w_{(-1)}).v)$$

for all $v, w \in V$. Note that Yetter–Drinfel'd module structures on V with different groups can give the same braiding σ and not all braidings of V are induced by groups as described above. A pair (V, σ) with a finite dimensional vector space V and an invertible braiding is called a *braided vector space*.

If G is an abelian group, then for all $g \in G$ the space V_g is invariant under the action of G. Moreover, if G is finite and k is an algebraically closed field of characteristic zero then for all $g \in G$ the subspace V_g decomposes into the direct sum of one-dimensional G-modules. Direct sums of one-dimensional Yetter–Drinfel'd modules over kG are called *of diagonal type* [2, Def. 5.8]. In this case the braiding σ has a particularly simple form. Assume that $V = \bigoplus_{i=1}^n V_i$ and each V_i is a one-dimensional Yetter–Drinfel'd module over kG. Then there are unique $g_i \in G$ such that $V_i \subset V_{g_i}$, and numbers $q_{ij} \in k^* = k \setminus \{0\}$ such that $g_i|_{V_j} = q_{ij}\mathrm{id}$. One obtains that $\sigma(v \otimes w) = q_{ij} w \otimes v$ for all $v \in V_i$ and $w \in V_j$. One says that σ is of diagonal type. All braided vector spaces of diagonal type can be obtained from Yetter–Drinfel'd modules over abelian groups as described above.

1.2 Definitions and Notation

From now on let k be a field of characteristic zero. A bicharacter on a group $(H,\cdot)$ (with values in k^*) is a map $\chi : H\times H \to k^*$ satisfying the equations

$$\begin{gathered}\chi(1,a)=\chi(a,1)=1,\\ \chi(a\cdot b,c)=\chi(a,c)\chi(b,c), \qquad \chi(a,b\cdot c)=\chi(a,b)\chi(a,c)\end{gathered} \tag{1.1}$$

for all $a,b,c\in H$. Then $\chi^{\mathrm{op}} : H\times H\to k^*$, $\chi^{\mathrm{op}}(a,b):=\chi(b,a)$ is again a bicharacter, and the notation $\chi\chi^{\mathrm{op}}(a,b):=\chi(a,b)\chi(b,a)$ will be used for $a,b\in H$.

Let I be an index set and χ a bicharacter on $\mathbb{Z}^I$. Choose a basis $E=\{\mathbf{e}_i\,|\,i\in I\}$ of $\mathbb{Z}^I$. The numbers $q_{ij}:=\chi(\mathbf{e}_i,\mathbf{e}_j)$ are called the *structure constants of* χ with respect to E. For given E they determine χ uniquely.

Definition 1.2.1. Let E be a basis of $\mathbb{Z}^I$, and χ a bicharacter on $\mathbb{Z}^I$. Let q_{ij}, $i,j\in I$, denote the structure constants of χ with respect to E. The *generalized Dynkin diagram* of the pair (χ,E) is a (non-directed) graph $\mathcal{D}_{\chi,E}$ with the following properties:
(i) there is a bijective map ϕ from I to the vertices of $\mathcal{D}_{\chi,E}$,
(ii) for all $i\in I$ the vertex $\phi(i)$ is labeled by q_{ii},
(iii) for all $i,j\in I$ the number n_{ij} of edges between $\phi(i)$ and $\phi(j)$ is either 0 or 1. If $i=j$ or $q_{ij}q_{ji}=1$ then $n_{ij}=0$, otherwise $n_{ij}=1$ and the edge is labeled by $q_{ij}q_{ji}$.

Let (V,σ) be a d-dimensional braided vector space of diagonal type, where $d\in\mathbb{N}$. For the purpose of the present thesis the following definition of a Nichols algebra will be the most appropriate one.

Definition 1.2.2. Let $S_m\in \mathrm{End}_k(V^{\otimes m})$ and $S_{1,j}\in\mathrm{End}_k(V^{\otimes j+1})$ denote the maps

$$\begin{aligned}S_m&=\prod_{j=1}^{m-1}(\mathrm{id}^{\otimes m-j-1}\otimes S_{1,j}),\\ S_{1,j}&=\mathrm{id}+\sigma_{12}^{-1}+\sigma_{12}^{-1}\sigma_{23}^{-1}+\dots+\sigma_{12}^{-1}\sigma_{23}^{-1}\cdots\sigma_{j,j+1}^{-1}\end{aligned}$$

(in leg notation) for $m\geq 2$ and $j\in\mathbb{N}$. Then the subspace $S=\bigoplus_{m=2}^{\infty}\ker S_m$ of the tensor algebra $V^{\otimes}=\bigoplus_{m=0}^{\infty}V^{\otimes m}$ is a two-sided ideal, and the algebra $\mathcal{B}(V)=V^{\otimes}/S$ is termed the *Nichols algebra* associated to (V,σ). The unique maximal ideal of $\mathcal{B}(V)$ is denoted by $\mathcal{B}(V)^+$.

Let $\{x_i\,|\,1\leq i\leq d\}$ be a basis of V such that there exist numbers $q_{ij}\in k^*$, $i,j\in\{1,2,\dots,d\}$, with $\sigma(x_i\otimes x_j)=q_{ij}x_j\otimes x_i$. Since V is a braided vector space of diagonal type, such a basis always exists and is called *a canonical basis* of V. For convenience choose an abelian group G (for example $\mathbb{Z}^d$) and elements $g_1,\dots,g_d\in G$ such that the assignments

$$\delta(x_i)=g_i\otimes x_i, \qquad g_i.x_j=q_{ij}x_j \tag{1.2}$$

define a Yetter–Drinfel'd module structure on V. Let V^* denote the Yetter–Drinfel'd module dual to V. More precisely, for $f\in V^*$ one has

$$(h.f)(v)=f(\kappa(h).v), \quad f_{(0)}(v)f_{(-1)}=f(v_{(0)})\kappa^{-1}(v_{(-1)}) \quad \forall v\in V, h\in kG,$$

where $\delta(f)=f_{(-1)}\otimes f_{(0)}$. Note that the antipode κ of group algebras satisfies $\kappa^2=\mathrm{id}$ and hence left and right duals of Yetter–Drinfel'd modules coincide. Let $\{y_i\,|\,1\leq i\leq d\}$ denote the dual basis of V^*. Then one has

$$\delta(y_i)=g_i^{-1}\otimes y_i, \qquad g_i.y_j=q_{ij}^{-1}y_j, \qquad \sigma(y_i\otimes y_j)=q_{ij}y_j\otimes y_i.$$

Thus for diagonal braidings the linear map $\iota : V\to V^*$, $\iota(x_i):=y_i$ for $1\leq i\leq d$, extends to an algebra isomorphism $\iota:\mathcal{B}(V)\to\mathcal{B}(V^*)$.

Let $E=\{\mathbf{e}_i\,|\,1\leq i\leq d\}$ denote a basis of the $\mathbb{Z}$-module $\mathbb{Z}^d$. The tensor algebra $V^{\otimes}$ admits a $\mathbb{Z}^d$-grading defined by

$$\deg x_i:=\mathbf{e}_i, \quad 1\leq i\leq d. \tag{1.3}$$

The corresponding total degree totdeg is the $\mathbb{Z}$-grading of $V^{\otimes}$ defined by $\mathrm{totdeg}(x_i)=1$ for $1\leq i\leq d$. Note that the map σ is $\mathbb{Z}^d$-graded and hence the Nichols algebra $\mathcal{B}(V)$ inherits the $\mathbb{Z}^d$- and the $\mathbb{Z}$-grading of $V^{\otimes}$. Let $\mathcal{B}(V)_n$, $n\in\mathbb{N}_0$, denote the set of homogeneous elements of $\mathcal{B}(V)$ of total degree n.

Let V be a Yetter–Drinfel'd module of diagonal type over an abelian group G, and let $\{x_1,\dots,x_d\}$ be a basis of V such that the equations in (1.2) hold for certain $g_i\in G$ and $q_{ij}\in k^*$. Choose a basis $E=\{\mathbf{e}_1,\dots,\mathbf{e}_d\}$ of $\mathbb{Z}^d$. Then there exist a unique group homomorphism $g:\mathbb{Z}^d\to G$ and a unique bicharacter $\chi:\mathbb{Z}^d\times\mathbb{Z}^d\to k^*$ satisfying

$$g(\mathbf{e}_i)=g_i, \qquad \chi(\mathbf{e}_i,\mathbf{e}_j)=q_{ij} \tag{1.4}$$

for $i,j\in\{1,2,\dots,d\}$. For notational convenience we will also write $g(x)$ and $\chi(x',x'')$ instead of $g(\deg x)$ and $\chi(\deg x',\deg x'')$ for homogeneous elements

$x, x', x'' \in V^{\otimes}$ and $x, x', x'' \in \mathcal{B}(V)$. The generalized Dynkin diagram of (χ, E) will also be called the generalized Dynkin diagram of V.

On the other hand, if $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ is a basis of $\mathbb{Z}^d$ and χ is a bicharacter on $\mathbb{Z}^d$ then there exists a unique (up to isomorphism) braided vector space (V, σ) such that the equations $\sigma(x_i \otimes x_j) = \chi(\mathbf{e}_i, \mathbf{e}_j) x_j \otimes x_i$, where $i, j \in \{1, 2, \ldots, d\}$, hold with respect to a canonical basis $\{x_1, \ldots, x_d\}$ of V. One says that (V, σ) is *the braided vector space associated to the pair* (χ, E).

1.3 Duality of Nichols Algebras

Now we are going to give a slightly modified version (see Lemma 1.3.1) of Lusztig's bilinear form [29], cf. [7]. It is closely related to differential operators on $\mathcal{B}(V)$ defined in [17], [16] and [7].

By Definition 1.2.2 there exists a bilinear map $\langle \cdot, \cdot \rangle : V^* \times \mathcal{B}(V) \to \mathcal{B}(V)$ such that $\langle f, 1 \rangle := 0$ and

$$\langle f, \rho \rangle := \sum_i f(a_i) \rho_i, \text{ where } S_{1,m-1}(\rho) = \sum_i a_i \otimes \rho_i \in V \otimes V^{\otimes m-1}$$

for $m > 0$ and $\rho \in V^{\otimes m}$. Since $(f \otimes \mathrm{id}) \sigma^{-1}(v \otimes w) = (f_{(-1)}.v) f_{(0)}(w)$ for all $v, w \in V$ and $f \in V^*$ the map $\langle \cdot, \cdot \rangle$ has the property

$$\langle f, \rho\rho' \rangle = \langle f, \rho \rangle \rho' + (f_{(-1)}.\rho) \langle f_{(0)}, \rho' \rangle$$

for all $f \in V^*$ and $\rho, \rho' \in \mathcal{B}(V)$. Note that $\rho = 0$ in $\mathcal{B}(V)^+$ if and only if $\langle f, \rho \rangle = 0$ for all $f \in V^*$. Moreover, the mappings D_i in [7] and [17] can be considered as mappings $\langle y_i, \cdot \rangle$ where $\{y_i\}$ is a canonical basis of V^*.

Let $\mathcal{B}(V^*) \# kG$ denote the set $\mathcal{B}(V^*) \otimes kG$ with the product

$$(f' \otimes g')(f'' \otimes g'') := f'(g'.f'') \otimes g'g''$$

for all $f', f'' \in \mathcal{B}(V^*)$ and $g', g'' \in G$. Then $\mathcal{B}(V^*) \# kG$ becomes a Hopf algebra with coproduct

$$\Delta(f) = f \otimes 1 + \delta(f), \qquad \Delta(g) = g \otimes g, \qquad f \in V^*, g \in G.$$

Lemma 1.3.1. *There exists a unique bilinear map* $\langle \cdot, \cdot \rangle : (\mathcal{B}(V^*) \# kG) \times \mathcal{B}(V) \to \mathcal{B}(V)$ *satisfying* $\langle f, v \rangle = f(v)$ *for* $f \in V^*$, $v \in V$, *and*

$$\langle f, \rho\rho' \rangle = \langle f, \rho \rangle \rho' + (f_{(-1)}.\rho) \langle f_{(0)}, \rho' \rangle, \qquad \langle g, \rho \rangle = g.\rho,$$
$$\langle h_1 h_2, \rho \rangle = \langle h_1, \langle h_2, \rho \rangle \rangle$$

for all $f \in V^*$, $g \in kG$, $h_1, h_2 \in \mathcal{B}(V^*) \# kG$, *and* $\rho, \rho' \in \mathcal{B}(V)$.

Since $\Delta(f) = f \otimes 1 + \delta(f)$ for $f \in V$ and $\Delta(g) = g \otimes g$ for $g \in G$ the following corollary is an immediate consequence of Lemma 1.3.1.

Corollary 1.3.2. *For any* $f \in \mathcal{B}(V^*) \# kG$ *and* $\rho, \rho' \in \mathcal{B}(V)$ *the formula* $\langle f, \rho\rho' \rangle = \langle f_{(1)}, \rho \rangle \langle f_{(2)}, \rho' \rangle$ *holds where* $\Delta(f) = f_{(1)} \otimes f_{(2)}$.

Proof of the Lemma. It is clear that the restriction of $\langle \cdot, \cdot \rangle$ to $V^* \times \mathcal{B}(V)$ has to be the map $\langle \cdot, \cdot \rangle$ introduced above. Therefore the uniqueness assertion immediately follows. Further, equation

$$\langle g.f, \rho \rangle = g^{-1}.\langle f, g.\rho \rangle, \quad g \in G, f \in V^*, \rho \in \mathcal{B}(V),$$

implies that there exists at least a map $\langle \cdot, \cdot \rangle : (\bigoplus_{m=0}^{\infty} V^{*\otimes m} \# kG) \times \mathcal{B}(V) \to \mathcal{B}(V)$ satisfying the above equations.

For $1 \le l \le m$ there exist linear maps $S_{l,m-l} \in \mathrm{End}_k(V^{\otimes m})$ such that $\prod_{j=1}^{l} (\mathrm{id}^{\otimes l-j} \otimes S_{1,m-l+j-1}) = (S_l \otimes \mathrm{id}^{\otimes m-l}) S_{l,m-l}$. Thus for $l, m \in \mathbb{N}_0$, $l \le m$, $f_i \in V^*$, $1 \le i \le l$, and $\rho \in V^{\otimes m}$ one has

$$\langle f_1, \langle f_2, \cdots \langle f_l, \rho \rangle \rangle \rangle = (f_l \otimes f_{l-1} \otimes \cdots f_1 \otimes \mathrm{id}^{m-l})((S_l \otimes \mathrm{id}^{m-l}) S_{l,m-l}(\rho)).$$

Moreover, direct computation shows that

$$(f \otimes g)(\sigma^{-1}(v \otimes w)) = \sigma^{-1}(g \otimes f)(w \otimes v)$$

for $f, g \in V^*$ and $v, w \in V$. Therefore $\langle \ker S_l^*, \mathcal{B}(V) \rangle = 0$ where the map $S_l^* \in \mathrm{End}_k(V^{*\otimes l})$ is defined analogously to S_l. ■

The skew-differential operators $\langle y_i, \cdot \rangle \in \mathrm{End}_k(\mathcal{B}(V))$ give immediately information about $\mathcal{B}(V)$. For $\rho \in \mathcal{B}(V)$ define $(\mathrm{ad}_\sigma x_i)(\rho) := x_i \rho - (g_i.\rho) x_i$. Moreover, for $n \in \mathbb{N}_0$ and $q \in k^*$ set $(n)_q := \sum_{i=0}^{n-1} q^i$ and $(n)_q^! := \prod_{i=1}^{n} (i)_q$.

Lemma 1.3.3. *[7, Lemma 3.7]*
(i) One has $x_i^n = 0$ *if and only if* $q_{ii} \in R_r$, *where* $1 < r \le n$.
(ii) If $i \ne j$ *then* $(\mathrm{ad}_\sigma x_i)^r (x_j) = 0$ *if and only if* $(r)_{q_{ii}}^! \prod_{l=0}^{r-1} (q_{ii}^l q_{ij} q_{ji} - 1) = 0$.

Proof. By induction one obtains the formulas

$$\langle y_j, x_j^r \rangle = (r)_{q_{jj}^{-1}} x_j^{r-1}, \qquad \langle y_l, x_j^r \rangle = 0,$$
$$\langle y_j, (\mathrm{ad}_\sigma x_i)^r (x_j) \rangle = \prod_{l=0}^{r-1} (q_{ji}^{-1} - q_{ii}^l q_{ij}) x_i^{r-1}, \quad \langle y_l, (\mathrm{ad}_\sigma x_i)^r (x_j) \rangle = 0 \qquad (1.5)$$

for all $l \ne j$. This implies the claim. ■

1.4 PBW Bases of Nichols Algebras of Diagonal Type

In [27] Kharchenko proves that certain Hopf algebras "of diagonal type" have a Poincaré–Birkhoff–Witt basis. He also gives very useful information on the relations of the algebra. It is worth to mention that (as was shown by Ufer in [37]) such results hold in a more general context, namely if the diagonal braiding is generalized to a triagonal one. In order to prove Theorem 1.6.1 results from [28] and [27] are recalled and adapted to our conventions. The symbol $>$ for the lexicographic order in [27] is replaced by $<$.

Set $X := \{\alpha_1, \dots, \alpha_d\}$ and consider the total order $<$ on X given by $\alpha_i < \alpha_j$ for $i < j$. Let $\mathbb{X}$ and $\mathbb{X}^+$ denote the set of words and nonempty words, respectively, in the letters of X. Then $<$ induces the lexicographic order on $\mathbb{X}$: $u, v \in \mathbb{X}$ satisfy $u < v$ if and only if either $v = uw$ for some $w \in \mathbb{X}^+$ or there exist $u_1, u_2, u_3 \in \mathbb{X}$ and $i < j$ such that $u = u_1\alpha_i u_2$ and $v = u_1\alpha_j u_3$. For $u, v \in \mathbb{X}$ we write $u \le v$ if $u = v$ or $u < v$. The length of a word u, i. e. the number of its letters, is denoted by $|u|$. A word $u \in \mathbb{X}^+$ is called *Lyndon* if for any decomposition $u = vw$ with $v, w \in \mathbb{X}^+$ the relation $vw < wv$ holds.

Proposition 1.4.1. *(i) [28, Prop. 5.1.2] A word $u \in \mathbb{X}^+$ is Lyndon if and only if $u = vw$ with $v, w \in \mathbb{X}^+$ implies $u < w$.*
(ii) [28, Prop. 5.1.3] A word $u \in \mathbb{X}^+$ is Lyndon if and only if either $u \in X$ or there exist Lyndon words $v, w \in \mathbb{X}$ such that $v < w$ and $u = vw$.

Any word $u \in \mathbb{X}$ has a unique decomposition into the product of a non-increasing sequence of Lyndon words [28, Thm. 5.1.5]. Further, any Lyndon word $u \notin X$ has a decomposition into the product of two Lyndon words $u = vw$ (which then satisfy $v < w$) such that $|v|$ is minimal. This is called the *Shirshow decomposition* of u.

Proposition 1.4.2. *[28, Prop. 5.1.4] Suppose that u, v, w are Lyndon words and $u = vw$. Then $u = vw$ is the Shirshow decomposition of u if and only if $v \in X$ or for the Shirshow decomposition $v = v_1v_2$ the relation $w \le v_2$ holds.*

Let k be a field of characteristic zero, G an abelian group, and $V \in {}^{kG}_{kG}\mathcal{YD}$ a d-dimensional Yetter–Drinfel'd module of diagonal type. Let $\mathcal{I}$ be an $\mathbb{N}_0^d$-graded ideal of $V^{\otimes}$, $\mathcal{A}$ the $\mathbb{N}_0^d$-graded algebra $V^{\otimes}/\mathcal{I}$, and $\mathcal{A}_n$ the subspace

of $\mathcal{A}$ of homogeneous elements of total degree n (where $n \in \mathbb{N}_0$). Let $\mathcal{A}^+$ denote the unique maximal ideal of $\mathcal{A}$. The following results will be needed for $\mathcal{A} = V^{\otimes}$ and for $\mathcal{A} = \mathcal{B}(V)$.

After fixing a canonical basis $\{x_1, x_2, \dots, x_d\}$ of V one can associate to any Lyndon word $u \in \mathbb{X}$ an element $[u] \in \mathcal{A}$ as follows. Set $[\alpha_i] := x_{d+1-i}$ for $1 \le i \le d$, and $[u] := [w][v] - \chi([w], [v])[v][w]$ if $u = vw$ is the Shirshow decomposition of u. Note that this definition differs from that in [27] by a constant for each Lyndon word u. However this is not relevant for the following assertions.

Lemma 1.4.3. *[27, Lemma 3] If u, v are Lyndon words with $u < v$ then $u^h < v$ for any $h > 0$.*

Lemma 1.4.4. *[27, Lemma 5] Let u be a Lyndon word with $|u| = m$. Then $[u] \in \mathcal{A}$ is a linear combination of monomials $[\alpha_{i_1}][\alpha_{i_2}]\cdots[\alpha_{i_m}] \in \mathcal{A}$, $\alpha_{i_j} \in X$, such that $u \le \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_m}$.*

Lemma 1.4.5. *[27, Lemma 6] If $u < v \in \mathbb{X}$ are Lyndon words then $[v][u] - \chi([v], [u])[u][v]$ is a linear combination of products $[u_1][u_2]\cdots[u_i]$ for certain $i \in \mathbb{N}$ and Lyndon words u_j with $u < u_j < v$ such that* $\deg([v][u]) = \deg([u_1][u_2]\cdots[u_i])$ *and* $uv \le u_1u_2\cdots u_i$.

For a Lyndon word $u \in \mathbb{X}$ define h_u as follows. If m is the smallest number such that $[u]^m$ can be written as a linear combination of products $[u_1][u_2]\cdots[u_i]$, $i \in \mathbb{N}$, where u_j are Lyndon words with $u < u_j$, and either $m = 1$ or $\chi([u], [u])$ is a primitive mth root of 1, then set $h_u := m$. Otherwise put $h_u := \infty$. By Lemma 1.4.3 the relation $u < u_1$ implies that $u^{h_u} < u_1$. Now since u^{h_u} is not the beginning of u_1 one obtains that $u^{h_u} < u_1u_2\dots u_i$ has to hold as well.

Define $B := \{u \in \mathbb{X} \mid u \text{ is Lyndon}, h_u > 1\}$. For each $u \in B$ let $S(u)_<$ and $S(u)$ denote the subalgebras of $\mathcal{A}$ generated by the sets $\{[v] \mid v \in B, u < v\}$ and $\{[v] \mid v \in B, u \le v\}$, respectively. Define $S(u)^+ := S(u) \cap \mathcal{A}^+$ and $S(u)^+_< := S(u)_< \cap \mathcal{A}^+$.

Theorem 1.4.6. *[27, Theorem 2] The set $\{[u_1]^{n_1}[u_2]^{n_2}\cdots[u_i]^{n_i} \mid i \in \mathbb{N}_0, u_j \in B, u_i < \cdots < u_2 < u_1, n_j < h_{u_j} \forall j\}$ is a basis of the vector space $\mathcal{A}$.*

Corollary 1.4.7. *For any $n \in \mathbb{N}$, $u \in B$ the sets*

$$\{[u_1]^{n_1}[u_2]^{n_2}\cdots[u_i]^{n_i} \mid i \in \mathbb{N}_0, u_j \in B, u < u_1 < u_2 < \cdots < u_i, n_j < h_{u_j} \forall j\}$$

and

$$\{[u_1]^{n_1}[u_2]^{n_2}\cdots[u_i]^{n_i}\,|\,i\in\mathbb{N}_0, u_j\in B, u\le u_1<u_2<\cdots<u_i, n_j<h_{u_j}\,\forall j\}$$

form a basis of the vector space $S(u)_<$ *and* $S(u)$, *respectively.*

Proof. Since $S(u)_< = \bigcup_{v\in B, u<v} S(v)$ it suffices to prove the assertion for $S(u)$. As $\mathcal{A}=\bigoplus_{n=0}^\infty \mathcal{A}_n$ and $S(u)=\bigoplus_{n=0}^\infty S(u)\cap\mathcal{A}_n$ the proof can be performed by induction on n. Note that for given $n\in\mathbb{N}_0$ the set $\mathcal{A}_n$ is finite dimensional and $\{v\in B\,|\,\deg([v])\le n\}$ is a finite set. Suppose that $\{u_1,u_2,\ldots,u_i\}=\{v\in B\,|\,u<v,\deg([v])\le n\}$ and $u_1<u_2<\ldots<u_i$. By Lemma 1.4.5 the elements $[u]^m[u_{j_1}][u_{j_2}]\cdots[u_{j_r}]$, $m<h_u$, $m\deg([u])+\sum_{s=1}^r\deg([u_{j_s}])=n$, span $S(u)\cap\mathcal{A}_n$. By induction hypothesis the elements $[u_1]^{n_1}[u_2]^{n_2}\cdots[u_i]^{n_i}$, $n_j<h_{u_j}$ for all j, where $\sum_{j=1}^i n_j\deg([u_j])=n-m\deg([u])$, span $S(u_1)\cap\mathcal{A}_{n-m\deg([u])}$ for all $m>0$. Therefore the elements $[u]^m[u_1]^{n_1}[u_2]^{n_2}\cdots[u_i]^{n_i}$, $n_j<h_{u_j}$ for all j, $m<h_u$, where $m\deg([u])+\sum_{j=1}^i n_j\deg([u_j])=n$, span $S(u)\cap\mathcal{A}_n$. Then Theorem 1.4.6 and a simple dimension argument imply the assertion. ■

Later we will need the fact that for $w\in B$ one has

$$S(w)=(S(w)S(w)_<^+ + S(w)[w]^2)\oplus k[w]\oplus k1 \tag{1.6}$$

as graded vector spaces which is one of the consequences of Corollary 1.4.7 and Lemma 1.4.5.

Proposition 1.4.8. *[27, Corollary 2] For a Lyndon word* u *of length* l *one has* $h_u=1$ *if and only if* $[u]$ *can be written as a linear combination of monomials* $[\alpha_{i_1}][\alpha_{i_2}]\cdots[\alpha_{i_l}]$, $\alpha_{i_j}\in X$ *for all* j, *such that* $u<\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_l}$.

1.5 Types of Rank 2 Nichols Algebras

For basic definitions in this section we refer to [12]. Recall that a binary tree T is a (nonempty finite) tree such that each node has at most two children. One says that T is *full* if each node of T has exactly zero or two children [12]. For examples see Appendix A. For the set of nodes of a full binary tree T which have zero and two children, respectively, we use the symbol $N_0(T)$ and $N_2(T)$, respectively. Let $\mathbf{r}(T)$ or simply $\mathbf{r}$ denote the root of the binary tree T. Further, we write $N(T)=N_0(T)\cup N_2(T)$ for the set of all nodes of T. Let $\{\text{‘}L\text{’},\text{‘}R\text{’}\}$ be a set with two elements and define $\bar N_2(T):=N_2(T)\cup\{\text{‘}L\text{’},\text{‘}R\text{’}\}$, $\bar N(T):=N(T)\cup\{\text{‘}L\text{’},\text{‘}R\text{’}\}$ (disjoint unions).

Definition 1.5.1. Let T be a full binary tree and $a\in N(T)$. A node $b\in\bar N_2(T)$ is called the *left godfather of* a, denoted by $b:=a^{\mathrm{L}}$, if one of the following conditions holds.

- $a=\mathbf{r}(T)$ and $b=\text{‘}L\text{’}$,
- a is the right child of b,
- a is the left child of its parent c and b is the left godfather of c.

Similarly one defines the *right godfather* a^{R} *of* a by replacing everywhere left by right and vice versa and setting $\mathbf{r}^{\mathrm{R}}:=\text{‘}R\text{’}$. If $a\in N_2(T)$ then let a_{L} and a_{R} denote the left and right child of a, respectively.

Note that $\cdot^{\mathrm{L}}$ and $\cdot^{\mathrm{R}}$ are well defined maps from $N(T)$ to $\bar N_2(T)$ and any $a\in N(T)$ is uniquely determined by a^{L} and a^{R}. Indeed, $a\in N(T)$, $b=a^{\mathrm{L}}$ implies that there is a subset $\{a_1,a_2,\ldots,a_m\}$ of $N(T)$ such that $m\in\mathbb{N}$, $a_m=a$, a_{i+1} is the left child of a_i for all $i>0$, and either $a_1=b_{\mathrm{R}}$ or $a_1=\mathbf{r}$, $b=\text{‘}L\text{’}$. Further, ${a_{i+1}}^{\mathrm{R}}=a_i$ for all $i>1$ and ${a_1}^{\mathrm{R}}\notin\{a_1,a_2,\ldots,a_m\}$. Thus a is uniquely determined by a^{L} and a^{R}.

We define functions $\ell^{\mathrm{L}},\ell^{\mathrm{R}},\ell_{\mathrm{L}}$, and $\ell_{\mathrm{R}}:N(T)\to\mathbb{N}$ recursively to denote lengths of certain branches of T. Set

$$\ell^{\mathrm{L}}(a):=\begin{cases}\ell^{\mathrm{L}}(a^{\mathrm{L}})+1 & \text{if } {a^{\mathrm{L}}}_{\mathrm{R}}=a,\\ 1 & \text{else,}\end{cases}\qquad \ell_{\mathrm{L}}(a):=\begin{cases}1 & \text{if } a\in N_0(T),\\ \ell_{\mathrm{L}}(a_{\mathrm{L}})+1 & \text{else,}\end{cases}$$

$$\ell^{\mathrm{R}}(a):=\begin{cases}\ell^{\mathrm{R}}(a^{\mathrm{R}})+1 & \text{if } {a^{\mathrm{R}}}_{\mathrm{L}}=a,\\ 1 & \text{else,}\end{cases}\qquad \ell_{\mathrm{R}}(a):=\begin{cases}1 & \text{if } a\in N_0(T),\\ \ell_{\mathrm{R}}(a_{\mathrm{R}})+1 & \text{else.}\end{cases}$$

Any full binary tree T can be identified with a subtree of the infinite *Stern–Brocot tree* [14, pp. 116–117], see also [35]. This means that there exists a map $\sigma_T:\bar N(T)\to\mathbb{Z}\times\mathbb{Z}$ defined recursively by $\sigma_T(\text{‘}L\text{’}):=(0,1)$, $\sigma_T(\text{‘}R\text{’}):=(1,0)$, and $\sigma_T(a):=\sigma_T(a^{\mathrm{L}})+\sigma_T(a^{\mathrm{R}})$ for any $a\in N(T)$. Note that $\sigma_T(a)\in\mathbb{N}\times\mathbb{N}$ for $a\in N(T)$ by the definition of $\cdot^{\mathrm{L}}$ and $\cdot^{\mathrm{R}}$ and since $\sigma_T(\mathbf{r})=(1,1)$. Thus the map $Q:\bar N(T)\to\mathbb{Q}\cup\{+\infty\}$,

$$Q(a):=\begin{cases}r/s & \text{if } \sigma_T(a)=(r,s),\ s\neq 0,\\ +\infty & \text{if } a=\text{‘}R\text{’},\end{cases}$$

and the total order $<$ on $\mathbb{Q}$ induce an order $<_Q$ on $\bar{N}(T)$ such that for all $a \in N(T)$ the relations 'L' $<_Q a <_Q$ 'R' hold. There is another natural map $|\sigma_T| : \bar{N}(T) \to \mathbb{Z}$ defined by $|\sigma_T|(a) = r + s$ whenever $\sigma_T(a) = (r, s)$. It will be used mainly for inductive proofs.

Proofs for assertions (i)–(iii) of the following Lemma can also be found for example in [14].

Lemma 1.5.1. *Let T be a full binary tree and $a, b \in N(T)$.*
(i) If $\sigma_T(a) = (r, s)$, $\sigma_T(a^{\mathrm{L}}) = (r_1, s_1)$, $\sigma_T(a^{\mathrm{R}}) = (r_2, s_2)$ then equations $rs_1 - r_1 s = r_2 s - r s_2 = r_2 s_1 - r_1 s_2 = 1$ hold.
(ii) The entries of $\sigma_T(c)$, $c \in \bar{N}(T)$, are relatively prime integers.
(iii) The map $Q : \bar{N}(T) \to \mathbb{Q} \cup \{+\infty\}$ is injective and hence $<_Q$ is a total order on $\bar{N}(T)$.
(iv) $a^{\mathrm{L}} <_Q a <_Q a^{\mathrm{R}}$.
(v) If $a <_Q b$ and $|\sigma_T|(a) < |\sigma_T|(b) + |\sigma_T|(b^{\mathrm{L}})$ then relation $a \leq_Q b^{\mathrm{L}}$ holds.
(vi) If $a <_Q b$ and $|\sigma_T|(b) < |\sigma_T|(a) + |\sigma_T|(a^{\mathrm{R}})$ then relation $a^{\mathrm{R}} \leq_Q b$ holds.
(vii) Relation $a^{\mathrm{L}} \in N(T)$ implies that $a <_Q a^{\mathrm{LR}}$.
(viii) Relation $a^{\mathrm{R}} \in N(T)$ implies that $a^{\mathrm{RL}} <_Q a$.
(ix) If $a \in N(T), b \in \bar{N}(T)$, $|\sigma_T|(b) \leq |\sigma_T|(a)$, and $a^{\mathrm{L}} <_Q b <_Q a^{\mathrm{R}}$ then one has $b = a$.
(x) If $a, b \in \bar{N}_2(T)$ such that $a <_Q b$ then there exists $c \in N(T)$ such that $a <_Q c <_Q b$.

Proof. (i) We prove this by induction on $|\sigma_T|(a)$. If $a = \mathbf{r}$ then $\sigma_T(a) = (1, 1)$ and the assertion holds. Otherwise let $c \in N(T)$ denote the parent of a. If a is the left child of c then $a^{\mathrm{R}} = c$, $a^{\mathrm{L}} = c^{\mathrm{L}}$, and $\sigma_T(c) = (r_2, s_2)$. Since $|\sigma_T|(c) < |\sigma_T|(a)$ equation $\sigma_T(c^{\mathrm{L}}) = (r_1, s_1)$ and induction hypothesis give $r_2 s_1 - r_1 s_2 = 1$. Thus $rs_1 - r_1 s = (r_1 + r_2)s_1 - r_1(s_1 + s_2) = 1$ and $r_2 s - r s_2 = r_2(s_1 + s_2) - (r_1 + r_2)s_2 = r_2 s_1 - r_1 s_2 = 1$. If $a = c_{\mathrm{R}}$ then one argues similarly.

(ii) This one gets from (i) using $\sigma_T(\text{'}L\text{'}) = (0, 1)$, $\sigma_T(\text{'}R\text{'}) = (1, 0)$.

(iv) This follows from (i) and the fact that $\sigma_T(c) \in \mathbb{N} \times \mathbb{N}$ for $c \in N(T)$.

(v) Suppose to the contrary that $b^{\mathrm{L}} <_Q a <_Q b$ and $|\sigma_T|(a) < |\sigma_T|(b) + |\sigma_T|(b^{\mathrm{L}})$. If $\sigma_T(a) = (r, s)$, $\sigma_T(b) = (r_2, s_2)$, and $\sigma_T(b^{\mathrm{L}}) = (r_1, s_1)$ then $r_2 s_1 - r_1 s_2 = 1$, $rs_1 - r_1 s \geq 1$ and $r_2 s - r s_2 \geq 1$. Therefore

$$(r_2 s)s_1 \geq (rs_1)s_2 + s_1 \geq r_1 s s_2 + s_2 + s_1 \Rightarrow s_1 + s_2 \leq s,$$
$$r_2(rs_1) \geq r_1(r_2 s) + r_2 \geq r_1 r s_2 + r_1 + r_2 \Rightarrow r_1 + r_2 \leq r.$$

This is a contradiction to $r+s = |\sigma_T|(a) < |\sigma_T|(b)+|\sigma_T|(b^{\mathrm{L}}) = r_1+r_2+s_1+s_2$.

(vi) Use arguments as in (v).

(iii) If $Q(a) = Q(b)$ then $\sigma_T(a) = \sigma_T(b)$ by (ii). Now $b^{\mathrm{L}} <_Q a$, $|\sigma_T|(b^{\mathrm{L}}) < |\sigma_T|(b) = |\sigma_T|(a)$, and (v) imply that $b^{\mathrm{L}} \leq_Q a^{\mathrm{L}}$. By symmetry one gets $Q(a^{\mathrm{L}}) = Q(b^{\mathrm{L}})$ and similarly $Q(a^{\mathrm{R}}) = Q(b^{\mathrm{R}})$. Thus using that a^{L} and a^{R} determine a uniquely induction on $|\sigma_T|(a)$ gives the assertion.

(vii) Again we use induction on $|\sigma_T|(a)$. Note that $a \neq \mathbf{r}$ since $a^{\mathrm{L}} \in N(T)$. If a is the left child of its parent c then $a^{\mathrm{L}} = c^{\mathrm{L}}$ and $a^{\mathrm{R}} = c$. By induction hypothesis $c <_Q c^{\mathrm{LR}}$ and hence (iv) implies that $a <_Q a^{\mathrm{R}} = c <_Q a^{\mathrm{LR}}$. If on the other hand a is the right child of its parent c then one gets $a^{\mathrm{L}} = c$ and $a^{\mathrm{R}} = c^{\mathrm{R}}$. Thus by (iv) one obtains $a <_Q a^{\mathrm{R}} = c^{\mathrm{R}} = a^{\mathrm{LR}}$.

(viii) The proof goes as for (vii).

(ix) Relation $b <_Q a^{\mathrm{R}}$ and the converse of (vi) imply that $b \leq_Q a$. Similarly, $a^{\mathrm{L}} <_Q b$ and the converse of (v) yield that $a \leq_Q b$. By (iii) one gets $a = b$.

(x) Set $c_1 := a_{\mathrm{R}}$ and $c_2 := b_{\mathrm{L}}$. Then $a <_Q c_1$ and $c_2 <_Q b$ by (iv). If $c_1 = b$ then $a = b^{\mathrm{L}} = {c_2}^{\mathrm{L}} <_Q c_2 <_Q {c_2}^{\mathrm{R}} = b$ again by (iv). In this case one can take $c = c_2$. Similarly if $a = c_2$ then $a = {c_1}^{\mathrm{L}} <_Q c_1 <_Q {c_1}^{\mathrm{R}} = a^{\mathrm{R}} = b$ and one can take $c = c_1$. Suppose now that $c_1 \neq b$ and $c_2 \neq a$. If $|\sigma_T|(a) \leq |\sigma_T|(b)$ then using $a <_Q {c_2}^{\mathrm{R}}$ the converse of (vi) implies that $a \leq_Q c_2$. Since $|\sigma_T|(c_2) > |\sigma_T|(a)$ one obtains $a <_Q c_2$ and hence one can set $c := c_2$. Similarly one gets $c := c_1 <_Q b$ if $|\sigma_T|(b) \leq |\sigma_T|(a)$. ∎

Suppose now that V is a Yetter–Drinfel'd module of diagonal type with $d := \dim_k V = 2$. Let $\{x_1, x_2\}$ denote a canonical basis of V. Then for a full binary tree T the following assignment defines a unique map $\tau_0 : \bar{N}(T) \to V^{\otimes}$.

- $\tau_0(\text{'}L\text{'}) := x_2$, $\tau_0(\text{'}R\text{'}) := x_1$.
- If $a \in N(T)$ then $\tau_0(a) := \tau_0(a^{\mathrm{R}})\tau_0(a^{\mathrm{L}}) - \chi(\tau_0(a^{\mathrm{R}}), \tau_0(a^{\mathrm{L}}))\tau_0(a^{\mathrm{L}})\tau_0(a^{\mathrm{R}})$.

Note that if $a \in \bar{N}(T)$ and $\sigma_T(a) = (r, s)$ then $\deg \tau_0(a) = r\mathbf{e}_1 + s\mathbf{e}_2$. To shorten notation we will write $\chi(a, b)$ and $g(a)$ instead of $\chi(\tau_0(a), \tau_0(b))$ and $g(\tau_0(a))$, respectively, for any $a, b \in \bar{N}(T)$. Let $\tau : \bar{N}(T) \to \mathcal{B}(V)$ denote the composition of τ_0 with the canonical map $V^{\otimes} \to \mathcal{B}(V)$.

Definition 1.5.2. Let $n \in \mathbb{N}_0$, T a full binary tree, V a Yetter–Drinfel'd module of diagonal type with $\dim_k V = 2$, and let $\mathcal{B}(V)$ denote the corresponding Nichols algebra. We say that $\mathcal{B}(V)$ *is of type T in degree n* if there exists a canonical basis $\{x_1, x_2\}$ of V such that the sets

$$\Big\{ \prod_{a\in \bar{N}_2(T)} \tau(a)^{i_a} \,\Big|\, 0 \le i_a < \operatorname{ord}\chi(a,a) \quad \forall a \in \bar{N}_2(T), \quad \sum_{a\in\bar{N}_2(T)} i_a|\sigma_T|(a) \le m \Big\}$$

form a basis of the vector spaces $\bigoplus_{i=0}^m \mathcal{B}(V)_i$, respectively, for all $m \le n$. Here the elements of $\bar{N}_2(T)$ are ordered with respect to the order $<_Q$, and $\operatorname{ord}\chi(a,a)$ is the order of $\chi(a,a)$ in the multiplicative group k^* if $\chi(a,a) \ne 1$ is a root of 1, and ∞ otherwise. Further, we say that $\mathcal{B}(V)$ *is of type* T if $\mathcal{B}(V)$ is of type T in degree n for all $n \in \mathbb{N}$.

Note that $\mathcal{B}(V)$ is of type T in degree 0 for any full binary tree T. Further, if $\mathcal{B}(V)$ is of type T for a full binary tree T and $\operatorname{ord}\chi(a,a) < \infty$ for all $a \in \bar{N}_2(T)$ then $\mathcal{B}(V)$ is finite dimensional. More exactly, one gets $\dim_k \mathcal{B}(V) = \prod_{a\in\bar{N}_2(T)} \operatorname{ord}\chi(a,a)$.

Lemma 1.5.2. *Let T be a full binary tree.*
(i) There exists a unique map $\gamma : \bar{N}(T) \to \mathbb{X}$ such that $\gamma(`L') = \alpha_1$, $\gamma(`R') = \alpha_2$, and $\gamma(a) = \gamma(a^{\mathrm{L}})\gamma(a^{\mathrm{R}})$ for all $a \in N(T)$.
(ii) For $a \in \bar{N}(T)$ the equation $|\gamma(a)| = |\sigma_T|(a)$ holds.
(iii) Any word $\gamma(a)$, $a \in \bar{N}(T)$, is Lyndon and $\gamma(a^{\mathrm{L}})\gamma(a^{\mathrm{R}})$ is the Shirshow decomposition of $\gamma(a)$ for $a \in N(T)$.
(iv) For any $a, b \in \bar{N}(T)$ the relation $\gamma(a) < \gamma(b)$ is equivalent to $a <_Q b$.

Proof. (i) Existence and uniqueness of γ follow from the facts that $(\cdot)^{\mathrm{L}}$ and $(\cdot)^{\mathrm{R}}$ are well defined maps from $N(T)$ to $\bar{N}(T)$ and $|\sigma_T|(a^{\mathrm{L}}) < |\sigma_T|(a)$, $|\sigma_T|(a^{\mathrm{R}}) < |\sigma_T|(a)$ for all $a \in N(T)$.

(ii) This follows immediately from the definition of γ and $|\sigma_T|$.

(iii), (iv) We use induction on $|\sigma_T|(a)$ and $\max\{|\sigma_T|(a), |\sigma_T|(b)\}$, respectively. If $a = `L'$ or $a = `R'$ then $\gamma(a)$ is Lyndon. Further, if $|\sigma_T|(a) = |\sigma_T|(b) = 1$ then $a, b \in \{`L', `R'\}$ and hence $a <_Q b$ is equivalent to $a = `L', b = `R'$ which holds if and only if $\gamma(a) = \alpha_1 < \alpha_2 = \gamma(b)$.

Assume now that (iii) and (iv) hold whenever $a, b \in \bar{N}(T)$, $|\sigma_T|(a) \le n$, and $|\sigma_T|(b) \le n$ for some $n \in \mathbb{N}$. If $a \in N(T)$ then by induction hypothesis $\gamma(a^{\mathrm{L}})$ and $\gamma(a^{\mathrm{R}})$ are Lyndon words. Since $a^{\mathrm{L}} <_Q a^{\mathrm{R}}$ we also have $\gamma(a^{\mathrm{L}}) < \gamma(a^{\mathrm{R}})$. Thus $\gamma(a^{\mathrm{L}})\gamma(a^{\mathrm{R}})$ is Lyndon by Proposition 1.4.1(ii). This proves the induction step of the first part of (iii).

Now we prove (iv) in the case $|\sigma_T|(a) = n+1$, $|\sigma_T|(b) \le n$. The proof for $|\sigma_T|(b) = n+1$ is completely analogous. Let $(a_1, a_2, \dots, a_m)$ denote the

set of nodes of T with $|\sigma_T|(a_i) \le n$ in increasing order with respect to $<_Q$. By Lemma 1.5.1(ix) the node $a \in N(T)$ is the unique $c \in \bar{N}(T)$ such that $|\sigma_T|(c) \le n+1$ and $a^{\mathrm{L}} <_Q c <_Q a^{\mathrm{R}}$. Thus there exists $i \in \mathbb{N}$ such that $a^{\mathrm{L}} = a_i$ and $a^{\mathrm{R}} = a_{i+1}$. On the other hand, the induction hypothesis gives that $\gamma(a_j) < \gamma(a_l)$ if and only if $j < l$. Now note that $\gamma(a_i) < \gamma(a) = \gamma(a_i)\gamma(a_{i+1}) < \gamma(a_{i+1})$ as $\gamma(a)$ is Lyndon.

It remains to show that $\gamma(a^{\mathrm{L}})\gamma(a^{\mathrm{R}})$ is the Shirshow decomposition of $\gamma(a)$ where $|\sigma_T|(a) = n+1$. If $a^{\mathrm{L}} = `L'$ then $|\gamma(a^{\mathrm{L}})| = 1$ and we are done. Otherwise $a^{\mathrm{L}} \in N(T)$ and Lemma 1.5.1(vi),(vii) gives $a^{\mathrm{R}} \le_Q a^{\mathrm{LR}}$. Therefore $\gamma(a^{\mathrm{R}}) \le \gamma(a^{\mathrm{LR}})$ by the induction hypothesis for (iv). Further, the induction hypothesis of (iii) tells us that $\gamma(a^{\mathrm{LL}})\gamma(a^{\mathrm{LR}})$ is the Shirshow decomposition of $\gamma(a^{\mathrm{L}})$. Thus Proposition 1.4.2 for $u = \gamma(a)$ together with the last two relations give the claim. ∎

Our aim in this section is to give a computable criterion which ensures that the Nichols algebras in Theorem 1.6.1 are of the given type. To do so we have to introduce additional notation which will be needed only for $\mathcal{A} = \mathcal{B}(V)$.

For a Lyndon word u and $n \in \mathbb{N}$ let $\mathcal{F}(u)_n$ denote the kG-module

$$\mathcal{F}(u)_n := \Big(S(u) \cap \big(\mathcal{B}(V)S(u)^+_< + \mathcal{B}(V)[u]^2\big) \cap \bigoplus_{m=1}^n \mathcal{B}(V)_m\Big)\#kG$$

and set $\mathcal{F}(u)_0 = \{0\}$. By Corollary 1.4.7 one obtains that

$$\mathcal{F}(u)_n = \Big(\big(S(u)S(u)^+_< + S(u)[u]^2\big) \cap \bigoplus_{m=1}^n \mathcal{B}(V)_m\Big)\#kG.$$

Let $n \in \mathbb{N}$. Suppose that there exists a full binary tree T such that for any Lyndon word u with $|u| \le n$ the relation $h_u > 1$ is equivalent to $u = \gamma(a)$ for some $a \in \bar{N}_2(T)$. The definition of τ and $[\cdot]$ and Lemma 1.5.2(iii) imply that for any $a \in \bar{N}_2(T)$ one has $\tau(a) = [\gamma(a)]$. Then by Corollary 1.4.7 with $u = \alpha_1$ and by Lemma 1.5.2(iv) the set

$$\Big\{ \prod_{a\in\bar{N}_2(T)} \tau(a)^{i_a} \,\Big|\, i_a < h_{\gamma(a)}, \sum_{a\in\bar{N}_2(T)} i_a \operatorname{totdeg}(\tau(a)) \le n \Big\}$$

where the product is taken with respect to the order $<_Q$ of $\bar{N}_2(T)$ forms a basis of $\bigoplus_{m=0}^n \mathcal{B}(V)_m$.

For $a \in \bar{N}(T)$ define $p_a := \chi(a,a)^{-1}$ and

$$\lambda(a) := \begin{cases} 0 & \text{if } a \notin N(T), \\ \chi(\text{`}L\text{'}, \text{`}R\text{'})^{-1} - \chi(\text{`}R\text{'}, \text{`}L\text{'}) & \text{if } a = \mathbf{r}, \\ \chi(a^{\mathrm{L}}, a^{\mathrm{R}})^{-1} - \chi(a^{\mathrm{R}}, a^{\mathrm{L}}) + \lambda(b) & \text{otherwise,} \end{cases} \tag{1.7}$$

where b is the parent of a. Further, for any $b \in N(T)$ with $b^{\mathrm{L}} \in N(T)$ set

$$\mu(b) := \begin{cases} \lambda(b) & \text{if } b = b^{\mathrm{L}}{}_{\mathrm{R}}, \\ \lambda(b)\mu(b^{\mathrm{R}}) & \text{otherwise.} \end{cases}$$

Finally, for any $b \in N(T)$ with $c := b^{\mathrm{L}} \in N(T)$ and $\ell^{\mathrm{R}}(b) \leq 2$ set

$$\nu(b) := \begin{cases} \chi(c^{\mathrm{L}}, b)^{-1} - \chi(b, c^{\mathrm{L}}) + \lambda(b)\lambda(c)\Big((2)_{p_f}^{-1} - (2)_{p_c}^{-1}\Big) & \text{if } \ell^{\mathrm{R}}(b) = 1, \\ \chi(c^{\mathrm{L}}, c_{\mathrm{R}})^{-1} + \lambda(c)\lambda(c_{\mathrm{R}})(2)_{p_c}^{-1}\Big((2)_{p_f}^{-1} - (3)_{p_c}^{-1}\Big) & \text{if } \ell^{\mathrm{R}}(b) = 2, \end{cases}$$

where $f = c^{\mathrm{R}}$, whenever all denominators are nonzero.

1.6 Explicit Examples

The following theorem is the main result of this chapter.

Theorem 1.6.1. *Let k be a field of characteristic zero and V a braided vector space of diagonal type with $\dim_k V = 2$. Suppose that the generalized Dynkin diagram of V appears in Table A.1 and is marked with the tree Tn, where $n \in \{1, 2, \ldots 22\}$ and the full binary tree Tn is given in Appendix A. Then $\mathcal{B}(V)$ is of type Tn. Moreover, all relations of $\mathcal{B}(V)$ are elements of the ideal of $V^{\otimes}$ generated by the set*

$$\begin{aligned} &\{\tau_0(a) \mid a \in N_0(T)\} \cup \{\tau_0(a)^{\operatorname{ord} p_a} \mid a \in \bar{N}_2(T), 2 \leq \operatorname{ord} p_a < \infty\} \cup \\ &\quad \{\tau_0(b)\tau_0(c^{\mathrm{L}}) - \chi(b, c^{\mathrm{L}})\tau_0(c^{\mathrm{L}})\tau_0(b) - \mu(b)/(\ell^{\mathrm{R}}(b)+1)^!_{p_c}\tau_0(c)^{\ell^{\mathrm{R}}(b)+1} \mid \\ &\qquad b \in N_2(T), c := b^{\mathrm{L}} \in N_2(T)\}. \end{aligned} \tag{1.8}$$

In particular, if q_{11}, q_{22}, and $q_{12}q_{21}$ are roots of 1 (and q_{11}, q_{22} are different from 1 for type T1) then the Nichols algebra $\mathcal{B}(V)$ is finite dimensional.

The proof of Theorem 1.6.1 will be given in Section 1.7.

Remark. Many of the rank 2 Nichols algebras in Theorem 1.6.1 appeared in the literature previously. Those with $q_{12}q_{21} = q_{11}^{-n}$, $q_{22} = q_{11}^{n}$, $n \in \{0, 1, 2, 3\}$, are called *of finite Cartan type* in [7]. They have type T1, T2, T3, and T8. Other examples (cf. Section 3.3 in [7]) cover essentially all Nichols algebras of type T2 and T3. Further, there exist recent computations on Nichols algebras by M. Graña and Ch. Heaton [15] which give a PBW basis for the examples of type T5 and T9. ■

1.7 The Finiteness Results

From now on let T be a full binary tree such that

$$\min\{\ell_{\mathrm{R}}(b_{\mathrm{L}}), \ell_{\mathrm{L}}(b_{\mathrm{R}})\} \leq 3 \quad \text{for all } b \in N_2(T), \tag{1.9}$$

i.e. either $b_{\mathrm{LRRR}} \notin N(T)$ or $b_{\mathrm{RLLL}} \notin N(T)$. Note that all binary trees in Appendix A satisfy this condition.

Definition 1.7.1. We call a triple (T, V, n) where $n \in \mathbb{N}_0$ and $V \in {}^{kG}_{kG}\mathcal{YD}$ is a two-dimensional Yetter–Drinfel'd module of diagonal type *admissible* if the following conditions are satisfied.

- For $a \in N(T)$ with $|\sigma_T|(a) \leq n$ relation $a \in N_2(T)$ is equivalent to $\lambda(a) \neq 0$,
- the numbers p_a for $a \in \bar{N}_2(T)$, $|\sigma_T|(a) \leq n$, are either different from 1, or $N_2(T) = \{\}$,
- if $a \in N_2(T)$, $a_{\mathrm{L}} \in N_2(T)$, and $|\sigma_T|(a_{\mathrm{L}}) \leq n$ then $p_a \neq -1$, $p_{a^{\mathrm{R}}} \neq -1$,
- for all $b \in N_2(T)$ with $c := b^{\mathrm{L}} \in N_2(T)$ and $|\sigma_T|(b) + |\sigma_T|(b^{\mathrm{LL}}) \leq n$ one has $(\ell^{\mathrm{R}}(b)+1)^!_{p_c} \neq 0$ and either $\ell_{\mathrm{R}}(c_{\mathrm{L}}) \leq \ell^{\mathrm{R}}(b)$ or $\ell^{\mathrm{R}}(b) \leq 2$, $\nu(b) = 0$.

Note that if $b \in N_2(T)$ and $c := b^{\mathrm{L}} \in N_2(T)$ then $\ell^{\mathrm{R}}(b) < \ell_{\mathrm{L}}(c_{\mathrm{R}})$. Thus by (1.9) for c one has either $\ell^{\mathrm{R}}(b) \leq 2$ or $\ell_{\mathrm{R}}(c_{\mathrm{L}}) \leq \ell^{\mathrm{R}}(b)$.

Proposition 1.7.1. *Let T denote a full binary tree and $V \in {}^{kG}_{kG}\mathcal{YD}$ a two-dimensional Yetter–Drinfel'd module of diagonal type. Let $n \in \mathbb{N}_0$. If (T, V, n) is admissible then $\mathcal{B}(V)$ is of type T in degree n and all homogeneous* *relations of* $\mathcal{B}(V)$ *of degree at most* n *are elements of the ideal of* $V^{\otimes}$ *generated by the set*

$$\begin{aligned}&\{\tau_0(a)\mid a\in N_0(T), |\sigma_T|(a)\le n\}\cup\\ &\quad\{\tau_0(a)^{\operatorname{ord}p_a}\mid a\in \bar N_2(T), |\sigma_T|(a)\cdot \operatorname{ord}p_a\le n\}\cup\\ &\quad\{\tau_0(b)\tau_0(c^{\mathrm{L}})-\chi(b,c^{\mathrm{L}})\tau_0(c^{\mathrm{L}})\tau_0(b)-\mu(b)/(\ell^{\mathrm{R}}(b)+1)^!_{p_c}\tau_0(c)^{\ell^{\mathrm{R}}(b)+1}\mid\\ &\qquad\qquad b\in N_2(T), c:=b^{\mathrm{L}}\in N_2(T), |\sigma_T|(b)+|\sigma_T|(c^{\mathrm{L}})\le n\}.\end{aligned}$$

The following corollary is an immediate consequence of Proposition 1.7.1. It will be our main tool to prove Theorem 1.6.1.

Corollary 1.7.2. *Let* T *denote a full binary tree and* $V\in {}^{kG}_{kG}\mathcal{YD}$ *a two-dimensional Yetter–Drinfel'd module of diagonal type. If* (T,V,n) *is admissible for all* $n\in\mathbb{N}$ *then* $\mathcal{B}(V)$ *is of type* T *and all relations of* $\mathcal{B}(V)$ *are elements of the ideal of* $V^{\otimes}$ *generated by the set*

$$\begin{aligned}&\{\tau_0(a)\mid a\in N_0(T)\}\cup\{\tau_0(a)^{\operatorname{ord}p_a}\mid a\in\bar N_2(T), 2\le \operatorname{ord}p_a<\infty\}\cup\\ &\quad\{\tau_0(b)\tau_0(c^{\mathrm{L}})-\chi(b,c^{\mathrm{L}})\tau_0(c^{\mathrm{L}})\tau_0(b)-\mu(b)/(\ell^{\mathrm{R}}(b)+1)^!_{p_c}\tau_0(c)^{\ell^{\mathrm{R}}(b)+1}\mid\\ &\qquad\qquad b\in N_2(T), c:=b^{\mathrm{L}}\in N_2(T)\}.\end{aligned}$$

Proof of Proposition 1.7.1. We proceed by induction over n. As noted previously the assertion is true for $n=0$. Assume that Proposition 1.7.1 is valid for $(T,V,n-1)$ and that (T,V,n) is admissible. By Corollary 1.4.7 for $\mathcal{A}=V^{\otimes}$ and with $u=\alpha_1$ and by Lemma 1.5.2 it suffices to prove that the following assertions hold.

(a) If $a\in\bar N_2(T)$ and $|\sigma_T|(a)\le n$ then $h_{\gamma(a)}\ge\min\{\operatorname{ord}p_a,(n+1)/|\sigma_T|(a)\}$. If $n\ge|\sigma_T|(a)\cdot\operatorname{ord}p_a$ then $[\gamma(a)]^{\operatorname{ord}p_a}=0$.

(b) If $a\in N_0(T)$ and $|\sigma_T|(a)\le n$ then $[\gamma(a)]=0$.

(c) If u is Lyndon, $2\le|u|\le n$, and $u=\gamma(a)\gamma(b)$ is the Shirshow decomposition of u with $a,b\in\bar N_2(T)$, $a=c^{\mathrm{L}}$, where $c=b^{\mathrm{L}}$, then $[u]=\mu(b)/(\ell^{\mathrm{R}}(b)+1)^!_{p_c}\tau(c)^{\ell^{\mathrm{R}}(b)+1}$ and $u<\gamma(c)$.

(d) If u is Lyndon, $|u|\le n$, $u\notin\{\gamma(a)\mid a\in\bar N(T)\}$, and u is not as in (c) then $h_u=1$ and the relation corresponding to u follows from those given in Proposition 1.7.1 for $(T,V,n-1)$.

In order to prove assertions (a)–(d) we additionally use the following induction hypotheses which will be proven after the proof of (a)–(d).

(e) If $a\in N(T)$, $m:=|\sigma_T|(a)\le n$, and $u=\gamma(a)$ has Shirshow decomposition $u=vw$ then

$$\begin{aligned}&\Delta(\iota([u]))-\iota([u])\otimes 1-g([u])^{-1}\otimes\iota([u])\\ &\qquad-\lambda(a)g([v])^{-1}\iota([w])\otimes\iota([v])\in\iota(\mathcal{F}(w)_{m-1})\otimes\mathcal{B}(V^*).\end{aligned}\tag{1.10}$$

(f) If $a,b\in\bar N(T)$ such that $|\sigma_T|(a)\le n$, $|\sigma_T|(b)\le n$, and $a<_Q b$ then $\langle\iota(\tau(b)),\tau(a)\rangle=0$.

(g) If $a\in N(T)$ and $|\sigma_T|(a)\le n$ then the following equations hold.

$$\begin{aligned}\langle\iota(\tau(a^{\mathrm{L}})),\tau(a)\rangle&=\lambda(a)\langle\iota(\tau(a^{\mathrm{L}})),\tau(a^{\mathrm{L}})\rangle\tau(a^{\mathrm{R}}),\\ \langle\iota(\tau(a)),\tau(a)\rangle&=\lambda(a)\langle\iota(\tau(a^{\mathrm{L}})),\tau(a^{\mathrm{L}})\rangle\langle\iota(\tau(a^{\mathrm{R}})),\tau(a^{\mathrm{R}})\rangle.\end{aligned}$$

(h) Suppose that $b\in N_2(T)$, $c:=b^{\mathrm{L}}\in N(T)$, and $|\sigma_T|(b)+|\sigma_T|(c^{\mathrm{L}})\le n$. Set $a:=c^{\mathrm{L}}$. If $\ell^{\mathrm{R}}(b)=1$ then $b=c_{\mathrm{R}}$ (see Figure 1.1) and one has

$$\begin{aligned}\Big\langle\iota(\tau(a)),\tau(b)\tau(a)-\chi(b,a)\tau(a)\tau(b)-\frac{\lambda(b)}{(2)_{p_c}}\tau(c)^2\Big\rangle&\\ =\langle\iota(\tau(a)),\tau(a)\rangle\nu(b)\tau(b).&\end{aligned}$$

If $\ell^{\mathrm{R}}(b)=2$ then $b=c_{\mathrm{RL}}$ and with $d:=c_{\mathrm{L}}$ (see Figure 1.2) one has

$$\begin{aligned}\Big\langle\iota(\tau(d)),\tau(b)\tau(a)-\chi(b,a)\tau(a)\tau(b)-\frac{\mu(b)}{(3)^!_{p_c}}\tau(c)^3\Big\rangle&\\ =\lambda(b)\langle\iota(\tau(d)),\tau(d)\rangle\nu(b)\tau(c_{\mathrm{R}}).&\end{aligned}$$

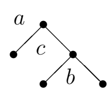


Figure 1.1[1]

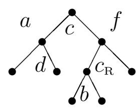


Figure 1.2[1]

[1]The labels belong to the nodes above them.

Note that for $n = 0$ all assertions (a)–(h) are trivially fulfilled and hence we may start with the induction step.

To (d). Suppose that u is a Lyndon word with $|u| = n \geq 2$ and Shirshow decomposition $u = vw$ and u is as in (d). If $h_v = 1$ or $h_w = 1$ then $h_u = 1$ by Lemma 1.4.4 and Proposition 1.4.8. More precisely, in order to ensure $h_u = 1$ one does not need a new relation. Otherwise by induction hypothesis there exist $a, b \in \bar{N}_2(T)$ such that $\gamma(a) = v$, $\gamma(b) = w$. Note that since $\sigma_T(a) = \deg([v])$ and $\sigma_T(b) = \deg([w])$ Lemma 1.5.1(iii) implies that a and b are uniquely determined by $\gamma(a)$ and $\gamma(b)$, respectively.

Since u is Lyndon we have also $v < w$ and hence $a <_Q b$ by Lemma 1.5.2(iv). Further, Proposition 1.4.2 and Lemma 1.5.2(iv) imply that either $a =$ 'L' or $a \in N_2(T)$ and $b \leq_Q a^{\mathrm{R}}$. Assume for a moment that $a \in N_2(T)$ and $b = a^{\mathrm{R}}$ and set $c := a_{\mathrm{R}}$. Then $a = c^{\mathrm{L}}$, $b = c^{\mathrm{R}}$, and $|\sigma_T|(c) = n$ and hence $u = \gamma(c)$. This case is covered by (a) and (b). Similarly, if $a =$ 'L' and $b =$ 'R' then $n = 2$ and $u = \gamma(\mathbf{r})$ which again belongs to (a) or (b). Therefore one has either $a =$ 'L' and $b \in N_2(T)$ or $a, b \in N_2(T)$ and $a <_Q b <_Q a^{\mathrm{R}}$.

Now we prove that $a <_Q b^{\mathrm{L}}$. First note that since $a <_Q b$ Lemma 1.5.1(vi) implies the relation $|\sigma_T|(a) < |\sigma_T|(b)$ in the case $a \in N_2(T)$. If $a =$ 'L' then this relation is trivial. Therefore one gets $a \leq_Q b^{\mathrm{L}}$ by Lemma 1.5.1(v). Assume for a moment that $a = b^{\mathrm{L}}$ and set $c := b_{\mathrm{L}}$. Then $a = c^{\mathrm{L}}$ and $b = c^{\mathrm{R}}$ and hence $u = \gamma(c)$ which is covered by (a) and (b). Thus we are arrived at the situation that $b \in N_2(T)$, $a <_Q b^{\mathrm{L}}$, and either $a =$ 'L' or $a \in N_2(T)$ and $b <_Q a^{\mathrm{R}}$.

If $a =$ 'L' then $a <_Q b^{\mathrm{L}}$ implies that $|\sigma_T|(a) < |\sigma_T|(b^{\mathrm{L}})$. If $a \in N_2(T)$ then the same relation follows from $a <_Q b^{\mathrm{L}}$ and Lemma 1.5.1(vi) as $b^{\mathrm{L}} <_Q b <_Q a^{\mathrm{R}}$. Therefore Lemma 1.5.1(v) gives $a \leq_Q b^{\mathrm{LL}}$. The case $a = b^{\mathrm{LL}}$ can be omitted as it is exactly the situation in (c). Otherwise set $u_1 = \gamma(a)\gamma(b^{\mathrm{L}})$. Note that u_1 is Lyndon by Proposition 1.4.1(ii). We show that $h_{u_1} = 1$ holds which proves (d) by Lemma 1.4.4 and Proposition 1.4.8.

Suppose that $h_{u_1} > 1$. As the length of u_1 is less than n but at least 2 by induction hypothesis (a) there exists $c \in N_2(T)$ such that $u_1 = \gamma(c)$. Since $\gamma(a) < u_1 < \gamma(b^{\mathrm{L}})$ one obtains $a <_Q c <_Q b^{\mathrm{L}}$. By Lemma 1.5.1(iv),(v) this implies that $a \leq_Q c^{\mathrm{L}}$. As $\gamma(c^{\mathrm{L}})\gamma(c^{\mathrm{R}})$ is the Shirshow decomposition of $\gamma(c)$ by Lemma 1.5.2(iii) the equation $\gamma(a)\gamma(b^{\mathrm{L}}) = \gamma(c^{\mathrm{L}})\gamma(c^{\mathrm{R}})$ implies that $|\gamma(b^{\mathrm{L}})| \leq |\gamma(c^{\mathrm{R}})|$ and hence $\gamma(c^{\mathrm{L}}) \leq \gamma(a)$. Thus $a \leq_Q c^{\mathrm{L}}$, Lemma 1.5.2(iv), and Lemma 1.5.1(iii) imply that $c^{\mathrm{L}} = a$ and $c^{\mathrm{R}} = b^{\mathrm{L}}$. Now we have $c^{\mathrm{L}} = a <_Q b^{\mathrm{LL}} <_Q b^{\mathrm{L}} = c^{\mathrm{R}}$, $|\sigma_T|(b^{\mathrm{LL}}) < |\sigma_T|(b^{\mathrm{L}})$, and $|\sigma_T|(c) > |\sigma_T|(b^{\mathrm{L}})$. Then Lemma 1.5.1(ix) applied to the pair (c, b^{LL}) gives a contradiction.

To (c). The relation $\gamma(a)\gamma(b) < \gamma(c)$ follows from $\gamma(c) = \gamma(a)\gamma(c^{\mathrm{R}})$ and $b <_Q b^{\mathrm{LR}} = c^{\mathrm{R}}$ (see Lemma 1.5.1(vii)).

Set $U := \tau(b)\tau(a) - \chi(b,a)\tau(a)\tau(b) - \mu(b)/(\ell^{\mathrm{R}}(b)+1)^!_{p_c}\tau(c)^{\ell^{\mathrm{R}}(b)+1}$. First note that using (e), (f), and (g) one obtains

$$\begin{aligned}
&\langle \iota(\tau(c)), \tau(b)\tau(a) - \chi(b,a)\tau(a)\tau(b) - \mu(b)/(\ell^{\mathrm{R}}(b)+1)^!_{p_c}\tau(c)^{\ell^{\mathrm{R}}(b)+1}\rangle = \\
&\quad = \langle \iota(\tau(c)), \tau(b)\rangle \tau(a) - \chi(b^{\mathrm{R}}, a)\tau(a)\langle \iota(\tau(c)), \tau(b)\rangle \\
&\qquad - (\mu(b)/(\ell^{\mathrm{R}}(b)+1)^!_{p_c})(\ell^{\mathrm{R}}(b)+1)_{p_c}\langle \iota(\tau(c)), \tau(c)\rangle \tau(c)^{\ell^{\mathrm{R}}(b)} \\
&\quad = \langle \iota(\tau(c)), \tau(c)\rangle \big(\lambda(b)\tau(b^{\mathrm{R}})\tau(a) - \lambda(b)\chi(b^{\mathrm{R}}, a)\tau(a)\tau(b^{\mathrm{R}}) \\
&\qquad - \mu(b)/(\ell^{\mathrm{R}}(b))^!_{p_c}\tau(c)^{\ell^{\mathrm{R}}(b)}\big).
\end{aligned}$$

If $\ell^{\mathrm{R}}(b) = 1$ then $b^{\mathrm{R}} = c^{\mathrm{R}}$, $\tau(b^{\mathrm{R}})\tau(a) - \chi(b^{\mathrm{R}}, a)\tau(a)\tau(b^{\mathrm{R}}) = \tau(c)$, and $\mu(b) = \lambda(b)$. Otherwise $\mu(b) = \lambda(b)\mu(b^{\mathrm{R}})$ and we can use induction hypothesis (c). In both cases one obtains $\langle \iota(\tau(c)), U\rangle = 0$. Thus by (f) one gets $\langle \iota(\tau(d)), U\rangle = 0$ for all $d \in \bar{N}_2(T)$ with $c \leq_Q d$. Hence it suffices to show that $\langle \iota([u']), U\rangle = 0$ for all Lyndon words u' with $\deg([u']) = \deg(U)$ and that $\langle \iota(\tau(a_1)\tau(a_2)\cdots\tau(a_m)), U\rangle = 0$ for $a_i \in \bar{N}_2(T)$, $a_1 \leq_Q a_2 \leq_Q \cdots \leq_Q a_m <_Q c$, $m \geq 2$, $\sum_i \sigma_T(a_i) = (\ell^{\mathrm{R}}(b)+1)\sigma_T(c)$. In the second case set $\sigma_T(a_i) := (r_i, s_i)$. Then $a_i <_Q c$ implies $r_i < Q(c)s_i$ and hence $\sum_i r_i < Q(c)\sum_i s_i$ which is a contradiction to $\sum_i \sigma_T(a_i) = (\ell^{\mathrm{R}}(b)+1)\sigma_T(c)$. In the first case $u' \neq \gamma(d)$ for all d since the entries of $\deg([u'])$ are not relatively prime. Thus by (d) one can reduce to the case where $u' = v'w'$ is the Shirshow decomposition of u', $v' = \gamma(a')$, $w' = \gamma(b')$, and $a' = b'^{\mathrm{LL}}$. Since $(\ell^{\mathrm{R}}(b)+1)\sigma_T(c) = \deg(U) = \deg([v'][w']) = (\ell^{\mathrm{R}}(b')+1)\sigma_T(b'^{\mathrm{L}})$ and both the entries of $\sigma_T(c)$ and those of $\sigma_T(b'^{\mathrm{L}})$ are relatively prime one gets from Lemma 1.5.1(iii) the equation $c = b'^{\mathrm{L}}$. Again by Lemma 1.5.1(iii) this yields $a = c^{\mathrm{L}} = b'^{\mathrm{LL}} = a'$ and hence $b = b'$ and $u' = u$.

Note that $\langle \iota(\tau(b)), U\rangle = 0$ and hence it remains to check that the relation $\langle \iota(\tau(b)\tau(a)), U\rangle = 0$ holds. Using $\tau(b) = \tau(b^{\mathrm{R}})\tau(c) - \chi(b^{\mathrm{R}}, c)\tau(c)\tau(b^{\mathrm{R}})$ this implies that it remains to check the equation $\langle \iota(\tau(b^{\mathrm{R}})\tau(c_{\mathrm{L}})), U\rangle = 0$. Set $c_1 := c_{\mathrm{L}}$ and $c_{i+1} := c_{i\mathrm{R}}$ for $1 \leq i < \ell_{\mathrm{R}}(c_1)$. Further, set $b_1 := b^{\mathrm{R}}$ and $b_{i+1} := b_i{}^{\mathrm{R}}$ for $1 \leq i < \ell^{\mathrm{R}}(b)$. Then using (c) for $(T, V, n-1)$, the definition of τ, and Lemma 1.5.2(iii) one can show by induction over i that for all $i \leq \min\{\ell^{\mathrm{R}}(b), \ell_{\mathrm{R}}(c_1)\}$ the assertions $\langle \iota(\tau(b)\tau(a)), U\rangle = 0$ and $\langle \iota(\tau(b_i)\tau(c_i)), U\rangle = 0$ are equivalent. Now we use that (T, V, n) is admissible. More exactly, we have either $\ell_{\mathrm{R}}(c_1) \leq \ell^{\mathrm{R}}(b)$ or $\ell^{\mathrm{R}}(b) \leq 2$. In the first case one has $\tau(c_i) = 0$ for $i = \ell_{\mathrm{R}}(c_1)$. Otherwise (h) gives $\langle \iota(\tau(c_i)), U\rangle = 0$ for $i = \ell^{\mathrm{R}}(b) - 1$ where

$c_0 = a$. Therefore in both cases we get $\langle \iota(\tau(b)\tau(a)), U\rangle = 0$ and hence $U = 0$.

To (b). By (c), (d), (e), and (f) it suffices to show that the equations $\langle \iota(\tau(a_1)\tau(a_2)\cdots\tau(a_m)), \tau(a)\rangle = 0$ hold for all $a_1 \le_Q a_2 \le_Q \cdots \le_Q a_m \le_Q a$ with $\sum_i \sigma_T(a_i) = \sigma_T(a)$. Set $(r_i, s_i) := \sigma_T(a_i)$. Then $a_i \le_Q a$ implies $r_i \le Q(a)s_i$ for all i with equality if and only if $a_i = a$. Hence $\sum_i \sigma_T(a_i) = \sigma_T(a)$ implies $m = 1$ and $a_1 = a$. Note that (T, V, n) is admissible and hence $\lambda(a) = 0$. Thus the equation $\langle \iota(\tau(a)), \tau(a)\rangle = 0$ follows from (g).

To (a). Let $a \in \bar N_2(T)$ and set $u := \gamma(a)$. Suppose that $[u]^h$, $h \in \mathbb{N}$, is a linear combination of elements $[u_1][u_2]\cdots[u_m]$ with $u < u_i$ for $1 \le i \le m$. By Corollary 1.4.7 and the induction hypothesis one can assume that $u_i = \gamma(a_i)$ for some a_i with $a <_Q a_i$. Set $(r_i, s_i) := \sigma_T(a_i)$ for all i and $(r, s) := \sigma_T(a)$. Then Lemma 1.5.2 implies that $r_i > Q(a)s_i$ for all i and hence $(\sum_i r_i) > (\sum_i s_i)\, Q(a)$. On the other hand, since $\tau(a_1)\tau(a_2)\cdots\tau(a_m)$ must have the same degree as $\tau(a)^h$ it follows that $(hr, hs) = (\sum_i r_i, \sum_i s_i)$ and hence $\tau(a)^h$ has to be zero. In particular, $0 = \langle \iota(\tau(a)), \tau(a)^h\rangle = (h)_{p_a}\langle \iota(\tau(a)), \tau(a)\rangle\tau(a)^{h-1}$ by (e) and (f). If $n = 1$ then $\langle \iota(\tau(a)), \tau(a)\rangle = 1$. Otherwise since (T, V, n) is admissible one has $\lambda(a) \ne 0$. In this case (g) gives $\langle \iota(\tau(a)), \tau(a)\rangle \ne 0$ and hence the first part of (a) is proven. To show that $\tau(a)^{\operatorname{ord} p_a} = 0$ for $n \ge |\sigma_T|(a)\cdot \operatorname{ord} p_a$ by (c) and (d) it suffices to check that $\langle \iota(\tau(a_1)\tau(a_2)\cdots\tau(a_m)), \tau(a)^{\operatorname{ord} p_a}\rangle = 0$ whenever $a_1 \le_Q a_2 \le_Q \cdots \le_Q a_m <_Q a$, $a_i \in \bar N_2(T)$ for all i, and $\sum_i \sigma_T(a_i) = \sigma_T(a)\cdot \operatorname{ord} p_a$. However as arqued at the beginning of the proof of (a) such a choice of a_i is not possible.

It remains to prove the induction step for (e)–(h) $(n \to n+1)$ under the hypothesis (a)–(h) and admissibility of $(T, V, n+1)$.

Lemma 1.7.3. *For $a \in N(T)$ with $a^{\mathrm{R}} \in N(T)$, $|\sigma_T|(a) = n+1$ set $u := \gamma(a)$ and let $u = vw$ be the Shirshow decomposition of u. Let $\rho \in \mathcal{B}(V)^+ \cap \mathcal{F}(\gamma(a^{\mathrm{RR}}))_{|w|-1}$ be a homogeneous element with respect to the $\mathbb{Z}^2$-grading. Then $g([w])^{-1}g(\rho)\iota(\rho)\iota([v]) - \chi([w],[v])\iota([v])g([w])^{-1}g(\rho)\iota(\rho) \in \iota(\mathcal{F}(w)_n)$.*

Proof of the Lemma. First note that

$$\begin{aligned} g([w])^{-1}g(\rho)\iota(\rho)\iota([v]) &- \chi([w],[v])\iota([v])g([w])^{-1}g(\rho)\iota(\rho) \\ &= g([w])^{-1}g(\rho)\iota(\rho[v] - \chi(\rho,[v])[v]\rho), \end{aligned}$$

$$\begin{aligned} \rho_1\rho_2[v] - &\chi(\rho_1\rho_2,[v])[v]\rho_1\rho_2 \\ =&\rho_1(\rho_2[v] - \chi(\rho_2,[v])[v]\rho_2) + \chi(\rho_2,[v])(\rho_1[v] - \chi(\rho_1,[v])[v]\rho_1)\rho_2 \end{aligned} \qquad (*)$$

for $\mathbb{Z}^2$-homogeneous elements $\rho_i \in \mathcal{B}(V)$, $i = 1, 2$. As $\rho \in S(v)_<$ Lemma 1.4.5

implies that $\rho[v] - \chi(\rho,[v])[v]\rho \in S(v)_<$. Further, $\rho[v], [v]\rho \in \mathcal{B}(V)_m$ where $m = \operatorname{totdeg}(\rho) + |v| \le n$. By Lemma 1.5.1(ix) there exists no $b \in \bar N_2(T)$ such that $|\sigma_T|(b) \le n$ and $a^{\mathrm{L}} <_Q b <_Q a^{\mathrm{R}}$. Thus $S(v)_< \cap \mathcal{B}(V)_m = S(w) \cap \mathcal{B}(V)_m$. By $(*)$ and since $\rho \in S(\gamma(a^{\mathrm{RR}}))^+ \subset S(w)^+_<$ it suffices to show that

$$\rho[v] - \chi(\rho,[v])[v]\rho \in S(w)S(w)^+_< + S(w)[w]^2 \qquad (**)$$

for $\rho = \tau(a^{\mathrm{RR}})^2$ and for $\rho = \tau(b)$, $a^{\mathrm{RR}} <_Q b$. By Equation (1.6) relation $(**)$ is obviously true if $|v| \ge |w|$. Otherwise a is the left child of a^{R} and $a^{\mathrm{RL}} = a^{\mathrm{L}}$. If $\rho = \tau(a^{\mathrm{RR}})^2$ then $\operatorname{totdeg}(\rho[v]) = |v| + 2|\sigma_T|(a^{\mathrm{RR}}) = |w| + |\sigma_T|(a^{\mathrm{RR}}) > |w|$ and hence $(**)$ holds. On the other hand $\deg([v]\tau(a^{\mathrm{RR}})) = \deg([w])$ and Lemma 1.5.1(iii) imply that $\deg(\tau(b)) \ne \deg([w]) - \deg([v])$ for $b \ne a^{\mathrm{RR}}$. Thus if $\rho = \tau(b)$, $a^{\mathrm{RR}} <_Q b$ then again $(**)$ is valid. ■

To (e). Suppose that $|\sigma_T|(a) = n+1$. One has $a <_Q a^{\mathrm{R}}$ and if $a^{\mathrm{L}} \in N(T)$ then $a^{\mathrm{R}} \le_Q a^{\mathrm{LR}}$ by Lemma 1.5.1(vii),(vi). Therefore the induction hypothesis (e) for v and w and Lemma 1.7.3 give

$$\begin{aligned} &\Delta(\iota([v])) - \iota([v]) \otimes 1 - g(a^{\mathrm{L}})^{-1} \otimes \iota([v]) \\ &\quad -\lambda(a^{\mathrm{L}})g(a^{\mathrm{LL}})^{-1}\iota(\tau(a^{\mathrm{LR}})) \otimes \iota(\tau(a^{\mathrm{LL}})) \in \iota(\mathcal{F}(w)_{|v|-1}) \otimes \mathcal{B}(V^*), \\ &\Delta(\iota([u])) = \Delta(\iota([w]))\Delta(\iota([v])) - \chi(a^{\mathrm{R}}, a^{\mathrm{L}})\Delta(\iota([v]))\Delta(\iota([w])) \\ &\quad = \iota([u]) \otimes 1 + (\chi(a^{\mathrm{L}}, a^{\mathrm{R}})^{-1} - \chi(a^{\mathrm{R}}, a^{\mathrm{L}}))g(a^{\mathrm{L}})^{-1}\iota([w]) \otimes \iota([v]) \\ &\qquad + g(a)^{-1} \otimes \iota([u]) \\ &\qquad + \lambda(a^{\mathrm{L}})g([w]\tau(a^{\mathrm{LL}}))^{-1}\iota(\tau(a^{\mathrm{LR}})) \otimes \iota\big([w]\tau(a^{\mathrm{LL}}) - \chi(a^{\mathrm{R}}, a^{\mathrm{LL}})\tau(a^{\mathrm{LL}})[w]\big) \\ &\qquad + \lambda(a^{\mathrm{R}})g(a^{\mathrm{RL}})^{-1}\iota\big(\tau(a^{\mathrm{RR}})[v] - \chi(a^{\mathrm{RR}}, a^{\mathrm{L}})[v]\tau(a^{\mathrm{RR}})\big) \otimes \iota(\tau(a^{\mathrm{RL}})) \end{aligned}$$

up to terms in $\iota(\mathcal{F}(w)_n) \otimes \mathcal{B}(V^*)$. Thus if $a = \mathbf{r}(T)$ then $\lambda(a^{\mathrm{L}}) = \lambda(a^{\mathrm{R}}) = 0$ and one gets (1.10). If a is the left child of its parent then either $a^{\mathrm{L}} =$ 'L' or $a^{\mathrm{R}} <_Q a^{\mathrm{LR}}$ by Lemma 1.5.1(vi),(vii). Further, $a^{\mathrm{RL}} = a^{\mathrm{L}}$ and $\tau(a^{\mathrm{RR}})[v] - \chi(a^{\mathrm{RR}}, a^{\mathrm{L}})[v]\tau(a^{\mathrm{RR}}) = [w]$ and again we are done. Finally if a is the right child of its parent then either $a^{\mathrm{R}} =$ 'R' and $\lambda(a^{\mathrm{R}}) = 0$ or $|v| > |w|$ and $\tau(a^{\mathrm{RR}})[v] - \chi(a^{\mathrm{RR}}, a^{\mathrm{L}})[v]\tau(a^{\mathrm{RR}}) \in \mathcal{F}(w)_n$ by (1.6). Moreover $a^{\mathrm{LR}} = a^{\mathrm{R}}$ and $[w]\tau(a^{\mathrm{LL}}) - \chi(a^{\mathrm{R}}, a^{\mathrm{LL}})\tau(a^{\mathrm{LL}})[w] = [v]$. Thus (1.10) holds in this case as well.

To (f). If $|\sigma_T|(a) \le n$ and $|\sigma_T|(b) \le n$ then we are done by induction hypothesis. If $n+1 = |\sigma_T|(b) > |\sigma_T|(a)$ then $\langle \iota(\tau(b)), \tau(a)\rangle = 0$ clearly holds. Suppose now that $|\sigma_T|(a) = n+1$. If $n = 0$ then $a =$ 'L' and $b =$ 'R' and hence (f) holds by definition of $\langle\cdot,\cdot\rangle$. Otherwise $a^{\mathrm{R}} \le_Q b$ by Lemma 1.5.1(vi) and hence (e) and (f) imply that

$$\begin{aligned}\langle \iota(\tau(b)),\tau(a)\rangle =&\langle \iota(\tau(b)),\tau(a^{\mathrm{R}})\tau(a^{\mathrm{L}})-\chi(a^{\mathrm{R}},a^{\mathrm{L}})\tau(a^{\mathrm{L}})\tau(a^{\mathrm{R}})\rangle\\ =&\langle \iota(\tau(b)),\tau(a^{\mathrm{R}})\rangle a^{\mathrm{L}}-\chi(a^{\mathrm{R}},a^{\mathrm{L}})\chi(b,a^{\mathrm{L}})^{-1}a^{\mathrm{L}}\langle \iota(\tau(b)),\tau(a^{\mathrm{R}})\rangle.\end{aligned}$$

By (f) the last expression vanishes if $a^{\mathrm{R}}\le_Q b$.

To (g). By (f) and since $\tau(a)=\tau(a^{\mathrm{R}})\tau(a^{\mathrm{L}})-\chi(a^{\mathrm{R}},a^{\mathrm{L}})\tau(a^{\mathrm{L}})\tau(a^{\mathrm{R}})$ the second equation of (g) follows immediately from the first one.

If $a=\mathbf{r}(T)$ then one gets $\langle \iota(\tau(a^{\mathrm{L}})),\tau(a)\rangle=\langle y_2,x_1x_2-\chi(x_1,x_2)x_2x_1\rangle=(\chi(x_2,x_1)^{-1}-\chi(x_1,x_2))x_1=\lambda(a)\tau(a^{\mathrm{R}})$.

If $a={a^{\mathrm{L}}}_{\mathrm{R}}$ then $a^{\mathrm{LR}}=a^{\mathrm{R}}$ and $|\sigma_T|(a^{\mathrm{L}})>|\sigma_T|(a^{\mathrm{R}})$ and hence (e) and (f) gives

$$\begin{aligned}\langle \iota(\tau(a^{\mathrm{L}})),\tau(a)\rangle =&\langle \iota(\tau(a^{\mathrm{L}})),\tau(a^{\mathrm{R}})\tau(a^{\mathrm{L}})-\chi(a^{\mathrm{R}},a^{\mathrm{L}})\tau(a^{\mathrm{L}})\tau(a^{\mathrm{R}})\rangle\\ =&\chi(a^{\mathrm{L}},a^{\mathrm{R}})^{-1}\tau(a^{\mathrm{R}})\langle \iota(\tau(a^{\mathrm{L}})),\tau(a^{\mathrm{L}})\rangle\\ &+\lambda(a^{\mathrm{L}})\langle \iota(\tau(a^{\mathrm{R}})),\tau(a^{\mathrm{R}})\rangle\langle \iota(\tau(a^{\mathrm{LL}})),\tau(a^{\mathrm{L}})\rangle\\ &-\chi(a^{\mathrm{R}},a^{\mathrm{L}})\langle \iota(\tau(a^{\mathrm{L}})),\tau(a^{\mathrm{L}})\rangle\tau(a^{\mathrm{R}}).\end{aligned}$$

Since $a^{\mathrm{L}}\in N_2(T)$ in the second summand of the last expression one can use (g) for a^{L} and equation $a^{\mathrm{LR}}=a^{\mathrm{R}}$. One gets $\langle \iota(\tau(a^{\mathrm{L}})),\tau(a)\rangle=(\lambda(a^{\mathrm{L}})+\chi(a^{\mathrm{L}},a^{\mathrm{R}})^{-1}-\chi(a^{\mathrm{R}},a^{\mathrm{L}}))\langle \iota(\tau(a^{\mathrm{L}})),\tau(a^{\mathrm{L}})\rangle\tau(a^{\mathrm{R}})=\lambda(a)\langle \iota(\tau(a^{\mathrm{L}})),\tau(a^{\mathrm{L}})\rangle\tau(a^{\mathrm{R}})$.

Finally, if $a={a^{\mathrm{R}}}_{\mathrm{L}}$ then $a^{\mathrm{RL}}=a^{\mathrm{L}}$. Moreover Lemma 1.5.1(vii) implies that $a^{\mathrm{L}}=$ 'L' or $a^{\mathrm{R}}<_Q a^{\mathrm{RLR}}=a^{\mathrm{LR}}$. Thus using (e), (f), and (g) computations similar to the previous case lead to the desired assertion.

To (h). We need the following lemma.

Lemma 1.7.4. *Suppose that $(T,V,n+1)$ is admissible. Let $d\in N_2(T)$ such that $c:=d^{\mathrm{R}}\in N(T)$ and $d=c_{\mathrm{L}}$. Set $a:=c^{\mathrm{L}}=d^{\mathrm{L}}$ and $f:=c^{\mathrm{R}}$ (see Fig. 2).*
(i) If $|\sigma_T|(d)+|\sigma_T|(f)\le n$ then in $\mathcal{B}(V)$ one has

$$\tau(f)\tau(d)-\chi(f,d)\tau(d)\tau(f)=\frac{\lambda(d)}{(2)_{p_c}}\tau(c)^2.$$

(ii) If $|\sigma_T|(d)+|\sigma_T|(f)(=|\sigma_T|(a)+|\sigma_T|(c_{\mathrm{R}}))\le n+1$ then

$$\langle \iota(\tau(a)),\tau(c_{\mathrm{R}})\rangle=\frac{\lambda(c)\lambda(c_{\mathrm{R}})}{(2)_{p_f}}\langle \iota(\tau(a)),\tau(a)\rangle\tau(f)^2.$$

(iii) If $c_{\mathrm{R}}\in N_2(T)$ and $|\sigma_T|(c_{\mathrm{RL}})+|\sigma_T|(a)\le n+1$ then

$$\langle \iota(\tau(d)),\tau(c_{\mathrm{RL}})\rangle=\frac{\lambda(c_{\mathrm{R}})\lambda(c_{\mathrm{RL}})\lambda(c)\langle \iota(\tau(d)),\tau(d)\rangle}{(2)_{p_c}(2)_{p_f}}\tau(f)^2.$$

Proof of the Lemma. To (i). Note that $c^{\mathrm{L}}=d^{\mathrm{L}}$ by assumption. Further, we may use the induction hypothesis (c). One computes

$$\begin{aligned}&\tau(f)\tau(d)-\chi(f,d)\tau(d)\tau(f)\\ &\quad=\tau(f)(\tau(c)\tau(a)-\chi(c,a)\tau(a)\tau(c))-\chi(f,d)\tau(d)\tau(f)\\ &\quad=(\tau(c_{\mathrm{R}})+\chi(f,c)\tau(c)\tau(f))\tau(a)\\ &\qquad-\chi(c,a)(\tau(c)+\chi(f,a)\tau(a)\tau(f))\tau(c)-\chi(f,d)\tau(d)\tau(f)\\ &\quad=\chi(c_{\mathrm{R}},a)\tau(a)\tau(c_{\mathrm{R}})+\lambda(c_{\mathrm{R}})/(2)_{p_c}\tau(c)^2\\ &\qquad+\chi(f,c)\tau(c)(\tau(c)+\chi(f,a)\tau(a)\tau(f))-\chi(c,a)\tau(c)^2\\ &\qquad-\chi(c_{\mathrm{R}},a)\tau(a)(\tau(c_{\mathrm{R}})+\chi(f,c)\tau(c)\tau(f))-\chi(f,d)\tau(d)\tau(f)\\ &\quad=\lambda(c_{\mathrm{R}})/(2)_{p_c}\tau(c)^2+\chi(f,c)\tau(c)^2\\ &\qquad+\chi(f,d)(\tau(d)+\chi(c,a)\tau(a)\tau(c))\tau(f)-\chi(c,a)\tau(c)^2\\ &\qquad-\chi(c_{\mathrm{R}},a)\chi(f,c)\tau(a)\tau(c)\tau(f)-\chi(f,d)\tau(d)\tau(f)\\ &\quad=(\lambda(c_{\mathrm{R}})+\chi(f,c)+\chi(a,c)^{-1}-\chi(c,a)-\chi(c,f)^{-1})/(2)_{p_c}\tau(c)^2.\end{aligned}$$

Then the defining recursion formulas for $\lambda(c_{\mathrm{R}})$ and $\lambda(d)$ give (i).

To (ii). Note that $(2)_{p_f}\ne 0$ by admissibility of $(T,V,n+1)$. Further, if $\tau(c_{\mathrm{R}})=0$ then $\lambda(c_{\mathrm{R}})=0$ by (a). Thus in this case we are done. Assume now that $c=\mathbf{r}$ or $a_{\mathrm{R}}=c$ (i. e. $a^{\mathrm{R}}=f$). Then one has $\langle \iota(\tau(a)),\tau(f)\rangle=0$. Using (e) and (f) one gets

$$\begin{aligned}\langle \iota(\tau(a)),\tau(c_{\mathrm{R}})\rangle&=\langle \iota(\tau(a)),\tau(f)\tau(c)-\chi(f,c)\tau(c)\tau(f)\rangle\\ &=\chi(a,f)^{-1}\tau(f)\langle \iota(\tau(a)),\tau(c)\rangle+\lambda(a)\langle \iota(\tau(f)),\tau(f)\rangle\langle \iota(\tau(a^{\mathrm{L}})),\tau(c)\rangle\\ &\quad-\chi(f,c)\langle \iota(\tau(a)),\tau(c)\rangle\tau(f).\end{aligned}$$

If one starts with a_{L} instead of d in Lemma 1.7.4 then (ii) gives a formula for the second summand in the last expression. Further, (g) can be used to compute $\langle \iota(\tau(a)),\tau(c)\rangle$ and $\langle \iota(\tau(a)),\tau(a)\rangle$. One obtains

$$\begin{aligned}\langle \iota(\tau(a)),\tau(c_{\mathrm{R}})\rangle&=\lambda(c)\langle \iota(\tau(a)),\tau(a)\rangle(\chi(a,f)^{-1}-\chi(f,c))\tau(f)^2\\ &\quad+\lambda(a)\langle \iota(\tau(f)),\tau(f)\rangle\frac{\lambda(c)\lambda(a)}{(2)_{p_f}}\langle \iota(\tau(a^{\mathrm{L}})),\tau(a^{\mathrm{L}})\rangle\tau(f)^2\end{aligned}$$

$$= \frac{\lambda(c)\langle\iota(\tau(a)),\tau(a)\rangle}{(2)_{p_f}}(\chi(a,f)^{-1}+\chi(c,f)^{-1}-\chi(f,c)-\chi(f,a)+\lambda(a))\tau(f)^2.$$

Note that these computations make sense also in the case when $a =$ 'L'. Thus the recursion formulas for $\lambda(c)$ and $\lambda(c_{\mathrm{R}})$ give (ii) in this case. The proof of (ii) in the remaining case (when $f_{\mathrm{L}} = c$, i. e. $c^{\mathrm{L}} = f^{\mathrm{L}}$,) is obtained similarly.

The proof of (iii) is by far the most complicated one. We give only a scetch of it. Set $b := c_{\mathrm{RL}}$. Using (e), (f), (g), and Lemma 1.7.4(ii) one obtains

$$\begin{aligned}\langle\iota(\tau(a)),\tau(b)\rangle =&\frac{\lambda(c)\langle\iota(\tau(a)),\tau(a)\rangle}{(2)_{p_f}}\Big((\lambda(c_{\mathrm{R}})-\chi(c_{\mathrm{R}},c)(2)_{p_f})\tau(f)\tau(c_{\mathrm{R}})\\&+(\chi(f,c)\lambda(c_{\mathrm{R}})+\chi(a,c_{\mathrm{R}})^{-1}(2)_{p_f})\tau(c_{\mathrm{R}})\tau(f)\Big),\\ \langle\iota(\tau(c)),\tau(f)\tau(c_{\mathrm{R}})\rangle =&\frac{\lambda(c_{\mathrm{R}})\langle\iota(\tau(c)),\tau(c)\rangle}{(2)_{p_f}}(\lambda(c)+\chi(c,f)^{-1}(2)_{p_f})\tau(f)^2,\\ \langle\iota(\tau(c)),\tau(c_{\mathrm{R}})\tau(f)\rangle =&\lambda(c_{\mathrm{R}})\langle\iota(\tau(c)),\tau(c)\rangle\tau(f)^2,\\ \langle\iota(\tau(a)\tau(c)),\tau(b)\rangle =&\frac{\lambda(c_{\mathrm{R}})\lambda(b)\lambda(c)}{(2)_{p_f}}\langle\iota(\tau(c)),\tau(c)\rangle\langle\iota(\tau(a)),\tau(a)\rangle\tau(f)^2.\end{aligned}$$

By (c) and (h) one obtains $\nu(c_{\mathrm{R}}) = 0$. Now insert the second, third, and fourth equation into the first one and replace the summand $\lambda(c_{\mathrm{R}})\lambda(c)(2)_{p_f}^{-1}$ of the product $(\lambda(c_{\mathrm{R}})-\chi(c_{\mathrm{R}},c)(2)_{p_f})(\lambda(c)+\chi(c,f)^{-1}(2)_{p_f})$ by the expression $\lambda(c_{\mathrm{R}})\lambda(c)(2)_{p_c}^{-1} - \chi(a,c_{\mathrm{R}})^{-1} + \chi(c_{\mathrm{R}},a)$. Then use the recursion formulas for $\lambda(d)$, $\lambda(b)$, and $\lambda(c_{\mathrm{R}})$ to obtain (iii). ∎

To show the first equation of (h) one can use (e) and (f) to obtain

$$\begin{aligned}&\Big\langle\iota(\tau(a)),\tau(b)\tau(a)-\chi(b,a)\tau(a)\tau(b)-\frac{\lambda(b)}{(2)_{p_c}}\tau(c)^2\Big\rangle =\\&=\langle\iota(\tau(a)),\tau(b)\rangle\tau(a)+\chi(a,b)^{-1}\tau(b)\langle\iota(\tau(a)),\tau(a)\rangle\\&\quad-\chi(b,a)\langle\iota(\tau(a)),\tau(a)\rangle\tau(b)-\chi(f,a)^2\tau(a)\langle\iota(\tau(a)),\tau(b)\rangle\\&\quad-\frac{\lambda(b)}{(2)_{p_c}}(\langle\iota(\tau(a)),\tau(c)\rangle\tau(c)+\chi(a,c)^{-1}\tau(c)\langle\iota(\tau(a)),\tau(c)\rangle).\end{aligned}$$

After inserting the formula in Lemma 1.7.4(ii) the latter expression becomes the sum of $\langle\iota(\tau(a)),\tau(a)\rangle\nu(b)\tau(b)$ and a multiple of $\tau(c)\tau(f)$. Apply $\langle\iota(\tau(f)),\cdot\rangle$ to this equation. Since one has $\langle\iota(\tau(f)),\tau(b)\rangle = 0$ and

$\langle\iota(\tau(f)),\tau(c)\tau(f)\rangle = \chi(f,c)^{-1}\tau(c)\langle\iota(\tau(f)),\tau(f)\rangle$ Lemma 1.7.4(i) gives that the coefficient of $\tau(c)\tau(f)$ is zero.

In order to prove the second equation of (h) note that $|\sigma_T|(d) = |\sigma_T|(c) + |\sigma_T|(a) < 2|\sigma_T|(c) < |\sigma_T|(b)$. Thus (e) gives that

$$\begin{aligned}&\varDelta(\iota(\tau(d)))-\iota(\tau(d))\otimes 1-g(d)^{-1}\otimes\iota(\tau(d))-\lambda(d)g(a)^{-1}\iota(\tau(c))\otimes\iota(\tau(a))\\&\qquad\in\Big(\iota\big(S(\gamma(c_{\mathrm{R}}))^+ + \tau(c)S(\gamma(c_{\mathrm{R}}))^+\big)\#kG\Big)\otimes\mathcal{B}(V^*).\end{aligned}$$

By this fact and assumption (f) one obtains that

$$\begin{aligned}&\Big\langle\iota(\tau(d)),\tau(b)\tau(a)-\chi(b,a)\tau(a)\tau(b)-\frac{\lambda(b)\lambda(c_{\mathrm{R}})}{(3)^!_{p_c}}\tau(c)^3\Big\rangle =\\&=\langle\iota(\tau(d)),\tau(b)\rangle\tau(a)+\lambda(d)\langle g(a)^{-1}\iota(\tau(c)),\tau(b)\rangle\langle\iota(\tau(a)),\tau(a)\rangle\\&\quad-\chi(b,a)\chi(d,a)^{-1}\tau(a)\langle\iota(\tau(d)),\tau(b)\rangle\\&\quad-\frac{\lambda(b)\lambda(c_{\mathrm{R}})}{(3)^!_{p_c}}\Big(\chi(d,c)^{-1}\tau(c)\langle\iota(\tau(d)),\tau(c)^2\rangle\\&\quad+\lambda(d)\langle\iota(\tau(c)),\tau(c)\rangle\langle\iota(\tau(a)),\tau(c)^2\rangle\Big).\end{aligned}$$

Using Lemma 1.7.4(iii) and arguments as in the previous case one obtains the required result. ∎

Proof of Theorem 1.6.1. The proof bases on Corollary 1.7.2 and consists of a case by case checking of admissibility of (T,V,n) for all $n\in\mathbb{N}$. The computations are elementary, but they still need a considerable amount of time. However, in order to prove the first condition of admissibility, it is possible to avoid the explicit computation of the $\lambda(a)$, where $a\in N_2(T)$, as follows.

Suppose that $a\in N_2(T)$. Let $b,c\in N_0(T)$ be the unique nodes such that $b^{\mathrm{R}} = c^{\mathrm{L}} = a$. Then by (1.7) one obtains that

$$\begin{aligned}\lambda(b) =&\lambda(a)+(\chi(a^{\mathrm{L}},a)^{-1}-\chi(a,b^{\mathrm{L}}))(\ell^{\mathrm{L}}(b))_{p_a},\\ \lambda(c) =&\lambda(a)+(\chi(a,a^{\mathrm{R}})^{-1}-\chi(c^{\mathrm{R}},a))(\ell^{\mathrm{R}}(c))_{p_a}.\end{aligned}$$

Thus one gets

$$\begin{aligned}\lambda(b)=\lambda(c)\Leftrightarrow&\chi(a^{\mathrm{L}},a)^{-1}(\ell^{\mathrm{L}}(b))_{p_a}+\chi(c^{\mathrm{R}},a)(\ell^{\mathrm{R}}(c))_{p_a} =\\&=\chi(a,b^{\mathrm{L}})(\ell^{\mathrm{L}}(b))_{p_a}+\chi(a,a^{\mathrm{R}})^{-1}(\ell^{\mathrm{R}}(c))_{p_a}\\ \Leftrightarrow&(\ell^{\mathrm{L}}(b)+\ell^{\mathrm{R}}(c))_{p_a}(\chi(c^{\mathrm{R}},a)-\chi(a,b^{\mathrm{L}}))=0\\ \Leftrightarrow&(\ell^{\mathrm{L}}(b)+\ell^{\mathrm{R}}(c))_{p_a}(p_{a^{\mathrm{R}}}-p_a^{\ell^{\mathrm{L}}(b)-\ell^{\mathrm{R}}(c)}p_{a^{\mathrm{L}}})=0.\end{aligned}\tag{1.11}$$

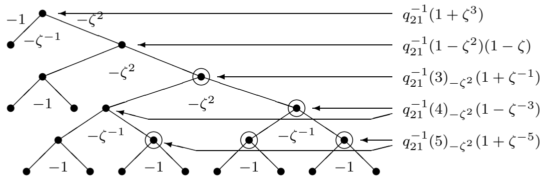


Figure 1.3: The example with tree T22.

Further, if $a \in N(T)$ and $a^{\mathrm{R}} =$ 'R' then

$$\lambda(a) = (q_{11}q_{12}q_{22})^{-1}(p_{\mathbf{r}} - q_{22}^{-1}/q_{11}^{-(\ell^{\mathrm{L}}(a)-2)})(\ell^{\mathrm{L}}(a))_{q_{11}^{-1}}, \tag{1.12}$$

and if $a \in N(T)$ and $a^{\mathrm{L}} =$ 'L' then

$$\lambda(a) = (q_{11}q_{12}q_{22})^{-1}(p_{\mathbf{r}} - q_{11}^{-1}/q_{22}^{-(\ell^{\mathrm{R}}(a)-2)})(\ell^{\mathrm{R}}(a))_{q_{22}^{-1}}. \tag{1.13}$$

Equations (1.11), (1.12), and (1.13) give an effective method to check the first condition of admissibility of (T, V, n) for all n. In fact, the equivalence between $\lambda(a) = 0$ and $a \in N_0(T)$ holds for all $a \in N(T)$ if and only if

$$\begin{aligned}
\ell_{\mathrm{R}}(\mathbf{r}) &= \min\{m \in \mathbb{N} \mid (m)_{q_{11}^{-1}}(q_{11}^{m-1}q_{12}q_{21} - 1) = 0\},\\
\ell_{\mathrm{L}}(\mathbf{r}) &= \min\{m \in \mathbb{N} \mid (m)_{q_{22}^{-1}}(q_{22}^{m-1}q_{12}q_{21} - 1) = 0\},\\
\ell_{\mathrm{R}}(a_{\mathrm{L}}) &= \min\{m \in \mathbb{N} \mid (m + \ell_{\mathrm{L}}(a_{\mathrm{R}}))_{p_a}(p_{a^{\mathrm{R}}}p_a^{\ell_{\mathrm{L}}(a_{\mathrm{R}})} - p_{a^{\mathrm{L}}}p_a^m) = 0\}
\end{aligned} \tag{1.14}$$

for all $a \in N_2(T)$.

As an example consider the last entry of Table A.1. Thus the generalized Dynkin diagram of V is $\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-1}{\circ}$ and T is the tree T22. In Figure 1.3 the numbers p_a, where $a \in N_2(T)$, are given as a label under the node a. Then Equations (1.14) can be easily checked. Moreover, the second and third condition of admissibility concerning the values of the p_a and the first relation of the fourth condition are also fulfilled by Figure 1.3. The relation $\ell_{\mathrm{R}}(c_{\mathrm{L}}) \le \ell^{\mathrm{R}}(b)$ in the fourth condition, where $c = b^{\mathrm{L}}$, is not fulfilled exactly for those nodes b which are marked with a big circle. In this case one can see that $\ell^{\mathrm{R}}(b) \le 2$, and for all five nodes the computation of $\nu(b)$ — using the values of $\lambda(\cdot)$ noted on the right hand side of Figure 1.3 — gives 0. ■

Chapter 2

Arithmetic Root Systems

2.1 Left and Right Skew-differential Operators

Let k be a field of characteristic zero and V a d-dimensional braided vector space of diagonal type. As noted in Section 1.1 there exists an abelian group G such that V is a Yetter–Drinfel'd module over kG. However, V is additionally a Yetter–Drinfel'd module over $k\mathbb{Z}^d$ where

$$e_i \triangleright x_j = q_{ji} x_j, \qquad \gamma(x_j) = e_j \otimes x_j. \tag{2.1}$$

Here $\{e_i \,|\, 1 \le i \le d\}$ is a fixed basis of the $\mathbb{Z}$-module $\mathbb{Z}^d$, and $\triangleright$ and γ denote the left action and left coaction of $k\mathbb{Z}^d$ on V. In order to avoid misunderstandings we will use the exponential notation for elements of $\mathbb{Z}^d$, thus e_i^{-1} is the inverse of e_i in the group $\mathbb{Z}^d$. The braiding σ of V commutes both with the action of g_i and the action of e_i: for all $i, j, m \in \{1, 2, \dots, d\}$ one has

$$\sigma(g_i.(x_j \otimes x_m)) = g_i.\sigma(x_j \otimes x_m), \quad \sigma(e_i \triangleright (x_j \otimes x_m)) = e_i \triangleright \sigma(x_j \otimes x_m). \tag{2.2}$$

The dual vector space V^* gives rise to *left* and *right skew-differential operators* y_i^{L} and y_i^{R} on $\mathcal{B}(V)$, where $1 \le i \le d$, in the following way. Let $\{y_i \,|\, 1 \le i \le d\}$ denote the basis of V^* dual to $\{x_i \,|\, 1 \le i \le d\}$. For $m \in \mathbb{N}$, $\rho \in \mathcal{B}(V)_m$ and $i \in \{1, 2, \dots, d\}$ set

$$y_i^{\mathrm{L}}(1) = y_i^{\mathrm{R}}(1) = 0, \qquad y_i^{\mathrm{L}}(\rho) = \rho_i', \qquad y_i^{\mathrm{R}}(\rho) = \rho_i'', \tag{2.3}$$

where

$$S_{m-1,1} = \mathrm{id} + \sigma_{m-1,m}^{-1} + \sigma_{m-1,m}^{-1}\sigma_{m-2,m-1}^{-1} + \dots + \sigma_{m-1,m}^{-1}\sigma_{m-2,m-1}^{-1} \cdots \sigma_{12}^{-1},$$

$$S_{1,m-1}(\rho) = \sum_{l=1}^{d} x_l \otimes \rho_l', \qquad S_{m-1,1}(\rho) = \sum_{l=1}^{d} \rho_l'' \otimes x_l.$$

One can consider the vector space $k\{y_i^{\mathrm{L}} \,|\, 1 \le i \le d\}$ as a Yetter–Drinfel'd module over kG dual to V, that is for $i, j \in \{1, 2, \dots, d\}$ one has

$$g_i.y_j^{\mathrm{L}} = q_{ij}^{-1} y_j^{\mathrm{L}}, \qquad \delta(y_j^{\mathrm{L}}) = g_j^{-1} \otimes y_j^{\mathrm{L}}. \tag{2.4}$$

Similarly, the vector space $k\{y_i^{\mathrm{R}} \,|\, 1 \le i \le d\}$ becomes a Yetter–Drinfel'd module over $k\mathbb{Z}^d$ dual to V, that is one has

$$e_i \triangleright y_j^{\mathrm{R}} = q_{ji}^{-1} y_j^{\mathrm{R}}, \qquad \gamma(y_j^{\mathrm{R}}) = e_j^{-1} \otimes y_j^{\mathrm{R}} \tag{2.5}$$

for all $i, j \in \{1, 2, \dots, d\}$. Moreover, the skew-differential operators y_i^{L} and y_i^{R}, where $1 \le i \le d$, and the braiding σ satisfy the equations

$$\begin{aligned}(y_i^{\mathrm{L}} \otimes \mathrm{id})(\sigma^{-1}(x_j \otimes x_m)) &= (g_i^{-1}.x_j)\, y_i^{\mathrm{L}}(x_m),\\ (\mathrm{id} \otimes y_i^{\mathrm{R}})(\sigma^{-1}(x_j \otimes x_m)) &= y_i^{\mathrm{R}}(x_j)\, e_i^{-1} \triangleright x_m\end{aligned} \tag{2.6}$$

for all $j, m \in \{1, 2, \dots, d\}$. Therefore equations (2.3) give that

$$\begin{aligned}y_i^{\mathrm{L}}(\rho_1\rho_2) &= y_i^{\mathrm{L}}(\rho_1)\rho_2 + g_i^{-1}.\rho_1\, y_i^{\mathrm{L}}(\rho_2),\\ y_i^{\mathrm{R}}(\rho_1\rho_2) &= \rho_1\, y_i^{\mathrm{R}}(\rho_2) + y_i^{\mathrm{R}}(\rho_1)\, e_i^{-1} \triangleright \rho_2\end{aligned} \tag{2.7}$$

for all $\rho_1, \rho_2 \in \mathcal{B}(V)$ and $i \in \{1, 2, \dots, d\}$. Note that by definition the skew-differential operators y_i^{L} and y_j^{R} commute:

$$y_i^{\mathrm{L}}(y_j^{\mathrm{R}}(\rho)) = y_j^{\mathrm{R}}(y_i^{\mathrm{L}}(\rho)) \quad \text{for all } i, j \in \{1, 2, \dots, d\},\ \rho \in \mathcal{B}(V). \tag{2.8}$$

Moreover, the skew-differential operators y_i^{L} and y_i^{R} and the group-like elements g_j, e_j, where $i, j \in \{1, 2, \dots, d\}$, satisfy the equations

$$\begin{aligned}g_j.(y_i^{\mathrm{L}}(g_j^{-1}.\rho)) &= q_{ji}^{-1} y_i^{\mathrm{L}}(\rho), & e_j \triangleright (y_i^{\mathrm{L}}(e_j^{-1} \triangleright \rho)) &= q_{ij}^{-1} y_i^{\mathrm{L}}(\rho),\\ g_j.(y_i^{\mathrm{R}}(g_j^{-1}.\rho)) &= q_{ji}^{-1} y_i^{\mathrm{R}}(\rho), & e_j \triangleright (y_i^{\mathrm{R}}(e_j^{-1} \triangleright \rho)) &= q_{ij}^{-1} y_i^{\mathrm{R}}(\rho)\end{aligned} \tag{2.9}$$

for all $\rho \in \mathcal{B}(V)$.

2.2 Finiteness Conditions

Retain the notation from the previous section. By Theorem 1.4.6 the algebra $\mathcal{B}(V)$ has a (restricted) Poincaré–Birkhoff–Witt basis consisting of homogeneous elements with respect to the $\mathbb{Z}^d$-grading of $\mathcal{B}(V)$. Moreover, one can assume that this basis has the additional property that

(P) the height of a Poincaré–Birkhoff–Witt generator of $\mathbb{Z}^d$-degree $\mathbf{e}$ is finite if and only if $2 \le \operatorname{ord}\chi(\mathbf{e}, \mathbf{e}) < \infty$ (ord means order with respect to multiplication), and in this case it coincides with $\operatorname{ord}\chi(\mathbf{e}, \mathbf{e})$.

Indeed, let $\mathbf{e}$ be the $\mathbb{Z}^d$-degree of a Poincaré–Birkhoff–Witt generator x of $\mathcal{B}(V)$ appearing in Theorem 1.4.6. By definition, if the height of x is finite, say m, then $\chi(\mathbf{e}, \mathbf{e})$ is a primitive m^{th} root of unity (with $m \ge 2$). Suppose now that the height of x is infinite, but $\chi(\mathbf{e}, \mathbf{e})$ is a primitive m^{th} root of unity. Then one can restrict the height of x to m and add x^m to the set of Poincaré–Birkhoff–Witt generators. Then the height of x^m is infinite and also equation $\chi(m\mathbf{e}, m\mathbf{e}) = \chi(\mathbf{e}, \mathbf{e})^{m^2} = 1$ holds.

Let $\boldsymbol{\Delta}^+(\mathcal{B}(V))$ denote the set of degrees of the (restricted) Poincaré–Birkhoff–Witt generators of $\mathcal{B}(V)$ counted with multiplicities. Note that this definition is independent of the choice of a $\mathbb{Z}^d$-graded Poincaré–Birkhoff–Witt basis satisfying property (P). By the definition of the $\mathbb{Z}^d$-degree of $\mathcal{B}(V)$ one clearly has $\boldsymbol{\Delta}^+(\mathcal{B}(V)) \subset \mathbb{N}_0^d$. Set $\boldsymbol{\Delta}(\mathcal{B}(V)) = \boldsymbol{\Delta}^+(\mathcal{B}(V)) \cup -\boldsymbol{\Delta}^+(\mathcal{B}(V))$.

Let us consider the following finiteness conditions on $\mathcal{B}(V)$.

(F1) $\dim_k \mathcal{B}(V) < \infty$,

(F2) the set $\boldsymbol{\Delta}^+(\mathcal{B}(V))$ is finite,

(F3) $\operatorname{Dim}_k \mathcal{B}(V) < \infty$,

where Dim_k denotes Gel'fand–Kirillov dimension. Obviously one has the implications (F1)⇒(F2)⇒(F3). Further, the condition (F1) holds if and only if (F2) is satisfied and the heights of all restricted Poincaré–Birkhoff–Witt generators of $\mathcal{B}(V)$ are finite. It is not known whether (F3) implies (F2).

As in [7] define

$$\operatorname{ad}_\sigma x_i(\rho) := x_i\rho - (g_i.\rho)x_i \tag{2.10}$$

for $\rho \in \mathcal{B}(V)$ and $i \in \{1, 2, \dots, d\}$. Consider the sets

$$M_{i,j} := \{(\operatorname{ad}_\sigma x_i)^m(x_j) \mid m \in \mathbb{N}_0\}$$

for $i, j \in \{1, 2, \dots, d\}$, $i \ne j$. By [33, Lemma 20], if one assumes that (F3) holds then all $M_{i,j}$ are finite sets. More generally, by Lemma 1.3.3 for given $i, j \in \{1, 2, \dots, d\}$ with $i \ne j$ the number

$$m_{ij} := \min\{m \in \mathbb{N}_0 \mid (m+1)_{q_{ii}}(q_{ii}^m q_{ij} q_{ji} - 1) = 0\} \tag{2.11}$$

is well-defined if and only if $M_{i,j}$ is finite. In this case one has the relations $(\operatorname{ad}_\sigma x_i)^{m_{ij}+1}(x_j) = 0$ and $(\operatorname{ad}_\sigma x_i)^{m_{ij}}(x_j) \ne 0$.

Fix $i \in \{1, 2, \dots, d\}$. By the above paragraph, if m_{ij} exists for all $j \in \{1, 2, \dots, d\}$ with $j \ne i$ (for example if (F3) holds) then one can introduce a linear map $s_i \in \operatorname{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$ as follows:

$$s_i(\mathbf{e}_j) := \begin{cases} -\mathbf{e}_i & \text{if } j = i,\\ \mathbf{e}_j + m_{ij}\mathbf{e}_i & \text{if } j \ne i.\end{cases} \tag{2.12}$$

The map s_i is a reflection [10, Ch. 5,§2], that is $s_i^2 = \mathrm{id}$ and the rank of the unique extension of $s_i - \mathrm{id}$ to a linear map in $\mathrm{End}_{\mathbb{Q}}(\mathbb{Q}^d)$ is 1.

The numbers m_{ij} have an important meaning also for $\boldsymbol{\Delta}(\mathcal{B}(V))$. Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ be a basis of $\mathbb{Z}^d$ and $\{x_1, \dots, x_d\}$ a canonical basis of V. Let $\mathbf{e}_i$ be the $\mathbb{Z}^d$-degree of x_i. Using Theorem 1.4.6, Proposition 1.4.8, and Lemma 1.3.3(ii) one obtains the following first result on $\boldsymbol{\Delta}^+(\mathcal{B}(V))$.

Lemma 2.2.1. *For any $i, j \in \{1, \dots, d\}$ with $i \neq j$ and for any $m \in \mathbb{Z}$ one has $\mathbf{e}_j + m\mathbf{e}_i \in \boldsymbol{\Delta}(\mathcal{B}(V))$ if and only if $0 \le m \le m_{ij}$.*

2.3 Transformations of Nichols Algebras

Suppose that $i \in \{1, 2, \dots, d\}$ is chosen such that $M_{i,j}$ is finite for all $j \in \{1, 2, \dots, d\}$ with $j \neq i$. We describe the construction of a Nichols algebra associated to i.

Since $ky_i^{\mathrm{R}} \in {}^{k\mathbb{Z}}_{k\mathbb{Z}}\mathcal{YD}$ one can construct the smash product

$$H_i := k[y_i^{\mathrm{R}}] \# k[e_i, e_i^{-1}].$$

It has a unique Hopf algebra structure satisfying the formulas

$$e_i y_i^{\mathrm{R}} = q_{ii}^{-1} y_i^{\mathrm{R}} e_i, \qquad \Delta(e_i) = e_i \otimes e_i, \qquad \Delta(y_i^{\mathrm{R}}) = 1 \otimes y_i^{\mathrm{R}} + y_i^{\mathrm{R}} \otimes e_i^{-1}. \tag{2.13}$$

By Equations (2.9) and (2.7) one obtains that $\mathcal{B}(V)$ is an H_i-module algebra, where e_i and e_i^{-1} act via $\triangleright$ and y_i^{R} acts by evaluation. By slight abuse of notation the symbol $\triangleright$ will also be used for the left action of H_i on $\mathcal{B}(V)$. Let $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$ denote the opposite algebra of the smash product of $\mathcal{B}(V)^{\mathrm{op}}$ and H_i^{cop}. Recall that it contains $\mathcal{B}(V)$ and $H_i^{\mathrm{op,cop}}$ as subalgebras and one has

$$\rho h = h_{(1)}(h_{(2)} \triangleright \rho), \tag{2.14}$$

and in particular

$$\rho y_i^{\mathrm{R}} = y_i^{\mathrm{R}} \cdot (e_i^{-1} \triangleright \rho) + y_i^{\mathrm{R}}(\rho) \tag{2.15}$$

for all $\rho \in \mathcal{B}(V)$ and $h \in H_i$, where $\Delta(h) = h_{(1)} \otimes h_{(2)}$ denotes the coproduct of $h \in H_i$. Further, $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$ is a Yetter–Drinfel'd module over kG where the left action . and left coaction δ on $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$ are given by

$$\begin{aligned} \delta(x_j) &= g_j \otimes x_j, & \delta(y_i^{\mathrm{R}}) &= g_i^{-1} \otimes y_i^{\mathrm{R}}, & \delta(e_i) &= 1 \otimes e_i, \\ g_j.x_m &= q_{jm} x_m, & g_j.y_i^{\mathrm{R}} &= q_{ji}^{-1} y_i^{\mathrm{R}}, & g_j.e_i &= e_i \end{aligned} \tag{2.16}$$

for all $j, m \in \{1, 2, \dots, d\}$.

Proposition 2.3.1. *Assume that $i \in \{1, 2, \dots, d\}$ such that $M_{i,j}$ is finite for all $j \in \{1, 2, \dots, d\}$ with $j \neq i$. Let V_i denote the subspace of the algebra $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$ generated by the set $\{(\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j) \mid j \neq i\} \cup \{y_i^{\mathrm{R}}\}$. The subalgebra $\mathcal{B}_i$ of $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$ generated by V_i is isomorphic to the Nichols algebra $\mathcal{B}(V_i)$, and the relation $\boldsymbol{\Delta}^+(\mathcal{B}_i) = (s_i(\boldsymbol{\Delta}^+(\mathcal{B}(V))) \setminus \{-\mathbf{e}_i\}) \cup \{\mathbf{e}_i\}$ holds.*

Remark. It will be clear from the construction that the transformation in Proposition 2.3.1 is invertible. This fact is in accord with the relation $s_i^2 = \mathrm{id}$. ∎

Proof. One easily checks that V_i is a Yetter–Drinfel'd submodule of $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$ over kG where the Yetter–Drinfel'd module structure of $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$ is described above the proposition.

In the following we will sometimes consider $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$ as a vector space and as such we identify it with $\mathcal{B}(V) \otimes H_i$.

We show that

$$\mathcal{B}_i = \ker y_i^{\mathrm{L}} \otimes k[y_i^{\mathrm{R}}] (\subset \mathcal{B}(V) \otimes H_i). \tag{2.17}$$

Recall that y_i^{R} may have finite order, but it is always the same as the order of x_i. Note also that by Equations (2.13), (2.14), and (2.8) the space $\ker y_i^{\mathrm{L}} \otimes k[y_i^{\mathrm{R}}]$ is a subalgebra of $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$. The inclusion $\mathcal{B}_i \subset \ker y_i^{\mathrm{L}} \otimes k[y_i^{\mathrm{R}}]$ holds by definition of $\mathcal{B}_i$. In order to prove that $\mathcal{B}_i \supset \ker y_i^{\mathrm{L}} \otimes k[y_i^{\mathrm{R}}]$ one first checks with standard methods [26, Lemma 2.2] that the multiplication map on the right hand side gives an isomorphism $\mathcal{B}(V) \cong \ker y_i^{\mathrm{L}} \otimes k[x_i]$ of graded vector spaces. Further, $\ker y_i^{\mathrm{L}}$ is generated as an algebra by $\bigcup_{j \mid j \neq i} M_{i,j}$. Now one proves by induction that

$$y_i^{\mathrm{R}}((\mathrm{ad}_\sigma x_i)^m(x_j)) = q_{ji}^{-1}(m)_{q_{ii}^{-1}}(1 - q_{ii}^{m-1} q_{ij} q_{ji})(\mathrm{ad}_\sigma x_i)^{m-1}(x_j) \tag{2.18}$$

for all $m \in \mathbb{N}_0$ and all $j \in \{1, 2, \dots, d\}$. Hence by Equation (2.15) one obtains that $\ker y_i^{\mathrm{L}} \otimes k[y_i^{\mathrm{R}}]$ is generated by V_i.

In order to prove that the subalgebra $\mathcal{B}_i$ of $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$ is a Nichols algebra it suffices to find d appropriate skew-differential operators on $\mathcal{B}_i$. More precisely, according to the fact that the Yetter–Drinfel'd module V_i generates $\mathcal{B}_i$, one has to find for all $m \in \{1, 2, \dots, d\}$ a map $Y_m \in \mathrm{End}_k(\mathcal{B}_i)$

such that

$$\begin{aligned} Y_j(\rho_1\rho_2) =& Y_j(\rho_1)\rho_2 + (g_i^{-m_{ij}} g_j^{-1}.\rho_1)\, Y_j(\rho_2),\\ Y_i(\rho_1\rho_2) =& Y_i(\rho_1)\rho_2 + (g_i.\rho_1)\, Y_i(\rho_2),\\ Y_l((\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j)) = \delta_{lj},& \quad Y_l(y_i^{\mathrm{R}}) = \delta_{li} \quad \text{(Kronecker's delta)}\end{aligned} \tag{2.19}$$

for all $\rho_1, \rho_2 \in \mathcal{B}_i$, and $j, l \in \{1, 2, \ldots, d\}$, $j \neq i$. For $\rho \in \mathcal{B}_i$ set

$$Y_i(\rho) := \mathrm{ad}_\sigma x_i(\rho) = x_i\rho - (g_i.\rho)x_i.$$

By definition of ad_σ, Y_i satisfies the second equation of (2.19) for all $\rho_1, \rho_2 \in (\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$. Moreover, by Equation (2.7) one obtains for all $\rho \in \ker y_i^{\mathrm{L}}$ the formula

$$y_i^{\mathrm{L}}(x_i\rho - (g_i.\rho)x_i) = \rho - g_i^{-1}.(g_i.\rho) = 0.$$

Hence Y_i maps $\ker y_i^{\mathrm{L}} \subset \mathcal{B}(V)$ onto itself. By Equations (2.16) and (2.15) one gets also

$$Y_i(y_i^{\mathrm{R}}) = x_i y_i^{\mathrm{R}} - (g_i.y_i^{\mathrm{R}})x_i = x_i y_i^{\mathrm{R}} - q_{ii}^{-1} y_i^{\mathrm{R}} x_i = y_i^{\mathrm{R}}(x_i) = 1.$$

Finally, the equation $Y_i((\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j)) = (\mathrm{ad}_\sigma x_i)^{m_{ij}+1}(x_j) = 0$ holds for all $j \in \{1, 2, \ldots, d\}$, $j \neq i$, by the definition of m_{ij}.

Let us now turn to the construction of Y_j, where $j \in \{1, 2, \ldots, d\}$, $j \neq i$. Set $\lambda_j := \langle \iota((\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j)), (\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j)\rangle$ (for the notation see Section 1.3). By Equation (1.5) and the definition of m_{ij} one obtains that $\lambda_j = \langle y_i^{m_{ij}} y_j, (\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j)\rangle \in k^*$. Define

$$Y_j(\rho \otimes (y_i^{\mathrm{R}})^m) := \frac{1}{\lambda_j}\langle \iota((\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j)), \rho\rangle \otimes (y_i^{\mathrm{R}})^m$$

for all $\rho \in \ker y_i^{\mathrm{L}}$, $m \in \mathbb{N}_0$, and $j \in \{1, 2, \ldots, d\}$, $j \neq i$. By induction one can show that the coproduct of $\iota((\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j)) \in \mathcal{B}(V^*) \# kG$ takes the form

$$\iota((\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j)) \otimes 1 + \sum_{m=0}^{m_{ij}} c_i (y_i^{\mathrm{L}})^{m_{ij}-m} g_i^{-m} g_j^{-1} \otimes \iota((\mathrm{ad}_\sigma x_i)^m(x_j))$$

for certain $c_i \in k$, where $c_{m_{ij}} = 1$. Since y_i^{L} vanishes on $\ker y_i^{\mathrm{L}}$, the first equation of (2.19) holds for all $\rho_1 \in \ker y_i^{\mathrm{L}}$, $\rho_2 \in \mathcal{B}_i$. It remains to show that the first equation of (2.19) is valid for $\rho_1 = y_i^{\mathrm{R}}$ and $\rho_2 \in \ker y_i^{\mathrm{L}}$. By Equations (2.15) and (2.8) one obtains that

$$\begin{aligned} Y_j(y_i^{\mathrm{R}}\rho_2) =& Y_j((e_i \triangleright \rho_2) \otimes y_i^{\mathrm{R}} - y_i^{\mathrm{R}}(e_i \triangleright \rho_2))\\ =& Y_j(e_i \triangleright \rho_2) \otimes y_i^{\mathrm{R}} - y_i^{\mathrm{R}}(Y_j(e_i \triangleright \rho_2)).\end{aligned} \tag{2.20}$$

On the other hand, by Equations (2.15) and (2.9) one gets

$$(g_i^{-m_{ij}} g_j^{-1}.y_i^{\mathrm{R}})\, Y_j(\rho_2) = q_{ii}^{m_{ij}} q_{ji}\Big((e_i \triangleright Y_j(\rho_2)) \otimes y_i^{\mathrm{R}} - y_i^{\mathrm{R}}(e_i \triangleright Y_j(\rho_2))\Big).$$

By Equation (2.9) the latter coincides with (2.20).

Finally, it has to be shown that $\boldsymbol{\Delta}^+(\mathcal{B}_i) = (s_i(\boldsymbol{\Delta}^+(\mathcal{B}(V))) \setminus \{-\mathbf{e}_i\}) \cup \{\mathbf{e}_i\}$. In the $\mathbb{Z}^d$-graded algebra $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$ the elements x_j and y_i^{R} have degree $\mathbf{e}_j$ and $-\mathbf{e}_i$, respectively, for all $j \in \{1, 2, \ldots, d\}$. Hence the elements $(\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j)$ have degree $\mathbf{e}_j + m_{ij}\mathbf{e}_i = s_i(\mathbf{e}_j)$ for all $j \in \{1, 2, \ldots, d\}$, $j \neq i$. Fix the $\mathbb{Z}^d$-degrees of the generators of $\mathcal{B}_i$ by

$$\deg\, y_i^{\mathrm{R}} := \mathbf{e}_i, \qquad \deg\, (\mathrm{ad}_\sigma x_i)^{m_{ij}}(x_j) := \mathbf{e}_j \tag{2.21}$$

for all $j \in \{1, 2, \ldots, d\}$, $j \neq i$. Then $s_i(\boldsymbol{\Delta}^+(\mathcal{B}_i))$ is exactly the set of degrees of the (restricted) Poincaré–Birkhoff–Witt generators of $\mathcal{B}_i$ in $(\mathcal{B}(V)^{\mathrm{op}} \# H_i^{\mathrm{cop}})^{\mathrm{op}}$.

Recall that any Nichols algebra $\mathcal{B}$ of rank d is isomorphic as a $\mathbb{Z}^d$-graded vector space to the algebra $k[x_r \,|\, r \in \boldsymbol{\Delta}^+(\mathcal{B})]/(x_r^{h_r} \,|\, r \in \boldsymbol{\Delta}^+(\mathcal{B}), h_r < \infty)$. Here h_r denotes the height of the (restricted) Poincaré–Birkhoff-Witt generator corresponding to $r \in \boldsymbol{\Delta}^+(\mathcal{B})$, and by property (P) it is uniquely determined by the $\mathbb{Z}^d$-degree of x_r. Therefore it suffices to know the multiplicities of the $\mathbb{Z}^d$-homogeneous components of $\mathcal{B}$ in order to determine $\boldsymbol{\Delta}^+(\mathcal{B}(V))$. This fact and the equation $\mathcal{B}(V) \cong \ker y_i^{\mathrm{L}} \otimes k[x_i]$ allows us to conclude that (with obvious interpretation) $\boldsymbol{\Delta}^+(\ker y_i^{\mathrm{L}} \otimes k[y_i^{\mathrm{R}}]) = (\boldsymbol{\Delta}^+(\mathcal{B}(V)) \setminus \{\mathbf{e}_i\}) \cup \{-\mathbf{e}_i\}$. Since the determination of $\boldsymbol{\Delta}^+$ can be performed using different total orderings of $\mathbb{Z}^d$, we conclude from Equation (2.17) that $s_i(\boldsymbol{\Delta}^+(\mathcal{B}_i)) = (\boldsymbol{\Delta}^+(\mathcal{B}(V)) \setminus \{\mathbf{e}_i\}) \cup \{-\mathbf{e}_i\}$. This proves the proposition. ■

Remark. 1. Proposition 2.3.1 has the following interpretation. Set $\boldsymbol{\Delta} := \boldsymbol{\Delta}(\mathcal{B}(V)) \subset \mathbb{Z}^d$. Then the set $\boldsymbol{\Delta}(\mathcal{B}_i)$ coincides with $\boldsymbol{\Delta}$ with respect to the basis $\{s_i(\mathbf{e}_j) \,|\, 1 \leq j \leq d\}$ of $\mathbb{Z}^d$. With other words, the transformation doesn't change $\boldsymbol{\Delta}(\mathcal{B}(V))$, it changes only the basis of $\mathbb{Z}^d$. Additionally, this base change is performed in such a way that the new basis is a subset of $\boldsymbol{\Delta}(\mathcal{B}(V))$.

2. The constants m_{ij} appearing in the definition of the map s_i, where $j \in \{1, 2, \dots, d\}$ and $j \neq i$, depend on the structure constants of the braiding. The latter usually change if one performs a transformation. However in a special case, namely if the braiding is of Cartan type, this change is not essential. This situation is the one which is best understood, and it will also be analyzed in more detail in Section 2.10. ∎

2.4 The Weyl Groupoid

Let $\mathcal{G}$ be a nonempty set, $D \subset \mathcal{G} \times \mathcal{G}$ a nonempty subset, and $\circ : D \to \mathcal{G}$ a map of sets. The pair $(\mathcal{G}, \circ)$ is called a *groupoid* if it satisfies the following conditions (see for example [11, Sect. 3.3]).

- If $(x, y) \in D$ then each of the three elements $x, y, x \circ y$ is uniquely determined by the other two.
- If $(x, y), (y, z) \in D$ then $(x \circ y, z), (x, y \circ z) \in D$ and $(x \circ y) \circ z = x \circ (y \circ z)$.
- If $(x, y), (x \circ y, z) \in D$ then $(y, z), (x, y \circ z) \in D$ and $(x \circ y) \circ z = x \circ (y \circ z)$.
- If $(y, z), (x, y \circ z) \in D$ then $(x, y), (x \circ y, z) \in D$ and $(x \circ y) \circ z = x \circ (y \circ z)$.
- For all $x \in \mathcal{G}$ there exist unique elements $e, f, y \in \mathcal{G}$ such that (e, x), $(x, f), (y, x) \in D$, $e \circ x = x \circ f = x$, and $y \circ x = f$.
- If $e \circ e = e$, $f \circ f = f$ for certain $e, f \in \mathcal{G}$ then there exists $x \in \mathcal{G}$ such that $e \circ x = x \circ f = x$.

Let $\widetilde{W}$ denote the groupoid consisting of all pairs (T, E) where $T \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$ and E is a basis of $\mathbb{Z}^d$, and the composition $(T_1, E_1) \circ (T_2, E_2)$ is defined (and is then equal to $(T_1 T_2, E_2)$) if and only if $T_2(E_2) = E_1$. The groupoid $\widetilde{W}$ is naturally acting on the set of all bases of $\mathbb{Z}^d$ via the rule

$$(T, E)(F) = \begin{cases} T(E) & \text{if } E = F, \\ \text{not defined} & \text{otherwise.} \end{cases} \tag{2.22}$$

Now a slight generalization of the construction of reflections in Section 2.2 will be given. Let χ be a bicharacter on $\mathbb{Z}^d$, F a basis of $\mathbb{Z}^d$, and fix

$\mathbf{f}' \in F$. Suppose that for all $\mathbf{f}'' \in F \setminus \{\mathbf{f}'\}$ the numbers

$$\begin{aligned} m(\mathbf{f}', \mathbf{f}'') := \min\{m \in \mathbb{N}_0 \,|\, \chi(\mathbf{f}', \mathbf{f}')^m \chi(\mathbf{f}', \mathbf{f}'')\chi(\mathbf{f}'', \mathbf{f}') = 1 \text{ or} \\ \chi(\mathbf{f}', \mathbf{f}')^{m+1} = 1,\ \chi(\mathbf{f}', \mathbf{f}') \neq 1\} \end{aligned} \tag{2.23}$$

exist. Set $m(\mathbf{f}', \mathbf{f}') = -2$ and define the linear map $s_{\mathbf{f}', F} \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$ by the formula

$$s_{\mathbf{f}', F}(\mathbf{f}'') := \mathbf{f}'' + m(\mathbf{f}', \mathbf{f}'')\mathbf{f}'. \tag{2.24}$$

Definition 2.4.1. Let E be a basis of $\mathbb{Z}^d$ and χ a bicharacter on $\mathbb{Z}^d$. Define $W_{\chi,E}$ as the smallest subgroupoid of $\widetilde{W}$ which contains (id, E), and if $(\mathrm{id}, F) \in W_{\chi,E}$ for a basis F of $\mathbb{Z}^d$ and $\mathbf{f} \in F$, then $(s_{\mathbf{f},F}, F), (\mathrm{id}, s_{\mathbf{f},F}(F)) \in W_{\chi,E}$ whenever $s_{\mathbf{f},F}$ is defined. It is called the *Weyl groupoid* associated to the pair (χ, E).

Let $W^{\mathrm{ext}}_{\chi,E}$ denote the smallest subgroupoid of $\widetilde{W}$ which contains $W_{\chi,E}$ and for each basis F of $\mathbb{Z}^d$ with $(\mathrm{id}, F) \in W_{\chi,E}$ and for each $\tau \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$ with $\tau(F) = F$ also $(\tau, F) \in W^{\mathrm{ext}}_{\chi,E}$. It is easily checked that

$$W^{\mathrm{ext}}_{\chi,E} = \{(\tau T, F) \,|\, (T, F) \in W_{\chi,E}, \tau \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d), \tau T(F) = T(F)\}. \tag{2.25}$$

Definition 2.4.2. The groupoid $W_{\chi,E}$ respectively $W^{\mathrm{ext}}_{\chi,E}$ is called *full* if $s_{\mathbf{f},F}$ is well-defined for all bases F of $\mathbb{Z}^d$ with $(\mathrm{id}, F) \in W_{\chi,E}$ and for all $\mathbf{f} \in F$.

Note that $W_{\chi,E}$ is full if and only if $W^{\mathrm{ext}}_{\chi,E}$ is full, and $W_{\chi,E}$ is finite if and only if $W^{\mathrm{ext}}_{\chi,E}$ is finite.

2.5 Arithmetic Root Systems

Let E be a basis of $\mathbb{Z}^d$ and χ a bicharacter on $\mathbb{Z}^d$ with values in k^*.

Definition 2.5.1. A triple $(\mathbf{\Delta}, \chi, E)$ is called an *arithmetic root system* if $W_{\chi,E}$ is full and finite and $\mathbf{\Delta} = \bigcup\{F \,|\, (\mathrm{id}, F) \in W_{\chi,E}\}$.

If $(\mathbf{\Delta}, \chi, E)$ is an arithmetic root system then for a basis F of $\mathbb{Z}^d$ with $(\mathrm{id}, F) \in W_{\chi,E}$ set $\mathbf{\Delta}^+_F = \mathbf{\Delta} \cap \mathbb{N}_0 F$.

Proposition 2.5.1. *If $E = \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ is a basis of $\mathbb{Z}^d$, $(\mathbf{\Delta}, \chi, E)$ is an arithmetic root system, and $F = \{\mathbf{f}_1, \dots, \mathbf{f}_d\}$ is a basis of $\mathbb{Z}^d$ such that $(\mathrm{id}, F) \in W_{\chi,E}$ then one has $\mathbf{\Delta} = \mathbf{\Delta}^+_F \cup -\mathbf{\Delta}^+_F$.*

Proof. Since $s_{\mathbf{f},F}(\mathbf{f}) = -\mathbf{f}$ whenever $(\mathrm{id}, F) \in W_{\chi,E}$ and $\mathbf{f} \in F$, one has $\mathbf{\Delta} = -\mathbf{\Delta}$.

Let V_E and V_F denote the braided vector spaces with structure constants $(\chi(\mathbf{e}_i, \mathbf{e}_j))_{i,j\in\{1,\dots,d\}}$ and $(\chi(\mathbf{f}_i, \mathbf{f}_j))_{i,j\in\{1,\dots,d\}}$, respectively, and consider the elements of E as the degrees of the generators of $\mathcal{B}(V_E)$. Then by the remark after Proposition 2.3.1 the elements of F are the degrees of the generators of the Nichols algebra $\mathcal{B}(V_F)$, and hence $\mathbf{\Delta}(\mathcal{B}(V_F)) = \mathbf{\Delta}^+(\mathcal{B}(V_F)) \cup -\mathbf{\Delta}^+(\mathcal{B}(V_F)) \subset \mathbb{N}_0 F \cup -\mathbb{N}_0 F$. Together with relations $\mathbf{\Delta} \subset \mathbf{\Delta}(\mathcal{B}(V_E)) = \mathbf{\Delta}(\mathcal{B}(V_F))$ the claim of the proposition follows. ∎

Proposition 2.5.2. *Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ be a basis of $\mathbb{Z}^d$ and let $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ denote the standard scalar product on $\mathbb{R}^d$ induced by E. If $(\mathbf{\Delta}, \chi, E)$ is an arithmetic root system then for any $\beta \in \mathbb{Z}^d$ such that $\langle \beta, \mathbf{e} \rangle_{\mathbb{R}^d} \neq 0$ for all $\mathbf{e} \in \mathbf{\Delta}$ there exists a unique basis $F = \{\mathbf{f}_1, \dots, \mathbf{f}_d\}$ of $\mathbb{Z}^d$ with $(\mathrm{id}, F) \in W_{\chi,E}$ and $\langle \mathbf{f}_i, \beta \rangle_{\mathbb{R}^d} > 0$ for all i.*

Proof. Without loss of generality assume that the numbers $\langle \beta, \mathbf{e}_i \rangle_{\mathbb{R}^d}$, where $1 \le i \le d$, are linearly independent over $\mathbb{Z}$. Choose $\alpha \in \mathbb{R}^d$ such that $\langle \alpha, \mathbf{e}_i \rangle_{\mathbb{R}^d} > 0$ for all i and the set $\{\langle \alpha, \mathbf{e}_i \rangle_{\mathbb{R}^d}, \langle \beta, \mathbf{e}_i \rangle_{\mathbb{R}^d} \,|\, 1 \le i \le d\}$ is linearly independent over $\mathbb{Z}$. For $i \in \{0, 1, \dots, d-1\}$ and $t \in [0,1] \subset \mathbb{R}$ define

$$\mathbf{n}\left(\frac{i+t}{d}\right) := (n_1, \dots, n_d), \qquad n_j := \begin{cases} \beta_j & \text{if } j \le i, \\ (1-t)\alpha_j + t\beta_j & \text{if } j = i+1, \\ \alpha_j & \text{if } j > i+1. \end{cases}$$

Then one has $\mathbf{n}(0) = \alpha$ and $\mathbf{n}(1) = \beta$, and $\mathbf{n}(t) \neq 0$ for all $t \in [0,1]$. Moreover, since for fixed $t \in [0,1]$ at least $d-1$ of the d numbers $\langle \mathbf{n}(t), \mathbf{e}_i \rangle_{\mathbb{R}^d}$ are linearly independent over $\mathbb{Z}$, the subspaces $H(t) \subset \mathbb{R}^d$ orthogonal to $\mathbf{n}(t)$ never contain more than one line $\mathbb{R}\alpha$ with $\alpha \in \mathbf{\Delta} \subset \mathbb{Z}^d \subset \mathbb{R}^d$. Note also that for all $\mathbf{e} \in \mathbb{Z}^d$ the function $\langle \mathbf{n}(\cdot), \mathbf{e} \rangle_{\mathbb{R}^d} : [0,1] \to \mathbb{R}$ is continuous and linear on the intervals $[i/d, (i+1)/d]$ for all $i \in \{0, 1, \dots, d-1\}$, and $\langle \mathbf{n}(i/d), \mathbf{e} \rangle_{\mathbb{R}^d} \neq 0$ for all $i \in \{0, 1, \dots, d\}$. Let $(t_1, \dots, t_m)$ denote the maximal sequence of numbers in $[0,1]$ such that $H(t_i) \cap \mathbf{\Delta} \neq \{\}$ and $t_i < t_j$ whenever $i < j$. We prove the existence of F by induction on m.

If $m = 0$ then take $F = E$. Otherwise by induction hypothesis one can assume that there exists a basis $F' = \{\mathbf{f}'_1, \dots, \mathbf{f}'_d\}$ of $\mathbb{Z}^d$ such that $(\mathrm{id}, F') \in W_{\chi,E}$ and $\langle \mathbf{f}_i, \mathbf{n}(t) \rangle_{\mathbb{R}^d} > 0$ for all i and all t in the open interval (t_{m-1}, t_m). Assume that $\mathbf{f}' \in \mathbf{\Delta}$ such that $\langle \mathbf{f}', \mathbf{n}(t) \rangle_{\mathbb{R}^d} > 0$ for $t \in (t_{m-1}, t_m)$ and $\langle \mathbf{f}', \mathbf{n}(t_m) \rangle_{\mathbb{R}^d} = 0$. By Proposition 2.5.1 one has $\mathbf{f}' \in \mathbb{N}_0 F'$. Moreover, $\mathbf{f}'$ has to be an element of F', since otherwise relations $\langle \mathbf{f}'_i, \mathbf{n}(t_m) \rangle_{\mathbb{R}^d} > 0$ for all i give a contradiction.

Set $F = s_{\mathbf{f}',F'}(F')$. One has to show that $\langle \mathbf{f}, \beta \rangle_{\mathbb{R}^d} > 0$ for all $\mathbf{f} \in F$. By piecewise linearity of the function $\langle \mathbf{n}(\cdot), \mathbf{f}' \rangle_{\mathbb{R}^d}$ relation $\langle \mathbf{f}', \mathbf{n}(t) \rangle_{\mathbb{R}^d} < 0$ holds for $t > t_m$, and hence for $-\mathbf{f}' = s_{\mathbf{f}',F'}(\mathbf{f}')$ one gets $\langle -\mathbf{f}', \beta \rangle_{\mathbb{R}^d} > 0$. Since $s_{\mathbf{f}',F'}(\mathbf{f}) \in \mathbf{\Delta}^+_{F'}$ and $s_{\mathbf{f}',F'}(\mathbf{f}) \cap \mathbb{R}\mathbf{f}' = \{\}$ for all $\mathbf{f} \in F' \setminus \{\mathbf{f}'\}$, one has $\langle s_{\mathbf{f}',F'}(\mathbf{f}), \mathbf{n}(t) \rangle_{\mathbb{R}^d} > 0$ for all $t > t_m$ and $\mathbf{f} \in F' \setminus \{\mathbf{f}'\}$.

It remains to prove the uniqueness of F. Assume that there is another basis $F' = \{\mathbf{f}'_1, \dots, \mathbf{f}'_d\}$ of $\mathbb{Z}^d$ such that $(\mathrm{id}, F') \in W_{\chi,E}$ and $\langle \beta, \mathbf{f}'_i \rangle_{\mathbb{R}^d} > 0$ for all i. By Proposition 2.5.1 and the latter relations one has $F' \subset \mathbf{\Delta}^+_F$, and similary $F \subset \mathbf{\Delta}^+_{F'}$. Thus there exist $a_{ij}, b_{ij} \in \mathbb{N}_0$ such that $\mathbf{f}_i = \sum_j a_{ij} \mathbf{f}'_j$ and $\mathbf{f}'_i = \sum_j b_{ij} \mathbf{f}_j$. From the equation $\mathbf{f}_i = \sum_j \sum_l a_{ij} b_{jl} \mathbf{f}_l$ one obtains that $F = F'$. ∎

Theorem 2.5.3. *If χ is a bicharacter on $\mathbb{Z}^d$ and $(\mathbf{\Delta}, \chi, E)$ is an arithmetic root system then for the braided vector space V of diagonal type with $\dim_k V = d$ and with structure constants $q_{ij} := \chi(\mathbf{e}_i, \mathbf{e}_j)$ one has $\mathbf{\Delta}(\mathcal{B}(V)) = \mathbf{\Delta}$. Conversely, if V is a braided vector space of diagonal type such that $\mathbf{\Delta}^+(\mathcal{B}(V))$ is finite then $(\mathbf{\Delta}(\mathcal{B}(V)), \chi, E)$ is an arithmetic root system, where $\chi : \mathbb{Z}^d \times \mathbb{Z}^d \to k^*$ is defined by (1.4). These correspondences are inverse to each other.*

Theorem 2.5.3 gives the following important facts about Nichols algebras of diagonal type.

Corollary 2.5.4. *If $\alpha \in (\mathbb{N} \setminus \{1\}) \cdot \mathbb{Z}^d$ then $\alpha \notin \mathbf{\Delta}$. In particular, if $\alpha \in \mathbf{\Delta}$ and $r \in \mathbb{R}$ then $r\alpha \in \mathbf{\Delta}$ if and only if $r \in \{1, -1\}$.*

Corollary 2.5.5. *For any braided vector space V of diagonal type such that $\mathbf{\Delta}(\mathcal{B}(V))$ is finite, the multiplicities of the elements of $\mathbf{\Delta}(\mathcal{B}(V))$ are one.*

Proof of the theorem. Let χ be a bicharacter on $\mathbb{Z}^d$, $(\mathbf{\Delta}, \chi, E)$ an arithmetic root system, and V as in the theorem. By Definition 2.5.1 the Weyl groupoid $W_{\chi,E}$ is full and finite. Moreover, Proposition 2.3.1 tells that $\mathbf{\Delta} \subset \mathbf{\Delta}(\mathcal{B}(V))$. One has to show that

1. any $\alpha \in \mathbf{\Delta}$ has multiplicity 1 in $\mathbf{\Delta}(\mathcal{B}(V))$,
2. if $\alpha \in \mathbf{\Delta}(\mathcal{B}(V))$, $\beta \in \mathbf{\Delta}$, $\lambda \in \mathbb{R}$, and $\alpha = \lambda\beta$, then $\lambda^2 = 1$, and
3. there is no $\alpha \in \mathbf{\Delta}(\mathcal{B}(V))$ such that $\alpha \notin \mathbb{R}\beta$ for all $\beta \in \mathbf{\Delta}$.

By definition of $\mathbf{\Delta}$ any $\alpha \in \mathbf{\Delta} \subset \mathbf{\Delta}(\mathcal{B}(V))$ is an element of some basis F of $\mathbb{Z}^d$ with $(\mathrm{id}, F) \in W_{\chi,E}$, and the multiplicity of α is invariant under the transformations $s_{\mathbf{f},F}$. Therefore assertion 1 is obvious and Lemma 1.3.3 implies the second assertion.

Let $\mathbf{e} \in \mathbf{\Delta}(\mathcal{B}(V)) \setminus \mathbf{\Delta}$, and choose $\alpha \in \mathbb{R}^d$ such that $\langle \mathbf{e}, \alpha \rangle_{\mathbb{R}^d} = 0$ but $\langle \mathbf{e}', \alpha \rangle_{\mathbb{R}^d} \neq 0$ for all $\mathbf{e}' \in \mathbf{\Delta}$. Since $\mathbf{\Delta}$ is finite, there exists $t \in \mathbb{R}$ such that

$$0 < \langle \mathbf{e} + t\alpha, \mathbf{e} \rangle_{\mathbb{R}^d} = \langle \mathbf{e}, \mathbf{e} \rangle_{\mathbb{R}^d} < |\langle \mathbf{e} + t\alpha, \mathbf{e}' \rangle_{\mathbb{R}^d}| \qquad (2.26)$$

for all $\mathbf{e}' \in \mathbf{\Delta}$. On the other hand, by Proposition 2.5.2 there exists a basis F of $\mathbb{Z}^d$ such that $(\mathrm{id}, F) \in W_{\chi,E}$ and $\langle \mathbf{e} + t\alpha, \mathbf{f} \rangle_{\mathbb{R}^d} > 0$ for all $\mathbf{f} \in F$. By Proposition 2.3.1 one obtains that $\mathbf{e} \in \mathbb{N}_0 F \cup -\mathbb{N}_0 F$ which is a contradiction to relation (2.26), and hence one gets $\mathbf{\Delta}(\mathcal{B}(V)) = \mathbf{\Delta}$.

Let V be a braided vector space of diagonal type with $\dim_k V = d$ such that $\mathbf{\Delta}^+(\mathcal{B}(V))$ is finite. Since $W_{\chi,E}$ can be interpreted as a set of pairs (F_1, F_2), where F_1, F_2 are subsets of $\mathbf{\Delta}(\mathcal{B}(V))$ of cardinality d, the groupoid $W_{\chi,E}$ is finite. Moreover, the sets $M_{i,j}$ in Section 2.2 have to be finite, and hence $W_{\chi,E}$ is also full. Therefore $(\mathbf{\Delta}, \chi, E)$, where χ is defined by (1.4) and $\mathbf{\Delta} = \bigcup\{F \subset \mathbb{Z}^d \,|\, (\mathrm{id}, F) \in W_{\chi,E}\}$, is an arithmetic root system. By the first part of the theorem one gets $\mathbf{\Delta}(\mathcal{B}(V)) = \mathbf{\Delta}$. This proves the second part.

The last assertion follows from the fact that $\mathbf{\Delta}$ and $\mathbf{\Delta}(\mathcal{B}(V))$ are uniquely determined by χ and V, respectively, and the latter are in one-to-one correspondence via (1.4). ∎

2.6 Weyl Equivalence

Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ and $F = \{\mathbf{f}_1, \dots, \mathbf{f}_d\}$ be bases of $\mathbb{Z}^d$ and χ, η bicharacters on $\mathbb{Z}^d$ with values in k^*.

Definition 2.6.1. The pairs (χ, E) and (η, F) are called *Weyl equivalent* if there exists $(T, E) \in W^{\mathrm{ext}}_{\chi,E}$ such that $\chi(T(e), T(e)) = \eta(\phi(e), \phi(e))$ for all $e \in \mathbb{Z}^d$, where $\phi \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$ is the unique linear map such that $\phi(\mathbf{e}_i) = \mathbf{f}_i$ for all $i \in \{1, 2, \dots, d\}$. The pairs (χ, E) and (η, F) are called *twist equivalent* if they are Weyl equivalent with $T(E) = E$.

One says that two arithmetic root systems respectively Yetter–Drinfel'd modules of diagonal type respectively Nichols algebras of diagonal type are Weyl equivalent (twist equivalent), if the corresponding pairs (χ, E) and (η, F) are Weyl equivalent (twist equivalent).

Remark. The above definition of twist equivalence of Nichols algebras of diagonal type coincides with [7, Definition 3.8]. Moreover, the pairs

(χ, E) and (η, F) are twist equivalent if and only if their generalized Dynkin diagrams coincide. ∎

Example 2.6.1. Let E be a basis of $\mathbb{Z}^d$, χ a bicharacter on $\mathbb{Z}^d$, and $(q_{ij})_{i,j}$ the set of structure constants of χ with respect to E. Choose $i \in \{1, 2, \dots, d\}$. For all $j \neq i$ set $m_{ij} := m(\mathbf{e}_i, \mathbf{e}_j)$ as in (2.11) and define $\mathbf{f}_j := s_{\mathbf{e}_i,E}(\mathbf{e}_j)$. One obtains that

$$\begin{aligned} \chi(\mathbf{f}_i, \mathbf{f}_i) &= q_{ii}, & \chi(\mathbf{f}_i, \mathbf{f}_j) &= q_{ij}^{-1} q_{ii}^{-m_{ij}}, \\ \chi(\mathbf{f}_j, \mathbf{f}_i) &= q_{ji}^{-1} q_{ii}^{-m_{ij}}, & \chi(\mathbf{f}_j, \mathbf{f}_l) &= q_{jl} q_{il}^{m_{ij}} q_{ji}^{m_{il}} q_{ii}^{m_{ij} m_{il}} \end{aligned}$$

for all $j, l \neq i$. Define

$$p_{ij} := \begin{cases} 1 & \text{if } q_{ii}^m q_{ij} q_{ji} = 1 \text{ for some } m \in \mathbb{Z}, \\ q_{ii}^{-1} q_{ij} q_{ji} & \text{otherwise.} \end{cases} \qquad (2.27)$$

Then the constants $q'_{ij} := \chi(\mathbf{f}_i, \mathbf{f}_j)$ satisfy the equations

$$q'_{ii} = q_{ii}, \quad q'_{jj} = p_{ij}^{m_{ij}} q_{jj}, \quad q'_{ij} q'_{ji} = p_{ij}^{-2} q_{ij} q_{ji}, \quad q'_{jl} q'_{lj} = p_{ij}^{m_{il}} p_{il}^{m_{ij}} q_{jl} q_{lj} \qquad (2.28)$$

for all $j, l \in \{1, 2, \dots, d\} \setminus \{i\}$. Thus (χ, E) is Weyl equivalent to any pair (χ', E) such that the structure constants $q'_{jl} := \chi'(\mathbf{e}_j, \mathbf{e}_l)$ of χ' with respect to E satisfy equations (2.28). ∎

Remark. In the theory of simple super Lie algebras it is a basic fact that (in contrast to semi-simple Lie algebras) for different choices of sets of simple roots one can obtain different Dynkin diagrams. Weyl equivalence defined above reflects the aim to consider objects which are independent of the choice of "sets of simple roots". ∎

Proposition 2.6.1. *Let V_1, V_2 be Weyl equivalent Yetter–Drinfel'd modules of diagonal type such that both $\mathbf{\Delta}^+(\mathcal{B}(V_1))$ and $\mathbf{\Delta}^+(\mathcal{B}(V_2))$ are finite. Then $\mathcal{B}(V_1)$ and $\mathcal{B}(V_2)$ have the same dimension and the same Gel'fand–Kirillov dimension.*

Proof. Let E be a basis of $\mathbb{Z}^d$ and let χ_1, χ_2 denote the bicharacters corresponding to V_1 and V_2 via Equation (1.4). By definition of Weyl equivalence it suffices to consider two special cases. First, assume that (χ_1, E) and (χ_2, E) are twist equivalent. Then by [7, Proposition 3.9] the corresponding Nichols algebras are isomorphic as $\mathbb{Z}^d$-graded vector spaces. In particular,

$\mathbf{\Delta}^+(\mathcal{B}(V_1)) = \mathbf{\Delta}^+(\mathcal{B}(V_2))$, and hence the claim holds. Second, if there exists $(T,E) \in W_{\chi_1,E}$ such that $\chi_1(T(\mathbf{e}_i),T(\mathbf{e}_j)) = \chi_2(\mathbf{e}_i,\mathbf{e}_j)$ for all $i,j \in \{1,2,\dots,d\}$ then by Proposition 2.3.1 one has again $\mathbf{\Delta}(\mathcal{B}(V_1)) = \mathbf{\Delta}(\mathcal{B}(V_2))$ (up to a change of basis in $\mathbb{Z}^d$). ■

2.7 Subsystems of Arithmetic Root Systems

In this section intersections of hyperplanes with arithmetic root systems are analyzed.

Fix $d \in \mathbb{N}$ and a basis $E = \{\mathbf{e}_i \,|\, 1 \le i \le d\}$ of $\mathbb{Z}^d$. Let $(\mathbf{\Delta},\chi,E)$ be an arithmetic root system, $H \subset \mathbb{R}^d$ a hyperplane containing 0, and set $\Gamma := \mathbb{R}(\mathbf{\Delta}\cap H)\cap\mathbb{Z}^d$ and $\mathbb{R}_+ := \{r \in \mathbb{R} \,|\, r \ge 0\}$.

Proposition 2.7.1. *There exists a unique basis E_H of the subspace $\mathbb{R}(\mathbf{\Delta}\cap H)$ of $\mathbb{R}^d$ such that the relations $E_H \subset \mathbf{\Delta}^+ \cap H \subset \mathbb{R}_+E_H$ hold. The triple $(\mathbf{\Delta}\cap H, \chi|_{\Gamma\times\Gamma}, E_H)$ is an arithmetic root system.*

Proof. The uniqueness of E_H follows from Corollary 2.5.4.

Choose $\alpha \in \mathbb{R}^d \setminus \{0\}$ such that $\langle\alpha,H\rangle_{\mathbb{R}^d} = 0$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ is the scalar product on $\mathbb{R}^d$ defined by $\langle\mathbf{e}_i,\mathbf{e}_j\rangle_{\mathbb{R}^d} = \delta_{ij}$. Set $\mu := \min\{|\langle\beta,\alpha\rangle_{\mathbb{R}^d}| \,\big|\, \beta \in \mathbf{\Delta}\setminus H\}$. Choose a positive number $r \in \mathbb{R}$ such that

$$|\langle\varepsilon,\beta\rangle_{\mathbb{R}^d}| < \mu/2 \text{ for all } \beta \in \mathbf{\Delta}, \tag{2.29}$$

where $\varepsilon := r\sum_{i=1}^d \mathbf{e}_i$. Then one gets

$$\begin{aligned} |\langle\alpha+\varepsilon,\beta\rangle_{\mathbb{R}^d}| &= |\langle\varepsilon,\beta\rangle_{\mathbb{R}^d}| > 0 \text{ for all } \beta \in \mathbf{\Delta}\cap H,\\ |\langle\alpha+\varepsilon,\beta\rangle_{\mathbb{R}^d}| &\ge |\langle\alpha,\beta\rangle_{\mathbb{R}^d}| - |\langle\varepsilon,\beta\rangle_{\mathbb{R}^d}| > \mu/2 \text{ for all } \beta \in \mathbf{\Delta}\setminus H. \end{aligned} \tag{2.30}$$

In particular, $\langle\alpha+\varepsilon,\beta\rangle_{\mathbb{R}^d} \ne 0$ for all $\beta \in \mathbf{\Delta}$. Thus by Proposition 2.5.2 there exists a basis $F = \{f_1,\dots,f_d\}$ of $\mathbb{Z}^d$ such that $F \subset \mathbf{\Delta}$, $\langle\alpha+\varepsilon,f\rangle_{\mathbb{R}^d} > 0$ for all $f \in F$, and $(\mathbf{\Delta},\chi,F)$ is an arithmetic root system. Without loss of generality assume that for an $l \in \{0,1,\dots,d\}$ one has $\langle\alpha+\varepsilon,f_i\rangle_{\mathbb{R}^d} < \mu/2$ if and only if $i \le l$. By (2.29) and (2.30) one has $f_i \in H$ if and only if $i \le l$, and in this case $f_i \in \mathbf{\Delta}^+$ since $\langle\varepsilon,f_i\rangle_{\mathbb{R}^d} = \langle\alpha+\varepsilon,f_i\rangle_{\mathbb{R}^d} > 0$. Moreover, if $\beta \in \mathbf{\Delta}^+\cap H$ then $0 < \langle\alpha+\varepsilon,\beta\rangle_{\mathbb{R}^d} < \mu/2$ by (2.29), and hence $\beta = \sum_{i=1}^l m_if_i$ with $m_i \in \mathbb{N}_0$ for $1 \le i \le l$. Set $E_H := \{f_1,\dots,f_l\}$. ■

Let $E_H = \{f_1,\dots,f_l\}$ be the basis appearing in Proposition 2.7.1. In the following the symbols $\mathbf{\Delta}(\chi;f_1,\dots,f_l)$ and $\mathbf{\Delta}^+(\chi;f_1,\dots,f_l)$ will be used for

the set of roots and positive roots, respectively, of the arithmetic root system $(\mathbf{\Delta}\cap H, \chi|_{\Gamma\times\Gamma}, E_H)$.

To apply Proposition 2.7.1 to hyperplanes H generated by roots, the following lemma will be useful.

Lemma 2.7.2. *Let $(\mathbf{\Delta},\chi,E)$ be an arithmetic root system of rank d and assume that $\{f_1,\dots,f_l\} \subset \mathbf{\Delta}$, where $l < d$, is a linearly independent set over $\mathbb{R}$. Set $H := \mathbb{R}\{f_1,\dots,f_l\}$. Then $\mathbf{\Delta}\cap H \subset \mathbb{R}_+\{f_1,\dots,f_l\} \cup -\mathbb{R}_+\{f_1,\dots,f_l\}$ if and only if for all $i < l$ and $(m_{i+1},\dots,m_l) \in \mathbb{N}_0^{l-i}\setminus\{0^{l-i}\}$ relations $f_i - \sum_{j=i+1}^l m_jf_j \notin \mathbb{N}\cdot\mathbf{\Delta}$ hold.*

Proof. The only if part of the assertion is obvious. The if part will be proven by induction over l, where for $l = 1$ the lemma clearly holds. Using the induction hypothesis and similar argumentations as in the proof of Proposition 2.7.1 one can assume that $f_i = \mathbf{e}_i$ for all $i \in \{2,\dots,l\}$ and $f_1 = m_1\mathbf{e}_1 + \sum_{j=2}^l m_j\mathbf{e}_j$ for certain $m_j \in \mathbb{N}_0$, $1 \le j \le l$. Since $\{f_1,\dots,f_l\}$ is linearly independent, one has also $m_1 > 0$. Moreover, $f_1 - \sum_{j=2}^l m_jf_j = m_1\mathbf{e}_1 \in \mathbb{N}\cdot\mathbf{\Delta}$, and hence $m_j = 0$ for all $j \in \{2,\dots,l\}$. Thus $f_1 = \mathbf{e}_1$ by Corollary 2.5.4, and the claim follows from the decomposition $\mathbf{\Delta} = \mathbf{\Delta}^+ \cup -\mathbf{\Delta}^+$ (see Lemma 2.2.1). ■

2.8 On the Structure of the Weyl Groupoid

In this section a condition for the finiteness of the Weyl groupoid will be given.

Definition 2.8.1. Let χ be a bicharacter on $\mathbb{Z}^d$ and E a basis of $\mathbb{Z}^d$. Let Q_0 denote the set of all generalized Dynkin diagrams $\mathcal{D}$ such that $\mathcal{D} = \mathcal{D}_{\chi',E}$ for a pair (χ',E) which is Weyl equivalent to (χ,E). For any diagram $\mathcal{D} = \mathcal{D}_{\chi',E} \in Q_0$ and any $\mathbf{e} \in E$ such that $s_{\mathbf{e},E}$ is defined, put an arrow $\rho_{\mathcal{D},\mathbf{e}}$ from $\mathcal{D}$ to the generalized Dynkin diagram $\mathcal{D}'$ of $(\chi', s_{\mathbf{e},E}(E))$, and label this arrow by the vertex of $\mathcal{D}$ corresponding to $\mathbf{e}$. The oriented graph determined in this way will be denoted by $\mathfrak{D}^{\chi,E}$.

Define

$$\begin{aligned} \mathcal{G}_{\chi,E} := \{T \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d) \,|\, &(T,E) \in W^{\mathrm{ext}}_{\chi,E},\\ &\chi(T(e),T(e)) = \chi(e,e) \text{ for all } e \in \mathbb{Z}^d\}. \end{aligned} \tag{2.31}$$

Lemma 2.8.1. *The set $\mathcal{G}_{\chi,E}$ is a subgroup of $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$.*

Proof. For a linear map $T \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^m, \mathbb{Z}^d)$ write $T_{E'',E'}$ for the matrix of T with respect to the bases E' of $\mathbb{Z}^m$ and E'' of $\mathbb{Z}^d$.

Suppose that $T, T' \in \mathcal{G}_{\chi,E}$ and set $E' = T(E)$. Then by definition of $W^{\mathrm{ext}}_{\chi,E}$, and since $\chi(T(e), T(e)) = \chi(e, e)$ for all $e \in \mathbb{Z}^d$, there exists $T'' \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$ such that $(T'', E') \in W^{\mathrm{ext}}_{\chi,E}$, $\chi(T''(e), T''(e)) = \chi(e, e)$ for all $e \in \mathbb{Z}^d$, and $T''_{E',E'} = T'_{E,E}$. Then $(T''T, E) \in W^{\mathrm{ext}}_{\chi,E}$ and $(T''T)_{E,E} = \mathrm{id}_{E,E'} T''_{E',E'} T_{E',E} = T_{E,E} T'_{E,E} = (TT')_{E,E}$, and hence $TT' \in W^{\mathrm{ext}}_{\chi,E}$. Similarly one shows that $T^{-1} \in W^{\mathrm{ext}}_{\chi,E}$. ■

Remark. (i) Let χ be a bicharacter on $\mathbb{Z}^d$. If for all $i, j \in \{1, 2, \ldots, d\}$ there exist numbers $n_{ij} \in \mathbb{N}$ such that $(q_{ij}q_{ji})^{n_{ij}} = 1$ then the graph $\mathfrak{D}^{\chi,E}$ is finite. Indeed, then there exists $m \in \mathbb{N}$ such that $\chi(e, e)^m = 1$ for all $e \in \mathbb{Z}^d$, and hence in Example 2.6.1 one has $p_{ij}^m = 1$ for all i, j with $i \neq j$. Therefore $\chi'(e, e)^m = 1$ for all $e \in \mathbb{Z}^d$, and hence for all pairs (χ', E) which are Weyl equivalent to (χ, E) equations $\chi'(e, e)^m = 1$ are satisfied for all $e \in \mathbb{Z}^d$. Now use the fact that there are only finitely many numbers ζ such that $\zeta^m = 1$.
(ii) If $d = 2$ then one can show that $\mathfrak{D}^{\chi,E}$ is finite. For bicharacters on $\mathbb{Z}^d$, where $d \geq 3$, the finiteness of $\mathfrak{D}^{\chi,E}$ is not known, even under the restriction that $W_{\chi,E}$ is full.
(iii) If (χ, E) and (χ', E) are Weyl equivalent then the groups $\mathcal{G}_{\chi,E}$ and $\mathcal{G}_{\chi',E}$ are isomorphic. Indeed, similarly to the proof of Lemma 2.8.1 one can show that if $(T, E) \in W^{\mathrm{ext}}_{\chi,E}$ then $S \in \mathcal{G}_{\chi,E}$ if and only if $S \in \mathcal{G}_{\chi,T(E)}$. ■

Choose a well-ordered set J with smallest element 0, and a subset $W_0 = \{(T_j, E) \mid j \in J\}$ of $W^{\mathrm{ext}}_{\chi,E}$ containing $(T_0 := \mathrm{id}, E)$ with the following properties.

1. For each $j \in J \setminus \{0\}$ there exist $i \in J$, $\mathbf{e}' \in T_i(E)$, and $\tau \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$, such that $i < j$, $\tau T_i(E) = T_i(E)$, and $T_j = s_{\mathbf{e}',T_i(E)} \tau T_i$.
2. If (χ', E) is Weyl equivalent to (χ, E) then there exists $j \in J$ such that $(\chi \circ (T_j \times T_j), E)$ is twist equivalent to (χ', E).
3. The pairs $(\chi \circ (T_i \times T_i), E)$ and $(\chi \circ (T_j \times T_j), E)$ are twist equivalent if and only if $i = j$.

Obviously such a set W_0 exists and its cardinality coincides with the cardinality of the set of vertices of $\mathfrak{D}^{\chi,E}$. The assertions of the next two lemmata follow from the definitions of $W^{\mathrm{ext}}_{\chi,E}$, $\mathfrak{D}^{\chi,E}$ and $\mathcal{G}_{\chi,E}$ using arguments as in the proof of Lemma 2.8.1.

Lemma 2.8.2. *The group $\mathcal{G}_{\chi,E}$ is generated by the elements $\tilde{T}_i^{-1} A T_j$, where $i, j \in J$, $E_j = T_j(E)$, $A \in \{\tau, s_{\mathbf{e},E_j}\tau \mid \tau \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d), \tau(E_j) = E_j, \mathbf{e} \in E_j\}$, $\chi(T_i(e), T_i(e)) = \chi(AT_j(e), AT_j(e))$ for all $e \in \mathbb{Z}^d$, and $(\tilde{T}_i^{-1}, A(E_j))$ is the unique element in $W^{\mathrm{ext}}_{\chi,E}$ satisfying the relation $(\tilde{T}_i^{-1})_{AT_j(E),AT_j(E)} = (T_i^{-1})_{T_i(E),T_i(E)}$.*

Lemma 2.8.3. *Let χ be a bicharacter on $\mathbb{Z}^d$ and E a basis of $\mathbb{Z}^d$. Then $W_{\chi,E}$ is finite if and only if the graph $\mathfrak{D}^{\chi,E}$ and the group $\mathcal{G}_{\chi,E}$ are finite.*

2.9 Connectedness

Let E be a basis of $\mathbb{Z}^d$ and χ a bicharacter on $\mathbb{Z}^d$.

Definition 2.9.1. The pair (χ, E) (or a braided vector space (V, σ) of diagonal type or a Nichols algebra $\mathcal{B}(V)$ of diagonal type) is called *connected* if for all $i, j \in \{1, \ldots, d\}$ there exists $m \in \mathbb{N}_0$ and a sequence $\{i = i_0, i_1, \ldots, i_{m-1}, i_m = j\}$ of elements of $\{1, \ldots, d\}$ such that the structure constants q_{ij} of χ with respect to E (or of V, respectively) satisfy the conditions $q_{i_l,i_{l+1}} q_{i_{l+1},i_l} \neq 1$ for all $l \in \{0, 1, \ldots, m-1\}$. An arithmetic root system $(\boldsymbol{\Delta}, \chi, E)$ is called *connected* if χ is connected. Note that a pair (χ, E) is connected if and only if its generalized Dynkin diagram $\mathcal{D}_{\chi,E}$ is a connected graph.

Obviously, the set $\{1, \ldots, d\}$ can be decomposed into the disjoint union of maximal subsets $I_\alpha \subset \{1, \ldots, d\}$ such that $(\chi|_{\mathbb{Z}I_\alpha \times \mathbb{Z}I_\alpha}, I_\alpha)$ is connected for all α. This induces a decomposition $\boldsymbol{\Delta} = \bigcup_\alpha \boldsymbol{\Delta}_\alpha$ into disjoint sets $\boldsymbol{\Delta}_\alpha$. Analogously, see [6, Lemma 4.2], a Nichols algebra of diagonal type can be written as the smash product of its connected components. Therefore only connected arithmetic root systems respectively Nichols algebras of diagonal type will be considered. Note that both twist equivalence and Weyl equivalence respect decompositions into connected components.

2.10 Nichols Algebras of Cartan Type

One says that a pair (χ, E) respectively an arithmetic root system $(\boldsymbol{\Delta}, \chi, E)$ is of *Cartan type* (see [2, Definition 5.18]) if for all $\mathbf{e}', \mathbf{e}'' \in E$ there exist integers $a_{\mathbf{e}',\mathbf{e}''}$ such that

$$\chi(\mathbf{e}',\mathbf{e}')^{a_{\mathbf{e}',\mathbf{e}''}} = \chi\chi^{\mathrm{op}}(\mathbf{e}',\mathbf{e}''). \tag{2.32}$$

In this case it will be assumed that
(i) $a_{\mathbf{e},\mathbf{e}} = 2$ for all $\mathbf{e} \in E$,
(ii) $a_{\mathbf{e}',\mathbf{e}''} = 0$ if $\chi(\mathbf{e}',\mathbf{e}') = 1$, and
(iii) $-\mathrm{ord}\,\chi(\mathbf{e}',\mathbf{e}') < a_{\mathbf{e}',\mathbf{e}''} \le 0$ whenever $\chi(\mathbf{e}',\mathbf{e}')$ is a root of unity different from 1.
The pair (χ, E) is called of *finite Cartan type* if the matrix $C = (a_{\mathbf{e}',\mathbf{e}''})_{\mathbf{e}',\mathbf{e}''\in E}$ is a Cartan matrix of finite type.

Lemma 2.10.1. *Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ be a basis of $\mathbb{Z}^d$ and χ a bicharacter on $\mathbb{Z}^d$. Assume that (χ, E) is of Cartan type and let $(a_{ij} := a_{\mathbf{e}_i,\mathbf{e}_j})_{i,j\in\{1,2,\dots,d\}}$ denote the corresponding Cartan matrix.*
(i) Suppose that $i \in \{1, 2, \dots, d\}$ and $m \in \mathbb{N}_0$ such that $m < -a_{ij}$ for at least one $j \in \{1, 2, \dots, d\}$. Then $q_{ii}^{m+1} \ne 1$.
(ii) For the numbers in Equation (2.11) one obtains $m_{ij} = -a_{ij}$ for all $i, j \in \{1, 2, \dots, d\}$.

Proof. To (i). Since V is of Cartan type, one has $q_{ii}^{a_{ij}} = q_{ij}q_{ji}$. Assume that $m + 1 \le -a_{ij}$ and $q_{ii}^{m+1} = 1$. One obtains that $q_{ii}^{m+1+a_{ij}} = q_{ij}q_{ji}$, and $m + 1 + a_{ij} \le 0$. This is a contradiction to the choice of a_{ij}.
To (ii). This follows from the definition of m_{ij} and from (i). ∎

There exist classification results of Rosso [33, Theorem 21] and Andruskiewitsch and Schneider [6, Theorem 1.1] on Nichols algebras of Cartan type with finite Gel'fand–Kirillov dimension (F3) and finite dimension (F1), respectively. The introduction of the Weyl groupoid $W_{\chi,E}$ in Section 2.4 allows to state a theorem without technical assumptions on the numbers q_{ij}.

Theorem 2.10.2. *Let $E = \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ be a basis of $\mathbb{Z}^d$ and χ a bicharacter on $\mathbb{Z}^d$. Let (V, σ) be the braided vector space associated to (χ, E) (see Section 1.2). Assume that (χ, E) is of Cartan type with Cartan matrix $C := (a_{ij})_{i,j\in\{1,2,\dots,d\}}$.*
(i) If C is not of finite type then $\boldsymbol{\Delta}(\mathcal{B}(V))$ is infinite.
(ii) If C is of finite type then $\boldsymbol{\Delta}(\mathcal{B}(V))$ can be identified with the set of roots of the semi-simple Lie algebra corresponding to C. Moreover, in any connected component the heights of the (restricted) Poincaré–Birkhoff–Witt generators depend only on the lengths of the roots corresponding to them.

Proof. To (i). Suppose that $\boldsymbol{\Delta}(\mathcal{B}(V))$ is finite. Then by Theorem 2.5.3 the Weyl groupoid $W_{\chi,E}$ is full and finite. Moreover, by Equation (2.28) one has $\chi(T(e), T(e)) = \chi(e, e)$ for all $e \in \mathbb{Z}^d$ and all $T \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$ such that $(T, E) \in W_{\chi,E}$. This means that the groups $\mathcal{G}_F := \{T \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d) \,|\, (T, F) \in W_{\chi,E}\}$ (see also Lemma 2.8.1) do not depend on the basis F of $\mathbb{Z}^d$. Moreover, the map $\mathcal{G}_E \times \mathcal{G}_E \to W_{\chi,E}$, $(T, T') \mapsto (T, T'(E))$, is a bijection. If C is symmetrizable then $\mathcal{G}$ is canonically isomorphic to the Weyl group W associated to C. It is well-known [24, Ch. 1, Theorem 4.8] that W is finite if and only if the symmetrizable Cartan matrix is of finite type. Thus for (i) it suffices to show that the groupoid $W_{\chi,E}$ is infinite whenever C is *not* symmetrizable. Note that if $W_{\chi',E'}$ is infinite, where $E' \subset E$ and $\chi' = \chi|_{\mathbb{Z}E'\times\mathbb{Z}E'}$, then $W_{\chi,E}$ is itself infinite. Further, if the Dynkin diagram associated to a Cartan matrix is simply-laced or has no cycles then it is symmetrizable. Thus we only have to show that $W_{\chi,E}$ is infinite if the corresponding Dynkin diagram is a cycle which is not simply-laced. Suppose now that $d \ge 4$, and $a_{ij} = 0$ if and only if $|i - j| > 1$ and $\{i, j\} \ne \{1, d\}$. Without loss of generality also $a_{21} = -1$ can be assumed. Then the set $E' := \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_d\} \subset \boldsymbol{\Delta}(\mathcal{B}(V))$ satisfies the equivalent conditions in Lemma 2.7.2 and hence Proposition 2.7.1 can be applied. By the equations

$$\begin{aligned}\chi(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2) &= \chi(\mathbf{e}_1, \mathbf{e}_1),\\ \chi(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3)\chi(\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2) &= \chi(\mathbf{e}_2, \mathbf{e}_3)\chi(\mathbf{e}_3, \mathbf{e}_2)\\ &= \chi(\mathbf{e}_2, \mathbf{e}_2)^{a_{23}} = \chi(\mathbf{e}_1, \mathbf{e}_1)^{-a_{12}a_{23}} \ne 1,\end{aligned}$$

the pair (χ', E') is again of Cartan type, and the corresponding Dynkin diagram is a cycle with $d - 1$ nodes. Therefore it remains to consider not symmetrizable cycles with 3 nodes.

The matrices t_i of $s_{\mathbf{e}_i,E}$, $i \in \{1, 2, 3\}$, with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ take the form

$$t_1 = \begin{pmatrix} -1 & -a_{12} & -a_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & 0 & 0\\ -a_{21} & -1 & -a_{23}\\ 0 & 0 & 1\end{pmatrix}, \quad t_3 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -a_{31} & -a_{32} & -1\end{pmatrix}.$$

Without loss of generality one can suppose that $a_{12} < -1$. Since we assumed (F2), by Theorem 2.5.3 the matrix

$$t_1t_2t_3 = \begin{pmatrix} a_{12}a_{21}+a_{13}a_{31}-1-a_{12}a_{23}a_{31} & a_{12}+a_{13}a_{32}-a_{12}a_{23}a_{32} & a_{13}-a_{12}a_{23}\\ a_{23}a_{31} - a_{21} & a_{23}a_{32} - 1 & a_{23}\\ -a_{31} & -a_{32} & -1\end{pmatrix}.$$

has to have finite order with respect to multiplication. This means in particular that all eigenvalues of the (invertible) matrix $t_1t_2t_3$ have to have absolute value 1, and hence its trace is not bigger than 3. Further, if the trace is 3 then the matrix has to be the identity. Since $a_{12} \leq -2$, for the trace of $t_1t_2t_3$ one obtains the relation

$$\mathrm{tr}(t_1t_2t_3) = a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32} - a_{12}a_{23}a_{31} - 3 \geq 2{+}1{+}1{+}2{-}3 = 3.$$

As $t_1t_2t_3$ is obviously not the identity this yields that $s_1s_2s_3$ does not have finite order and hence $W_{\chi,E}$ is infinite.

To (ii). Consider E as the set of simple roots of the root system Δ_C associated to C and the maps $s_i = s_{\mathbf{e}_i,E}$ as simple reflections of the set Δ_C with respect to the simple roots $\mathbf{e}_i$. If (χ, E) is of finite type then the Cartan matrix $C = (a_{ij})_{i,j\in\{1,2,\dots,d\}}$ is symmetrizable and hence there is a bijection between $W_{\chi,E}$ and $W \times W$ (see part (i) of the proof), where W is the (finite) Weyl group associated to C. Thus the assertion follows from Theorem 2.5.3 and the relations $\Delta_C = \bigcup_{i=1}^d W(\mathbf{e}_i)$ and $\boldsymbol{\Delta}(\mathcal{B}(V)) = \bigcup\{F \subset \mathbb{Z}^d \,|\, (\mathrm{id}, F) \in W_{\chi,E}\}$. ∎

Chapter 3

Classification of Arithmetic Root Systems

Our long term aim is to determine all finite dimensional Nichols algebras of diagonal type over an arbitrary base field of characteristic 0. By Theorem 2.5.3 this is in principle equivalent to the classification of arithmetic root systems $(\mathbf{\Delta}, \chi, E)$ where χ takes values in k^*. The rank 1 case is trivial, so we start our considerations with the rank 2 and rank 3 cases.

3.1 Arithmetic Root Systems of Rank 2

In Section 2.6 Weyl equivalence of rank 2 braided vector spaces of diagonal type was introduced. The main result of this section is that Table A.1 contains the generalized Dynkin diagrams of all rank 2 braided vector spaces V of diagonal type, such that $\mathbf{\Delta}(\mathcal{B}(V))$ is finite.

Theorem 3.1.1. *Let k be a field of characteristic 0. Then twist equivalence classes of arithmetic root systems of rank 2 are in one-to-one correspondence to generalized Dynkin diagrams appearing in Table A.1. Moreover, two such arithmetic root systems are Weyl equivalent if and only if their generalized Dynkin diagrams appear in the same row of Table A.1 and can be presented with the same set of fixed parameters.*

For the classification of arithmetic root systems of rank 2 several technical lemmata will be needed. The first one is a well-known classical result, which can be easily proved with help of eigenvalue considerations.

Lemma 3.1.2. *Let $A \in \mathrm{SL}(2,\mathbb{Z})$. Then $\mathrm{ord}\, A$ is finite if and only if either $A = \mathrm{id}$ or $A = -\mathrm{id}$ or $\det A = 1$, $\mathrm{tr} A \in \{-1, 0, 1\}$.*

Lemma 3.1.3. *For a given $M \in \mathbb{N}$ let S be a subsemigroup of $\mathrm{SL}(2,\mathbb{Z})$ generated by matrices of the form $\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$ with $0 < d < Mb < a$. Then all elements of S are of this form. In particular, $\mathrm{id} \notin S$.*

Proof. Assume that $A_1 := \begin{pmatrix} a_1 & -b_1 \\ c_1 & -d_1 \end{pmatrix} \in S$ and $A_2 := \begin{pmatrix} a_2 & -b_2 \\ c_2 & -d_2 \end{pmatrix} \in S$ such that $0 < d_1 < Mb_1 < a_1$ and $0 < d_2 < Mb_2 < a_2$. Since $\det A_1 = \det A_2 = 1$, one also has $c_1 = (a_1d_1 + 1)/b_1$ and $c_2 = (a_2d_2 + 1)/b_2$. Then $A_1A_2 = \begin{pmatrix} a_1a_2 - b_1c_2 & -(a_1b_2 - b_1d_2) \\ c_1a_2 - d_1c_2 & -(c_1b_2 - d_1d_2) \end{pmatrix}$. We obtain the following inequalities.

$$a_1b_2 - b_1d_2 = b_2(a_1 - Mb_1) + b_1(Mb_2 - d_2) > 0, \tag{3.1}$$

$$c_1b_2 - d_1d_2 = \frac{a_1d_1+1}{b_1}b_2 - d_1d_2 = \frac{d_1(a_1b_2 - b_1d_2) + b_2}{b_1} > 0 \text{ by (3.1)}, \tag{3.2}$$

$$\begin{aligned} M(a_1b_2 - b_1d_2) - (c_1b_2 - d_1d_2) &= (Mb_1 - d_1)(Mb_2 - d_2) \\ &+ b_2\left(Ma_1 - M^2b_1 + Md_1 - \frac{a_1d_1+1}{b_1}\right) \\ &= (Mb_1 - d_1)(Mb_2 - d_2) + \frac{b_2}{b_1}((Mb_1 - d_1)(a_1 - Mb_1) - 1) > 0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} a_1a_2 - b_1c_2 - M(a_1b_2 - b_1d_2) &= (a_1 - Mb_1)(a_2 - Mb_2) \\ &+ b_1\left(Ma_2 - M^2b_2 - \frac{a_2d_2+1}{b_2} + Md_2\right) \\ &= (a_1 - Mb_1)(a_2 - Mb_2) + \frac{b_1}{b_2}((Mb_2 - d_2)(a_2 - Mb_2) - 1) > 0. \end{aligned} \tag{3.4}$$

The inequalities (3.2)–(3.4) give the assertion. ■

For any basis $E = \{e_1, e_2\}$ of $\mathbb{Z}^2$ let $\tau_E \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^2)$ denote the linear map defined by $\tau_E(e_1) = e_2$, $\tau_E(e_2) = e_1$. A useful consequence of Definition 2.5.1 is the following.

Lemma 3.1.4. *Let $E = \{\mathbf{e}_1, \mathbf{e}_2\}$ be a basis of $\mathbb{Z}^2$ and χ a bicharacter on $\mathbb{Z}^2$. For $i \in \mathbb{N}_0$ and $j \in \{1, 2\}$ set inductively $E_0 := F_0 := E$, $\mathbf{e}_j^{(0)} := \mathbf{e}_j$, $\mathbf{f}_j^{(0)} := \mathbf{e}_{3-j}$, $P_i := s_{\mathbf{e}_1^{(i)}, E_i}\tau_{E_i}$, $Q_i := s_{\mathbf{f}_1^{(i)}, F_i}\tau_{F_i}$, $E_{i+1} := P_i(E_i)$, $F_{i+1} := Q_i(F_i)$, $\mathbf{e}_j^{(i+1)} := P_i(\mathbf{e}_j^{(i)})$, and $\mathbf{f}_j^{(i+1)} := Q_i(\mathbf{f}_j^{(i)})$. The triple $(\mathbf{\Delta}, \chi, E)$, where $\mathbf{\Delta} = \bigcup\{F \subset \mathbb{Z}^2 \mid (\mathrm{id}, F) \in W_{\chi,E}\}$, is an arithmetic root system if and only if the sequences $(P_iP_{i-1}\cdots P_0)_{i\in\mathbb{N}_0}$ and $(Q_iQ_{i-1}\cdots Q_0)_{i\in\mathbb{N}_0}$ of elements of $\mathrm{SL}(2,\mathbb{Z})$ are well-defined and periodic, or, equivalently, well-defined and contain the identity map.*

Proof. The if part is clear, since the equations $s_{\mathbf{e}_i, s_{\mathbf{e}_i,F}(F)}s_{\mathbf{e}_i,F} = \mathrm{id}$ hold for all F with $(\mathrm{id}, F) \in W_{\chi,E}$. On the other hand, if $(\mathbf{\Delta}, \chi, E)$ is an arithmetic root system then $W_{\chi,E}$ is full and hence the given sequences are well defined. Moreover, since $W_{\chi,E}$ is finite, the sets $\{\mathbf{e}_1^{(i)} \mid i \in \mathbb{N}_0\} \subset \mathbf{\Delta}$ and $\{E_i \mid i \in \mathbb{N}_0\}$ are finite. Since P_i is invertible and it depends only on $\mathbf{e}_1^{(i)}$ and E_i, one obtains that the sequence $(P_i)_{i\in\mathbb{N}_0}$ is periodic. Choose $j \in \mathbb{N}$ such that $P_i = P_{i+j}$ for all $i \in \mathbb{N}_0$. Then $P_{ij-1}P_{ij-2}\cdots P_0 = (P_{j-1}\cdots P_0)^i$ for all $i \in \mathbb{N}$. The finiteness of $W_{\chi,E}$ implies that also the set $\{(P_{j-1}\cdots P_0)^i \mid i \in \mathbb{N}_0\}$ is finite, and hence there exist $i_1 < i_2 \in \mathbb{N}_0$ such that $(P_{j-1}\cdots P_0)^{i_1} = (P_{j-1}\cdots P_0)^{i_2}$, that is $(P_{j-1}\cdots P_0)^{i_2-i_1} = \mathrm{id}$. The proof for the second sequence is similar. ■

Lemma 3.1.5. *Let V be a braided vector space of diagonal type with $\dim_k V = 2$, and let $(q_{ij})_{i,j\in\{1,2\}}$ be the matrix of its structure constants with respect to a basis $\{x_1, x_2\}$. If $\mathbf{\Delta}(\mathcal{B}(V))$ is finite then either*

$$(q_{12}q_{21} - 1)(q_{11}q_{12}q_{21} - 1)(q_{12}q_{21}q_{22} - 1)(q_{11} + 1)(q_{22} + 1) = 0 \qquad \textit{or}$$
$$q_{11}q_{12}^2q_{21}^2q_{22} = -1,\ (q_{11}^2 + q_{11} + 1)(q_{11}^2 + 1)(q_{11}^2q_{12}q_{21} - 1)(q_{11}^3q_{12}q_{21} - 1) = 0.$$

Proof. Let $E = \{\mathbf{e}_1, \mathbf{e}_2\}$ be a basis of $\mathbb{Z}^2$ and χ a bicharacter on $\mathbb{Z}^2$ such that V is associated to (χ, E) via $\deg x_i = \mathbf{e}_i$ for $i \in \{1, 2\}$. By Theorem 2.5.3 the triple $(\mathbf{\Delta}(\mathcal{B}(V)), \chi, E)$ is an arithmetic root system.

Set $z_1 := x_1x_2 - q_{12}x_2x_1$, $z_2 := x_1z_1 - q_{11}q_{12}z_1x_1$, and $z_3 := x_1z_2 - q_{11}^2q_{12}z_2x_1$. Note that one has $[\alpha_1] = x_2$, $[\alpha_2] = x_1$, $[\alpha_1\alpha_2] = z_1$, $[\alpha_1\alpha_2\alpha_2] = z_2$, and $[\alpha_1\alpha_2\alpha_2\alpha_2] = z_3$ by the definition above Lemma 1.4.3. Assume that $(q_{12}q_{21} - 1)(q_{11}q_{12}q_{21} - 1)(q_{12}q_{21}q_{22} - 1)(q_{11} + 1)(q_{22} + 1) \neq 0$. Then $\mathbf{e}_1 + \mathbf{e}_2, 2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + 2\mathbf{e}_2 \in \mathbf{\Delta}(\mathcal{B}(V))$ by Lemma 2.2.1, and the corresponding Lyndon words are $\alpha_1\alpha_2$, $\alpha_1\alpha_2\alpha_2$ and $\alpha_1\alpha_1\alpha_2$, respectively.

By Corollary 2.5.5 the finite set $\mathbf{\Delta}^+(\mathcal{B}(V))$ does not contain $2(\mathbf{e}_1 + \mathbf{e}_2)$. Therefore Theorem 1.4.6 implies that for the Lyndon word $u = \alpha_1\alpha_1\alpha_2\alpha_2$ one has $h_u = 1$. Thus Proposition 1.4.8 gives that $[\alpha_1\alpha_1\alpha_2\alpha_2] = z_2x_2 - \chi(\deg z_2, \deg x_2)x_2z_2 \in k\{z_2x_2, x_1[\alpha_1\alpha_1\alpha_2], z_1^2, x_1z_1x_2, x_1^2x_2^2\}$. Applying $\langle y_1^{\mathrm{L}}, \cdot\rangle$ and using the equations

$$\langle y_1^{\mathrm{L}}, z_1\rangle = 0, \qquad \langle y_1^{\mathrm{L}}, z_2\rangle = 0, \qquad \langle y_1^{\mathrm{L}}, [\alpha_1\alpha_1\alpha_2]\rangle = 0$$

and Theorem 1.4.6 one obtains that $[\alpha_1\alpha_1\alpha_2\alpha_2] \in k\{z_2x_2, z_1^2\}$. Equations

$$\langle y_2^{\mathrm{L}}, z_1\rangle = (q_{21}^{-1} - q_{12})x_1, \qquad \langle y_2^{\mathrm{L}}, z_2\rangle = (q_{21}^{-1} - q_{12})(q_{21}^{-1} - q_{11}q_{12})x_1^2$$

give for $a, b \in k$ the formula

$$\begin{aligned} &\langle y_2^{\mathrm{L}}, z_2x_2 - \chi(z_2, x_2)x_2z_2 - az_2x_2 - bz_1^2\rangle = (q_{21}^{-2}q_{22}^{-1} - q_{12}^2q_{22})z_2 \\ &\quad + (q_{21}^{-1} - q_{12})(q_{21}^{-1} - q_{11}q_{12})((1 - a)x_1^2x_2 - q_{12}^2x_2x_1^2) \end{aligned}$$

$$
\begin{aligned}
&- (q_{21}^{-1} - q_{12})b(x_1z_1 + q_{21}^{-1}q_{22}^{-1}z_1x_1)\\
&= -a(q_{21}^{-1} - q_{12})(q_{21}^{-1} - q_{11}q_{12})x_1^2x_2 \qquad (*)\\
&+ (q_{21}^{-1} - q_{12})\big((q_{21}^{-1} - q_{11}q_{12})(1 + q_{11}^{-1}) - b(1 + q_{11}^{-1}q_{12}^{-1}q_{21}^{-1}q_{22}^{-1})\big)x_1z_1\\
&+ \big(q_{21}^{-2}q_{22}^{-1} - q_{12}^2q_{22} + (q_{21}^{-1} - q_{12})(q_{12} - q_{11}^{-1}q_{21}^{-1} + q_{11}^{-1}q_{12}^{-1}q_{21}^{-1}q_{22}^{-1}b)\big)z_2,
\end{aligned}
$$

and hence $z_2x_2 - \chi(z_2, x_2)x_2z_2 - az_2x_2 - bz_1^2 = 0$ implies that $a = 0$ and $(q_{21}^{-1} - q_{11}q_{12})(1 + q_{11}^{-1}) = b(1 + q_{11}^{-1}q_{12}^{-1}q_{21}^{-1}q_{22}^{-1})$. Since $q_{11} \neq -1$ and $q_{22} \neq -1$, the latter equation gives that $q_{11}q_{12}q_{21}q_{22} \neq -1$. Therefore the vanishing of the coefficient of z_2 in equation $(*)$ implies that $q_{11}q_{12}^2q_{21}^2q_{22} = -1$.

If $z_3 = 0$ or $z_4 = 0$ then the proof is complete by Lemma 1.3.3. Otherwise, since $2(2\mathbf{e}_1+\mathbf{e}_2) \notin \mathbf{\Delta}^+(\mathcal{B}(V))$, the skew-commutator $z_3z_1-\chi(z_3, z_1)z_1z_3$ must not be a Poincaré–Birkhoff–Witt generator of $\mathcal{B}(V)$. Using again Proposition 1.4.8, Theorem 1.4.6 and the skew-differential $\langle y_1^{\mathrm{L}}, \cdot\rangle$ one obtains that $z_3z_1 - \chi(z_3, z_1)z_1z_3 \in k\{z_4x_2, z_3z_1, z_2^2\}$, where $z_4 = [\alpha_1\alpha_2\alpha_2\alpha_2\alpha_2] = x_1z_3 - q_{11}^3q_{12}z_3x_1$. Then relation $\langle y_1^{\mathrm{L}}y_2^{\mathrm{L}}, z_3z_1 - \chi(z_3, z_1)z_1z_3 - az_4x_2 - bz_3z_1 - cz_2^2\rangle = 0$ gives after a long computation the desired contradiction. ■

Let E be a basis of $\mathbb{Z}^2$ and χ a bicharacter on $\mathbb{Z}^2$. Assign to (χ, E) two sequences $(m_i)_{i\in\mathbb{N}_0}$ and $(n_i)_{i\in\mathbb{N}_0}$, such that $m_i = m(\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)})$ and $n_i = m(\mathbf{f}_1^{(i)}, \mathbf{f}_2^{(i)})$ using the notation in (2.23) and in Lemma 3.1.4. Further, for any $l \in \mathbb{N}_0$ set $q_{ij}^{(l)} := \chi(\mathbf{e}_i^{(l)}, \mathbf{e}_j^{(l)})$ and $r_{ij}^{(l)} := \chi(\mathbf{f}_i^{(l)}, \mathbf{f}_j^{(l)})$, and define

$$
p_l := \begin{cases} 1 & \text{if } (q_{11}^{(l)})^{m_l}q_{12}^{(l)}q_{21}^{(l)} = 1,\\ (q_{11}^{(l)})^{-1}q_{12}^{(l)}q_{21}^{(l)} & \text{otherwise,} \end{cases} \tag{3.5}
$$

$$
s_l := \begin{cases} 1 & \text{if } (r_{11}^{(l)})^{n_l}r_{12}^{(l)}r_{21}^{(l)} = 1,\\ (r_{11}^{(l)})^{-1}r_{12}^{(l)}r_{21}^{(l)} & \text{otherwise.} \end{cases} \tag{3.6}
$$

By Example 2.6.1 one obtains for all $l \in \mathbb{N}_0$ the relations

$$
q_{11}^{(l+1)} = p_l^{m_l}q_{22}^{(l)}, \qquad q_{12}^{(l+1)}q_{21}^{(l+1)} = p_l^{-2}q_{12}^{(l)}q_{21}^{(l)}, \qquad q_{22}^{(l+1)} = q_{11}^{(l)}, \tag{3.7}
$$

$$
r_{11}^{(l+1)} = s_l^{m_l}r_{22}^{(l)}, \qquad r_{12}^{(l+1)}r_{21}^{(l+1)} = s_l^{-2}r_{12}^{(l)}r_{21}^{(l)}, \qquad r_{22}^{(l+1)} = r_{11}^{(l)}. \tag{3.8}
$$

Proof of Theorem 3.1.1. Using Example 2.6.1 the second assertion of the theorem is easily checked. On the other hand, either from Theorem 1.6.1 or from Lemma 3.1.4 one can conclude that if $\mathcal{D}_{\chi,E}$ appears in Table A.1 then $(\mathbf{\Delta}, \chi, E)$ is an arithmetic root system, where $\mathbf{\Delta} = \bigcup\{F \subset \mathbb{Z}^2 \,|\, (\mathrm{id}, F) \in W_{\chi,E}\}$. Note that Theorem 1.6.1 implies this result immediately, but its

proof is much more complicated than the elementary calculations needed for Lemma 3.1.4.

It remains to show that for any rank 2 arithmetic root system $(\mathbf{\Delta}, \chi, E)$ the generalized Dynkin diagram $\mathcal{D}_{\chi,F}$ appears in Table A.1 for at least one F with $(\mathrm{id}, F) \in W_{\chi,E}$. We use Lemma 3.1.4 to obtain restrictions on the structure constants q_{ij}.

For all $i \in \mathbb{N}_0$ let $\tilde{P}_i$ and $\tilde{Q}_i$ denote the matrix of P_i and Q_i, respectively, with respect to the ordered basis $E_i = \{\mathbf{e}_1^{(i)}, \mathbf{e}_2^{(i)}\}$ and $F_i = \{\mathbf{f}_1^{(i)}, \mathbf{f}_2^{(i)}\}$, respectively. Then the matrix of $P_iP_{i-1}\cdots P_0$ with respect to the ordered basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ is $\tilde{P}_0\tilde{P}_1\cdots\tilde{P}_i$, and the matrix of $Q_iQ_{i-1}\cdots Q_0$ with respect to the ordered basis $\{\mathbf{e}_2, \mathbf{e}_1\}$ is $\tilde{Q}_0\tilde{Q}_1\cdots\tilde{Q}_i$.

Using Weyl equivalence, one can assume that m_0 is minimal among all m_i and n_i, where $i \in \mathbb{N}_0$. If $m_0 = 0$ then $q_{12}q_{21} = 1$ by (2.23), and then $\mathcal{D}_{\chi,E}$ appears in row 1 of Table A.1. Otherwise $m_i, n_i > 0$ for all $i \in \mathbb{N}_0$, and hence $q_{jj}^{(i)}, r_{jj}^{(i)} \neq 1$ for all $j \in \{1, 2\}$ and $i \in \mathbb{N}_0$. Now we have to consider several cases.

Step 1. *Assume that* $m_0 > 1$. Then $m_i > 1$ for all $i \in \mathbb{N}_0$. By Lemma 3.1.4 one of the maps $P_{2i+1}P_{2i}\cdots P_0$, where $i \in \mathbb{N}_0$, has to be the identity. For all $j \in \mathbb{N}_0$ one has

$$
\tilde{P}_j = \begin{pmatrix} -1 & m_j\\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1\\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m_j & -1\\ 1 & 0 \end{pmatrix}, \tag{3.9}
$$

and hence $\tilde{P}_{2i}\tilde{P}_{2i+1} = \begin{pmatrix} m_{2i}m_{2i+1} - 1 & -m_{2i}\\ m_{2i+1} & -1 \end{pmatrix}$ for all $i \in \mathbb{N}_0$. Since $m_j \geq 2$ for all $j \in \mathbb{N}_0$, these matrices satisfy the conditions of Lemma 3.1.3 with $M = 1$. Then Lemma 3.1.3 gives that $P_{2i+1}P_{2i}\cdots P_0$ is never the identity, which is a contradiction.

Step 2. *If* $m_0 = 1$ *and* $n_0 = 1$ then one has either $q_{11}q_{12}q_{21} = q_{12}q_{21}q_{22} = 1$ or $q_{11}q_{12}q_{21} = 1$, $q_{22} = -1$, or $q_{11} = -1$, $q_{12}q_{21}q_{22} = 1$ (which is twist equivalent to the previous case), or $q_{11} = q_{22} = -1$. In these cases $\mathcal{D}_{\chi,E}$ appears in rows 2–3 of Table A.1.

Step 3. *Suppose that* $m_{2i} = n_{2i+1} = 1$ *and* $m_{2i+1} > 1$, $n_{2i} > 1$ *for all* $i \in \mathbb{N}_0$. Again by Weyl equivalence one can assume that m_1 is minimal among all m_{2i+1} and n_{2i} for all $i \in \mathbb{N}_0$. Recall that one has $\tilde{P}_{2i}\tilde{P}_{2i+1} = \begin{pmatrix} m_{2i+1} - 1 & -1\\ m_{2i+1} & -1 \end{pmatrix}$, and hence if $m_1 > 3$ (that is $m_{2i+1} \geq 4$ for all $i \in \mathbb{N}_0$) then these matrices satisfy the conditions of Lemma 3.1.3 with $M = 2$. This gives again a contradiction. Thus one has $m_1 \in \{2, 3\}$, and since $m_0 = 1$, either $q_{11}q_{12}q_{21} = 1$ (see Step 3a) or $q_{11} = -1$, $q_{12}^2 q_{21}^2 \neq 1$ (Step 3b) holds by (2.23).

Step 3a. *Assume that additionally* $q_{11}q_{12}q_{21} = 1$ *holds.* Then (3.7) gives that $p_0 = 1$, $q_{11}^{(1)} = q_{22}$, $q_{12}^{(1)} q_{21}^{(1)} = q_{11}^{-1}$, $q_{22}^{(1)} = q_{11}$. If $m_1 = 2$ then (2.23) gives that either equation $q_{11} = q_{22}^2$ holds or relations $q_{22} \in R_3$, $q_{11}^3 \neq 1$ are valid. Then $\mathcal{D}_{\chi,E}$ appears in row 4 and 6, respectively, of Table A.1. If $m_1 = 3$ then (2.23) gives that either $q_{11} = q_{22}^3$ or $q_{22}^2 = -1$, $q_{11}^4 \neq 1$. In the first case $\mathcal{D}_{\chi,E}$ appears in row 11 of Table A.1. In the second, from (3.7) one gets $p_1 = q_{11}^{-1} q_{22}^{-1}$, $q_{11}^{(2)} = q_{11}^{-2} q_{22}$, $q_{12}^{(2)} q_{21}^{(2)} = -q_{11}$, $q_{22}^{(2)} = q_{22}$. Since $m_2 = 1$ and $q_{11}^4 \neq 1$, (2.23) yields $q_{11}^{(2)} = -1$, that is $q_{22} = -q_{11}^2$. This implies that $q_{11}^4 = -1$, and hence $\mathcal{D}_{\chi,E}$ coincides with the first entry of the 12th row of Table A.1.

Step 3b. *Consider the case* $q_{11} = -1$, $q_{12}^2 q_{21}^2 \neq 1$. Then

$$m_0 = 1, \quad p_0 = -q_{12}q_{21}, \quad q_{11}^{(1)} = -q_{12}q_{21}q_{22}, \quad q_{12}^{(1)} q_{21}^{(1)} = q_{12}^{-1} q_{21}^{-1}, \quad q_{22}^{(1)} = -1.$$

If $m_1 = 2$ then either $q_{12}q_{21}q_{22}^2 = 1$ or $-q_{12}q_{21}q_{22} \in R_3$, $q_{12}^3 q_{21}^3 \neq 1$. In the first case $\mathcal{D}_{\chi,E}$ appears in the 5th row of Table A.1. In the second one obtains

$$p_1 = -q_{12}^{-2} q_{21}^{-2} q_{22}^{-1}, \quad q_{11}^{(2)} = q_{12}^{-1} q_{21}^{-1} q_{22}, \quad q_{12}^{(2)} q_{21}^{(2)} = -q_{22}^{-1}, \quad q_{22}^{(2)} = -q_{12}q_{21}q_{22}.$$

From $m_2 = 1$ and from $q_{12}^2 q_{21}^2 \neq 1$ one gets $q_{12}q_{21} = -q_{22}$. Since $-q_{12}q_{21}q_{22} \in R_3$ and $q_{12}q_{21} \notin R_3$, this yields $q_{22} \in R_3$. Then $\mathcal{D}_{\chi,E}$ appears in row 7 of Table A.1.

If $m_1 = 3$ then one has again two possibilities: either $-q_{12}^2 q_{21}^2 q_{22}^3 = 1$ or $q_{12}^2 q_{21}^2 q_{22}^2 = -1$, $q_{12}^4 q_{21}^4 \neq 1$ (see Step 3b3 for the latter). In the first case

$$\begin{aligned} m_1 &= 3, \quad p_1 = 1, \quad q_{11}^{(2)} = -1, \quad q_{12}^{(2)} q_{21}^{(2)} = q_{12}^{-1} q_{21}^{-1}, \quad q_{22}^{(2)} = -q_{12}q_{21}q_{22}, \\ m_2 &= 1, \quad p_2 = -q_{12}^{-1} q_{21}^{-1}, \quad q_{11}^{(3)} = q_{22}, \quad q_{12}^{(3)} q_{21}^{(3)} = q_{12}q_{21}, \quad q_{22}^{(3)} = -1. \end{aligned}$$

By (2.23) either equation $q_{22}^{m_3} q_{12}q_{21} = 1$ has to hold (Step 3b1), or one has the relations $q_{22} \in R_{m_3+1}$, $q_{12}^{m_3+1} q_{21}^{m_3+1} \neq 1$ (Step 3b2), where $m_3 \geq 3$ in both cases.

Step 3b1. *The setting* $-q_{12}^2 q_{21}^2 q_{22}^3 = 1$, $q_{22}^{m_3} q_{12}q_{21} = 1$. One has

$$p_3 = 1, \quad q_{11}^{(4)} = -1, \quad q_{12}^{(4)} q_{21}^{(4)} = q_{12}q_{21}, \quad q_{22}^{(4)} = q_{22}.$$

Thus the sequence $(m_i)_{i \in \mathbb{N}_0}$ has period 4, and

$$\prod_{i=0}^{3} \tilde{P}_i = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_3 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m_3 - 2 & -1 \\ 2m_3 - 3 & -2 \end{pmatrix}.$$

By Lemma 3.1.4 this matrix has to have finite order. By Lemma 3.1.2 the latter happens if and only if $m_3 \in \{3, 4, 5\}$. If $m_3 = 3$ then equations $q_{22}^{m_3} q_{12}q_{21} = 1$ and $-q_{12}^2 q_{21}^2 q_{22}^3 = 1$ imply that $q_{12}q_{21} = q_{22}^{-3}$ and $-q_{22}^3 = 1$ which is a contradiction to $q_{12}^2 q_{21}^2 \neq 1$. If $m_3 = 4$ then one gets $q_{12}q_{21} = q_{22}^{-4}$ and $q_{22}^5 = -1$. Since $q_{12}^2 q_{21}^2 \neq 1$ this yields $q_{22} \in R_{10}$. Then $\mathcal{D}_{\chi,E}$ is the second entry in row 14 of Table A.1. If $m_3 = 5$ then $q_{12}q_{21} = q_{22}^{-5}$ and $-q_{22}^7 = 1$. Again relation $q_{12}^2 q_{21}^2 \neq 1$ implies that $q_{22} \in R_{14}$. In this case $\mathcal{D}_{\chi,E}$ is the second entry in row 17 of Table A.1.

Step 3b2. *Consider the setting* $-q_{12}^2 q_{21}^2 q_{22}^3 = 1$, $q_{22} \in R_{m_3+1}$, $q_{12}^2 q_{21}^2 \neq 1$, $q_{12}^{m_3+1} q_{21}^{m_3+1} \neq 1$, *where* $m_3 \geq 3$. One obtains

$$p_3 = q_{12}q_{21}q_{22}^{-1}, \quad q_{11}^{(4)} = -q_{12}^{m_3} q_{21}^{m_3} q_{22}, \quad q_{12}^{(4)} q_{21}^{(4)} = q_{12}^{-1} q_{21}^{-1} q_{22}^2, \quad q_{22}^{(4)} = q_{22}.$$

Since $m_4 = 1$, one has either $q_{12}^{m_3-3} q_{21}^{m_3-3} = 1$ (Step 3b2a) or $q_{12}^{m_3} q_{21}^{m_3} q_{22} = 1$, $q_{12}^{-1} q_{21}^{-1} q_{22}^2 \neq -1$ (Step 3b2b).

Step 3b2a. *As in Step 3b2, but additionally one has* $q_{12}^{m_3-3} q_{21}^{m_3-3} = 1$. Then

$$\begin{aligned} m_4 &= 1, \quad p_4 = 1, \quad q_{11}^{(5)} = q_{22}, \quad q_{12}^{(5)} q_{21}^{(5)} = q_{12}^{-1} q_{21}^{-1} q_{22}^2, \quad q_{22}^{(5)} = -q_{12}^3 q_{21}^3 q_{22}, \\ m_5 &= m_3, \quad p_5 = q_{12}^{-1} q_{21}^{-1} q_{22}, \quad q_{11}^{(6)} = -1, \quad q_{12}^{(6)} q_{21}^{(6)} = q_{12}q_{21}, \quad q_{22}^{(6)} = q_{22}. \end{aligned}$$

Thus the sequence $(m_i)_{i \in \mathbb{N}_0}$ has period 6. One has

$$\prod_{i=0}^{5} \tilde{P}_i = \prod_{i=0}^{5} \begin{pmatrix} m_i & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} {m_3}^2 - 4m_3 + 2 & 3 - m_3 \\ 2m_3^2 - 7m_3 + 3 & 5 - 2m_3 \end{pmatrix}$$

By Lemma 3.1.4 this matrix has to have finite order. By Lemma 3.1.2 if the latter happens then $-2 \leq (m_3 - 3)^2 - 2 \leq 2$, that is $(3 \leq) m_3 \leq 5$. If $m_3 = 3$ then $-q_{12}^2 q_{21}^2 = q_{22} \in R_4$, and hence $q_{12}q_{21} \in R_8$, and $\mathcal{D}_{\chi,E}$ is the second entry in row 12 of Table A.1. If $m_3 \in \{4, 5\}$ then relations $q_{12}^{m_3-3} q_{21}^{m_3-3} = 1$ and $q_{12}^2 q_{21}^2 \neq 1$ contradict to each other.

Step 3b2b. *As in Step 3b2, but additionally relations* $q_{12}^{m_3} q_{21}^{m_3} q_{22} = 1$, $q_{12}^{-1} q_{21}^{-1} q_{22}^2 \neq -1$ *hold.* Inserting equations $q_{12}^{m_3} q_{21}^{m_3} = q_{22}^{-1}$ and $q_{22}^{m_3} = q_{22}^{-1}$ into $(-q_{12}^2 q_{21}^2 q_{22}^3)^{2m_3} = 1$ one obtains $q_{22}^{10} = 1$. Since $q_{22} \in R_{m_3+1}$ and $m_3 \geq 3$, this means that $m_3 \in \{4, 9\}$. If $m_3 = 4$ then $q_{22} \in R_5$, $q_{22} = q_{12}^{-4} q_{21}^{-4}$, and hence equation $-q_{12}^2 q_{21}^2 q_{22}^3 = 1$ implies that $q_{12}^1 0 q_{21}^{10} = -1$ and $q_{12}^2 q_{21}^2 \neq -1$. Thus $q_{12}q_{21} \in R_{20}$. In this case $\mathcal{D}_{\chi,E}$ is the third entry in row 15 of Table A.1. Finally, if $m_3 = 9$ then equations $q_{22} = q_{12}^{-9} q_{21}^{-9}$ and $-q_{12}^2 q_{21}^2 q_{22}^3 = 1$ imply that

$q_{12}^{25}q_{21}^{25}=-1$. Since $q_{22}^5=-1$, one also has $q_{12}^{45}q_{21}^{45}=-1$, that is $q_{12}^5q_{21}^5=-1$ and hence $q_{22}=q_{12}q_{21}$. This is a contradiction to $q_{12}^{m_3+1}q_{21}^{m_3+1}\neq 1$.

Step 3b3. *The setting* $q_{11}=-1$, $q_{12}^2q_{21}^2q_{22}^2=-1$, $q_{12}^4q_{21}^4\neq 1$. One has

$$m_1=3,\quad p_1=q_{22},\quad q_{11}^{(2)}=-q_{22}^3,\quad q_{12}^{(2)}q_{21}^{(2)}=-q_{12}q_{21},\quad q_{22}^{(2)}=-q_{12}q_{21}q_{22}.$$

Since $m_2=1$, Equation (2.11) gives that either $q_{12}q_{21}q_{22}^3=1$ or $q_{22}^3=1$. In the first case $q_{12}^2q_{21}^2q_{22}^2=-1$ gives that $q_{12}q_{21}=-q_{22}$ and $q_{22}\in R_8$. Then $\mathcal{D}_{\chi,E}$ is the third entry in row 12 of Table A.1. In the second case again by equation $q_{12}^2q_{21}^2q_{22}^2=-1$ one concludes that $q_{22}=-q_{12}^2q_{21}^2$ and hence $q_{12}^6q_{21}^6=-1$. Since $q_{12}^4q_{21}^4\neq 1$ this yields $q_{12}q_{21}\in R_{12}$, so that $\mathcal{D}_{\chi,E}$ is the second entry in row 8 of Table A.1.

Step 4. If the setting is *different from steps 1–3* then there exists an $i\in\mathbb{N}_0$ such that $m_i=1$ and $m_{i+1}>1$, $m_{i+2}>1$. By Weyl equivalence one can assume that $n_0=1$ and $m_0>1$, $m_1>1$. Equations $n_0=1$ and (2.23) give that either $q_{12}q_{21}q_{22}=1$ (Step 4a) or $q_{22}=-1$, $q_{12}^2q_{21}^2\neq 1$ (Step 4b).

Step 4a. *The setting* $q_{12}q_{21}q_{22}=1$, $m_0>1$, $m_1>1$. By (2.23) one has either $q_{11}^{m_0}q_{12}q_{21}=1$ or $q_{11}\in R_{m_0+1}$, $q_{12}^{m_0+1}q_{21}^{m_0+1}\neq 1$. In the first case $p_0=1$ and $m_1=1$ which is a contradiction. In the second one has

$$p_0=q_{11}^{-1}q_{12}q_{21},\quad q_{11}^{(1)}=q_{11}q_{22}^{1-m_0},\quad q_{12}^{(2)}q_{21}^{(2)}=q_{11}^2q_{22},\quad q_{22}^{(2)}=q_{11}.\tag{3.10}$$

Lemma 3.1.5 and the inequalities $m_0>1$ and $m_1>1$ imply that $q_{11}^6q_{22}^{3-m_0}=-1$. By Equation (2.23) one again has to distinguish two cases: either $(q_{11}q_{22}^{1-m_0})^{m_1}q_{11}^2q_{22}=1$ (see Step 4a1) or $q_{11}q_{22}^{1-m_0}\in R_{m_1+1}$, $(q_{11}^2q_{22})^{m_1+1}\neq 1$ (see Step 4a2).

Step 4a1. *Assume that* $q_{12}q_{21}q_{22}=1$, $q_{11}^6q_{22}^{3-m_0}=-1$, $q_{11}^{m_1+2}q_{22}^{(1-m_0)m_1+1}=1$, $q_{11}\in R_{m_0+1}$, $q_{22}^{m_0+1}\neq 1$, $m_0>1$, *and* $m_1>1$. Then from (3.10) and (3.7) one concludes that

$$\begin{aligned}
&p_1=1, && q_{11}^{(2)}=q_{11}, && q_{12}^{(2)}q_{21}^{(2)}=q_{11}^2q_{22}, && q_{22}^{(2)}=q_{11}q_{22}^{1-m_0},\\
m_2=m_0,\quad &p_2=q_{11}q_{22}, && q_{11}^{(3)}=q_{22}, && q_{12}^{(3)}q_{21}^{(3)}=q_{22}^{-1}, && q_{22}^{(3)}=q_{11},\\
m_3=1,\quad &p_3=1, && q_{11}^{(4)}=q_{11}, && q_{12}^{(4)}q_{21}^{(4)}=q_{22}^{-1}, && q_{22}^{(4)}=q_{22}.
\end{aligned}$$

Hence the sequence $(m_i)_{i\in\mathbb{N}_0}$ is periodic with period 4. The matrix of the linear map $P_3P_2P_1P_0$ in Lemma 3.1.4 takes the form

$$\prod_{i=0}^{3}\tilde{P}_i=\prod_{i=0}^{3}\begin{pmatrix}m_i & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}{m_0}^2m_1-m_0m_1-2m_0+1 & m_0(2-m_0m_1)\\ m_0m_1-m_1-1 & 1-m_0m_1\end{pmatrix}.\tag{3.11}$$

By Lemma 3.1.4 this matrix has to have finite order. By Lemma 3.1.2 this gives that $0\leq(m_0-2)(m_0m_1-2)\leq 4$. Since $m_0\geq 2$ and $m_1\geq 2$, the latter means that $m_0=2$ or $m_0=3$, $m_1=2$. In the first case the matrix in (3.11) has trace -2, but it is not equal to $-\mathrm{id}$, and in the second its trace is 2, but the matrix is not the identity. Thus in both cases one obtains a contradiction to Lemma 3.1.2.

Step 4a2. *Consider the setting* $q_{12}q_{21}q_{22}=1$, $q_{11}^6q_{22}^{3-m_0}=-1$, $q_{11}\in R_{m_0+1}$, $q_{11}q_{22}^{1-m_0}\in R_{m_1+1}$, $q_{22}^{m_0+1}\neq 1$, $(q_{11}^2q_{22})^{m_1+1}\neq 1$, $m_0>1$, *and* $m_1>1$. Since $m_0>1$ and $m_1>1$, Lemma 3.1.5 implies that $m_0,m_1\in\{2,3\}$.

If $m_0=2$ then $q_{11}\in R_3$ and hence equation $q_{11}^6q_{22}^{3-m_0}=-1$ implies that $q_{22}=-1$. In this case the relation $q_{11}q_{22}^{1-m_0}\in R_{m_1+1}$ is a contradiction to $m_1\in\{2,3\}$.

If $m_0=3$ and $m_1=2$ then $q_{11}^2=-1$ and $q_{11}q_{22}^{-2}\in R_3$. The sixth power of the latter relation gives that $q_{22}^{12}=-1$. Thus $q_{22}\in R_8\cup R_{24}$. If $q_{22}\in R_8$ then $q_{22}^2\in R_4=\{q_{11},-q_{11}\}$ which is a contradiction to $q_{11}q_{22}^{-2}\in R_3$. On the other hand, if $q_{22}\in R_{24}$ then $q_{11}\in\{q_{22}^6,-q_{22}^6\}$. Again $q_{11}=q_{22}^6$ is a contradiction to $q_{11}q_{22}^{-2}\in R_3$. For the remaining case, where $q_{22}\in R_{24}$ and $q_{11}=-q_{22}^6$, $\mathcal{D}_{\chi,E}$ appears as the second entry in line 13 of Table A.1.

If $m_0=m_1=3$ then again $q_{11}^2=-1$. Relation $q_{11}q_{22}^{1-m_0}\in R_{m_1+1}$ gives that $(q_{11}q_{22}^{-2})^2=-1$, that is $q_{22}^4=1$. This is a contradiction to $q_{22}^{m_0+1}\neq 1$.

Step 4b. *Assume now that* $q_{22}=-1$, $q_{12}^2q_{21}^2\neq 1$, $m_0>1$, *and* $m_1>1$. There are again two cases: $q_{11}^{m_0}q_{12}q_{21}=1$ or $q_{11}\in R_{m_0+1}$, $q_{12}^{m_0+1}q_{21}^{m_0+1}\neq 1$. In the first one gets $p_0=1$ and $q_{11}^{(1)}=-1$ which is a contradiction $m_1>1$. Therefore one has $q_{11}\in R_{m_0+1}$, $q_{12}^{m_0+1}q_{21}^{m_0+1}\neq 1$, and hence

$$p_0=q_{11}^{-1}q_{12}q_{21},\quad q_{11}^{(1)}=-q_{11}q_{12}^{m_0}q_{21}^{m_0},\quad q_{12}^{(1)}q_{21}^{(1)}=q_{11}^2q_{12}^{-1}q_{21}^{-1},\quad q_{22}^{(1)}=q_{11}.\tag{3.12}$$

From Lemma 3.1.5 and the above equations one obtains that $m_0,m_1\in\{2,3\}$ and that $-q_{11}q_{12}^{m_0}q_{21}^{m_0}(q_{11}^2q_{12}^{-1}q_{21}^{-1})^2q_{11}=-1$. Further, by Equation (2.23) one has to have $(-q_{11}q_{12}^{m_0}q_{21}^{m_0})^{m_1}q_{11}^2q_{12}^{-1}q_{21}^{-1}=1$ (see Step 4b1) or $-q_{11}q_{12}^{m_0}q_{21}^{m_0}\in R_{m_1+1}$, $(q_{11}^2q_{12}^{-1}q_{21}^{-1})^{m_1+1}\neq 1$ (see Step 4b2).

Step 4b1. *Assume that the following relations hold:* $q_{11}\in R_{m_0+1}$, $q_{22}=-1$, $(-1)^{m_1}q_{11}^{m_1+2}(q_{12}q_{21})^{m_0m_1-1}=1$, $q_{11}^6q_{12}^{m_0-2}q_{21}^{m_0-2}=1$, $q_{12}^2q_{21}^2\neq 1$, *and* $m_0,m_1\in\{2,3\}$. If $m_0=3$ then $q_{11}^2=-1$ and $q_{11}^6q_{12}q_{21}=1$, which gives a contradiction to $q_{12}^2q_{21}^2\neq 1$.

If $m_0=2$ and $m_1=2$ then $q_{11}\in R_3$ and $q_{11}q_{12}^3q_{21}^3=1$, and hence $q_{12}q_{21}\in R_9$, $q_{11}=q_{12}^{-3}q_{21}^{-3}$. Then $\mathcal{D}_{\chi,E}$ is the second entry in row 10 of Table A.1.

If $m_0 = 2$ and $m_1 = 3$ then $q_{11} \in R_3$ and $q_{11}^{-1} q_{12}^5 q_{21}^5 = -1$. Thus $(q_{12}q_{21})^{15} = -1$ and $(q_{12}q_{21})^5 \neq -1$, and hence $q_{12}q_{21} \in R_{30} \cup R_6$. If $q_{12}q_{21} \in R_{30}$ then $\mathcal{D}_{\chi,E}$ is the third entry in row 16 of Table A.1. Otherwise one has $q_{12}q_{21} \in R_6$, $q_{11}q_{12}q_{21} = -1$, and hence the equations $n_0 = 1$, $r_{11}^{(0)} = -1$, $r_{12}^{(0)} r_{21}^{(0)} = q_{12}q_{21}$, $r_{22}^{(0)} = q_{11}$, and (3.6), (3.8) imply that $s_0 = -q_{12}q_{21}$ and $r_{11}^{(1)} = -q_{12}q_{21}q_{11} = 1$, $r_{12}^{(1)} r_{21}^{(1)} = q_{12}^{-1} q_{21}^{-1}$. In this case (2.23) has no solution for n_1, which is a contradiction by Lemma 3.1.4.

Step 4b2. *The setting* $q_{11} \in R_{m_0+1}$, $q_{22} = -1$, $q_{11}^6 q_{12}^{m_0-2} q_{21}^{m_0-2} = 1$, $-q_{11} q_{12}^{m_0} q_{21}^{m_0} \in R_{m_1+1}$, $(q_{11}^2 q_{12}^{-1} q_{21}^{-1})^{m_1+1} \neq 1$, $q_{12}^2 q_{21}^2 \neq 1$, $m_0, m_1 \in \{2,3\}$. If $m_0 = 3$ then $q_{11}^2 = -1$, and equation $q_{11}^6 q_{12}^{m_0-2} q_{21}^{m_0-2} = 1$ is a contradiction to $q_{12}^2 q_{21}^2 \neq 1$.

If $m_0 = 2$ and $m_1 = 3$ then $q_{11} \in R_3$ and $-q_{11} q_{12}^2 q_{21}^2 \in R_4$. Therefore $q_{11}^2 q_{12}^4 q_{21}^4 = -1$, and hence $q_{11} = -q_{12}^4 q_{21}^4$. Using Relation $q_{11} \in R_3$ one gets $(q_{12}q_{21})^{12} = -1$ and $q_{12}^4 q_{21}^4 \neq -1$, and hence $q_{12}q_{21} \in R_{24}$. In this case $\mathcal{D}_{\chi,E}$ is the third entry in line 13 of Table A.1.

Finally, if $m_0 = m_1 = 2$ then $q_{11} \in R_3$ and $-q_{11} q_{12}^2 q_{21}^2 \in R_3$. Since there are two elements in R_3, there are two possibilities. The first case is when $q_{11} = -q_{11} q_{12}^2 q_{21}^2$, that is $q_{12}q_{21} \in R_4$. Set $\zeta := -q_{11}^{-1} q_{12}q_{21} \in R_{12}$. Then $q_{11} = -\zeta^2$ and $q_{12}q_{21} = \zeta^3$, and hence $\mathcal{D}_{\chi,E}$ is the second entry in row 9 of Table A.1. On the other hand, if $q_{11}^2 = -q_{11} q_{12}^2 q_{21}^2$ then $q_{11} = -q_{12}^2 q_{21}^2$. Since $q_{11} \in R_3$, this gives that $q_{12}^6 q_{21}^6 = -1$ and $q_{12}^2 q_{21}^2 \neq -1$, that is $q_{12}q_{21} \in R_{12}$. Then $\mathcal{D}_{\chi,E}$ is the second entry in line 8 of Table A.1. ∎

Remark. Theorem 3.1.1 gives also a classification of finite dimensional Nichols algebras $\mathcal{B}(V)$ of diagonal type. As mentioned in Section 2.2, $\mathcal{B}(V)$ is finite dimensional if and only if $\mathbf{\Delta}^+(\mathcal{B}(V))$ is finite (that is $(\mathbf{\Delta}(\mathcal{B}(V)), \chi, E)$ is an arithmetic root system, where V is associated to (χ, E)), and all PBW generators have finite height. Moreover, the latter happens if and only if

$$2 \le \operatorname{ord} \chi(\mathbf{e},\mathbf{e}) < \infty \quad \text{for all } \mathbf{e} \in \mathbf{\Delta}^+(\mathcal{B}(V)). \tag{$*$}$$

By Theorem 2.5.3 any $\mathbf{e} \in \mathbf{\Delta}(\mathcal{B}(V))$ appears as an element of a basis F of $\mathbb{Z}^2$ such that $(\mathrm{id}, F) \in W_{\chi,E}$. Hence $\chi(\mathbf{e},\mathbf{e})$ appears as the label of a node of a representant of the Weyl equivalence class of (χ, E), and therefore it can be easily checked whether $(*)$ is fulfilled or not. ∎

Table A.1 allows to tell additional facts about rank 2 arithmetic root systems. In what follows some of them are stated because they will be used in the next section for the classification of rank 3 arithmetic root systems.

Proposition 3.1.6. *Let $E = \{\mathbf{e}_1, \mathbf{e}_2\}$ be a basis of $\mathbb{Z}^2$ and χ a bicharacter on $\mathbb{Z}^2$. Set $q_{ij} := \chi(\mathbf{e}_i, \mathbf{e}_j)$ for all $i, j \in \{1,2\}$. If $W_{\chi,E}$ is full and finite then the following assertions hold.*
(i) Either equation $(q_{12}q_{21}-1)(q_{11}+1)(q_{22}+1)(q_{11}q_{12}q_{21}-1)(q_{12}q_{21}q_{22}-1) = 0$ is satisfied or one has $q_{11} q_{12}^2 q_{21}^2 q_{22} = -1$, and in the latter case also one of the relations $q_{11} \in R_3$, $q_{22} \in R_3$ is fulfilled.
(ii) If the arithmetic root system $(\mathbf{\Delta}, \chi, E)$ determined by χ and E is connected then it is Weyl equivalent to an arithmetic root system $(\mathbf{\Delta}', \chi', E)$ such that the constants $q'_{ij} := \chi'(\mathbf{e}_i, \mathbf{e}_j)$, $i, j \in \{1,2\}$, satisfy either equation $q'_{22} = -1$ or equations $q'_{12}q'_{21}q'_{22} = 1$, ${q'_{11}}^a q'_{12}q'_{21} = 1$, where $a \in \{1,2,3\}$, or relations $q'_{12}q'_{21}q'_{22} = 1$, $q'_{11} \in R_3$.

Lemma 3.1.7. *Under the conditions of Proposition 3.1.6 the following assertions are true.*
(i) If $q_{12}q_{21} = -1$ and $q_{11} \neq -1$ then $q_{22} = -1$ and $q_{11} \in R_3 \cup R_4 \cup R_6$.
(ii) If $q_{12}q_{21} = -q_{11}$ and $q_{22} = -1$ then $q_{11} \in R_2 \cup R_3 \cup R_4 \cup R_6 \cup R_8 \cup R_{10}$.
(iii) If $q_{12}q_{21} = -q_{11}$ and $q_{22} = -q_{11}^{-1}$ then $q_{11} \in R_2 \cup R_3 \cup R_4 \cup R_6 \cup R_8$.

Lemma 3.1.8. *If $q, r \in k^*$, $q, r \neq 1$, and $(\mathbf{\Delta}_1, \chi_1, E)$ and $(\mathbf{\Delta}_2, \chi_2, E)$ are arithmetic root systems of rank two with generalized Dynkin diagrams $\overset{q}{\circ}\overset{r}{\text{———}}\overset{r^{-1}}{\circ}$ and $\overset{q}{\circ}\overset{r}{\text{———}}\overset{-1}{\circ}$, respectively, then either $q \in R_2 \cup R_3$ or $q^m r = 1$ with $m \in \{1,2,3\}$.*

Proof. If (χ_1, E) is of Cartan type then the claim of the lemma follows from Theorem 2.10.2. Otherwise $q \in R_{m+1}$ and $r^{m+1} \neq 1$ with $m \in \{1,2,3,4\}$. If $m = 4$ then $r \in R_{30}$ and $q = -r^{-3}$, and if $m = 3$ then either $r \in R_8, q = r^2$ or $r \in R_{24}, q = r^6$. In all three cases (χ_2, E) does not give an arithmetic root system. ∎

3.2 Arithmetic Root Systems of Rank 3

The main result of this section is the following theorem.

Theorem 3.2.1. *Let k be a field of characteristic 0. Then twist equivalence classes of connected arithmetic root systems of rank 3 are in one-to-one correspondence to generalized Dynkin diagrams appearing in Table A.2. Moreover, two such arithmetic root systems are Weyl equivalent if and only if their generalized Dynkin diagrams appear in the same row of Table A.2* *and can be presented with the same set of fixed parameters.*

Comparing the entries in Table A.2 and Table A.1 one obtains the following.

Corollary 3.2.2. *The generalized Dynkin diagram of a rank 2 subsystem of a connected arithmetic root system of rank $d \geq 3$ appears in one of the rows 1–7 and 11 in Table A.1.*

Proof. Using Proposition 2.5.2 and Weyl equivalence one can assume that the basis of the rank 2 subsystem is a subset of the basis of the rank d arithmetic root system. Thus it suffices to check all subdiagrams of the generalized Dynkin diagrams appearing in Table A.2. ∎

The remaining part of this thesis is devoted to the proof of Theorem 3.2.1.

Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a fixed basis of $\mathbb{Z}^3$ and χ a bicharacter on $\mathbb{Z}^3$ with values in k^*. Set $q_{ij} := \chi(\mathbf{e}_i, \mathbf{e}_j)$. Suppose that $(\boldsymbol{\Delta}, \chi, E)$ is a connected arithmetic root system.

Lemma 3.2.3. *If $q_{12}q_{21} \neq 1$ and $q_{13}q_{31} \neq 1$ then either $q_{23}q_{32} = 1$ or $q_{12}q_{21}q_{13}q_{31}q_{23}q_{32} = 1$.*

Proof. Assume that $q_{23}q_{32} \neq 1$. If (χ, E) is of Cartan type then Theorem 2.10.2 gives a contradiction. Thus without loss of generality $q_{11} \in R_{m+1}$ with $m \geq 1$ and $(q_{12}q_{21})^{m+1} \neq 1$. By Lemma 2.2.1(i) one has $a\mathbf{e}_1 + \mathbf{e}_2 \in \boldsymbol{\Delta}^+$ for $0 \leq a \leq m$. By the same reason $\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3 \in \boldsymbol{\Delta}^+$.

By Proposition 2.7.1 $\boldsymbol{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_3)$ is finite, and hence Lemmata 2.7.2 and 2.2.1(i) imply that $(m+1)\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \notin \boldsymbol{\Delta}^+$. Since $(m+1)\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = (\mathbf{e}_1 + \mathbf{e}_3) + (m\mathbf{e}_1 + \mathbf{e}_2)$, Lemma 2.2.1(i) implies that

$$1 = \chi\chi^{\mathrm{op}}(\mathbf{e}_1 + \mathbf{e}_3, m\mathbf{e}_1 + \mathbf{e}_2) = q_{11}^{-2} q_{12}q_{21}(q_{13}q_{31})^m q_{23}q_{32}. \tag{3.13}$$

If either $m = 1$ or $m \geq 2$, $q_{11}q_{13}q_{31} = 1$ then the claim holds. Otherwise $2\mathbf{e}_1 + \mathbf{e}_3 \in \boldsymbol{\Delta}^+$, and $(m+2)\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \notin \boldsymbol{\Delta}^+$ implies that

$$1 = \chi\chi^{\mathrm{op}}(2\mathbf{e}_1 + \mathbf{e}_3, m\mathbf{e}_1 + \mathbf{e}_2) = q_{11}^{-4}(q_{12}q_{21})^2(q_{13}q_{31})^m q_{23}q_{32}.$$

Together with (3.13) one obtains that $q_{11}^2(q_{12}q_{21})^{-1} = 1$ which is a contradiction to the assumption $(q_{12}q_{21})^{m+1} \neq 1$. ∎

Corollary 3.2.4. *If $q_{12}q_{21} \neq 1$ and $q_{13}q_{31} \neq 1$ then $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \in \boldsymbol{\Delta}^+$.*

Proof. By Proposition 2.7.1 the set $\boldsymbol{\Delta}(\chi; \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2)$ is finite. By Lemma 3.2.3 one has $\chi\chi^{\mathrm{op}}(\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2) \neq 1$ and hence the claim follows from Lemma 2.2.1(i). ∎

Lemma 3.2.5. *Suppose that $q_{12}q_{21} \neq 1$, $q_{13}q_{31} \neq 1$, and $(2a+1)\mathbf{e}_1 + 2\mathbf{e}_2 \in \boldsymbol{\Delta}^+$ for some $a \in \mathbb{N}_0$. Then $(2b+1)\mathbf{e}_1 + 2\mathbf{e}_3 \notin \boldsymbol{\Delta}^+$ for all $b \in \mathbb{N}_0$. In particular, either the generalized Dynkin diagram of $(\chi|_{\mathbb{Z}\{\mathbf{e}_1,\mathbf{e}_3\} \times \mathbb{Z}\{\mathbf{e}_1,\mathbf{e}_3\}}, \{\mathbf{e}_1, \mathbf{e}_3\})$ appears in one of rows 2–7 of Table A.1, or one of the following is true:*

(i) $q_{13}q_{31} \in R_{12}$, $q_{11} = -(q_{13}q_{31})^3$, $q_{33} = -1$,

(ii) $q_{11} \in R_{12}$, $q_{11}^3 q_{13}q_{31} = 1$, $q_{33} = -1$,

(iii) $q_{11} \in R_{18}$, $q_{11}^4 q_{13}q_{31} = 1$, $q_{33} = -1$.

Proof. Set $\alpha_a := (2a+1)\mathbf{e}_1 + 2\mathbf{e}_2$ and assume that $\beta_b := (2b+1)\mathbf{e}_1 + 2\mathbf{e}_3 \in \boldsymbol{\Delta}^+$ for some $b \in \mathbb{N}_0$. Then by Lemma 2.7.2 and Proposition 2.7.1 one obtains that $\{\alpha_a, \beta_b\}$ is a basis of $\boldsymbol{\Delta}(\chi; \alpha_a, \beta_b)$ and hence

$$(a + b + 1)\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = \alpha_a/2 + \beta_b/2 \notin \boldsymbol{\Delta}(\chi; \alpha_a, \beta_b). \tag{3.14}$$

Moreover $2(a+b+1)\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 = \alpha_a + \beta_b \notin \boldsymbol{\Delta}$ by Corollary 2.5.4. However $(a+1)\mathbf{e}_1 + \mathbf{e}_2, (b+1)\mathbf{e}_1 + \mathbf{e}_3 \in \boldsymbol{\Delta}^+$, and hence (again by Proposition 2.7.1 and Lemmata 2.7.2 and 2.2.1(i))

$$\begin{aligned}\chi\chi^{\mathrm{op}}(a\mathbf{e}_1 + \mathbf{e}_2, (b+1)\mathbf{e}_1 + \mathbf{e}_3) &= \chi\chi^{\mathrm{op}}((a+1)\mathbf{e}_1 + \mathbf{e}_2, b\mathbf{e}_1 + \mathbf{e}_3)\\ &= \chi\chi^{\mathrm{op}}(\alpha_a, \beta_b) = 1.\end{aligned}$$

The first equation gives $q_{11}^{2(a-b)} q_{12}q_{21}(q_{13}q_{31})^{-1} = 1$, and the third expression divided by the fourth power of the first one yields $q_{11}^{-4a+4b+2}(q_{12}q_{21})^{-2}(q_{13}q_{31})^2 = 1$. Thus $q_{11}^2 = 1$, and hence $a = b = 0$. This is a contradiction to Corollary 3.2.4 and relation (3.14).

The last assertion follows from Theorem 3.1.1 using the full binary trees in Appendix A. ∎

Lemma 3.2.6. *Assume that $q_{12}q_{21} \neq 1$, $q_{13}q_{31} \neq 1$, and $q_{23}q_{32} \neq 1$. Then $q_{11} = -1$ or $q_{22} = -1$ or $q_{33} = -1$.*

Proof. Suppose that the claim is false. If (χ, E) is of Cartan type then $\boldsymbol{\Delta}$ is not finite by Theorem 2.10.2. Thus without loss of generality one can assume that $q_{11} \in R_{m+1}$ and $(q_{12}q_{21})^{m+1} \neq 1$ for some $m \geq 2$. The proof of

Lemma 3.2.3 shows that $q_{11}q_{13}q_{31}=1$. Further, relations $q_{33}\neq -1$, $q_{11}\neq -1$ and $q_{11}q_{12}q_{21}\neq 1$ and Lemma 3.2.5 imply that $q_{33}q_{23}q_{32}=1$. By Lemma 3.2.3 one obtains that $q_{12}q_{21}=(q_{13}q_{31}q_{23}q_{32})^{-1}=q_{11}q_{33}$. Again by Lemma 3.2.5 and since $q_{22},q_{33}\neq -1$ one has $(q_{12}q_{21}q_{22}-1)(q_{13}q_{31}q_{33}-1)=0$. If $q_{13}q_{31}q_{33}=1$ then $q_{33}=(q_{13}q_{31})^{-1}=q_{11}$ and hence $q_{12}q_{21}=q_{11}^2$ which contradicts to the assumptions $q_{11}\in R_{m+1}$, $(q_{12}q_{21})^{m+1}\neq 1$. Therefore $q_{22}=(q_{12}q_{21})^{-1}=q_{11}^{-1}q_{33}^{-1}$. By Proposition 2.7.1 the set $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3)$ $\overset{q_{11}}{\circ}\!\overset{q_{11}^{-1}q_{33}^{-1}}{\text{———}}\!\overset{q_{33}}{\circ}$ is finite. This is a contradiction to Proposition 3.1.6(i), Lemma 3.1.7(i), and the conditions $q_{ii}\neq -1$ for all $i\in\{1,2,3\}$. ■

Lemma 3.2.7. *If $q_{12}q_{21}\neq 1$, $q_{13}q_{31}\neq 1$, and $q_{23}q_{32}=1$ then the equation*

$$(q_{11}+1)(q_{11}q_{12}q_{21}-1)(q_{11}q_{13}q_{31}-1)(q_{11}q_{12}q_{21}q_{13}q_{31}+1)=0$$

holds.

Proof. Suppose that the claim is false. By Lemma 3.2.5 one can assume that $(2b+1)\mathbf{e}_1+2\mathbf{e}_3\notin\boldsymbol{\Delta}$ for all $b\in\mathbb{N}_0$. In particular, one has

$$(q_{13}q_{31}q_{33}-1)(q_{33}+1)=0. \tag{3.15}$$

Further, if $2\mathbf{e}_1+\mathbf{e}_2\notin\boldsymbol{\Delta}^+$ or $2\mathbf{e}_1+\mathbf{e}_3\notin\boldsymbol{\Delta}^+$ then Lemma 2.2.1(i) gives a contradiction.

If $2\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_3\notin\boldsymbol{\Delta}$ then $2\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_3\notin\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2)$, and hence $(q_{11}q_{13}q_{31}q_{33}+1)(q_{11}q_{13}q_{31}q_{33}q_{12}q_{21}-1)=0$. Inserting (3.15) one gets again a contradiction. Therefore $2\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_3\in\boldsymbol{\Delta}(\chi;2\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3)$, and hence $q_{33}\neq -1$. By (3.15) one obtains that

$$q_{13}q_{31}q_{33}=1,\quad q_{33}^2\neq 1. \tag{3.16}$$

Under this restriction the second sentence of the proof is by Table A.1 equivalent to the equation

$$(q_{11}^2-q_{33})(q_{11}^2+q_{11}+1)=0. \tag{3.17}$$

Since $\boldsymbol{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2)$ is finite, Proposition 3.1.6(i) implies that

$$(q_{22}+1)(q_{12}q_{21}q_{22}-1)(q_{11}q_{12}^2q_{21}^2q_{22}+1)=0. \tag{3.18}$$

Assume that $2\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3\notin\boldsymbol{\Delta}$. Then it is not in $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3)$, and hence $(q_{11}q_{12}q_{21}q_{22}+1)(q_{11}q_{12}q_{21}q_{22}q_{13}q_{31}-1)=0$. If $q_{22}=-1$ or $q_{12}q_{21}q_{22}=1$ then the claim of the Lemma would hold. If $q_{11}q_{12}^2q_{21}^2q_{22}=-1$ then one gets $q_{12}q_{21}=-q_{13}q_{31}$. On the other hand, $2\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3\notin\boldsymbol{\Delta}(\chi;2\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2)$, and hence Lemma 2.2.1(i) implies that $(q_{12}^2q_{21}^2-1)(q_{22}+1)(q_{12}^2q_{21}^2q_{22}-1)=0$. Therefore one obtains again a contradiction. Thus one has $2\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3\in\boldsymbol{\Delta}^+(\chi;2\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2)$, and hence $q_{22}\neq -1$.

According to (3.17) consider first the case $q_{11}\in R_3$. Then one has $3\mathbf{e}_1+(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)\notin\boldsymbol{\Delta}$. This implies that $\chi\chi^{\mathrm{op}}(2\mathbf{e}_1+\mathbf{e}_2,2\mathbf{e}_1+\mathbf{e}_3)=1$, that is $q_{11}q_{12}q_{21}q_{13}q_{31}=1$. By (3.18) and since $q_{22}\neq -1$, one has either $q_{12}q_{21}q_{22}=1$ or $q_{11}q_{12}^2q_{21}^2q_{22}=-1$. In the first case $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1)$ is of infinite Cartan type, and in the second it is not finite by Proposition 3.1.6(i).

Finally suppose that $q_{11}^3\neq 1$. Then $q_{11}^2=q_{33}$ by (3.17). Recall that (3.18) holds and $q_{22}\neq -1$. If $q_{12}q_{21}q_{22}=1$ then $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1)$ $\overset{q_{11}}{\circ}\!\overset{q_{12}^{-1}q_{21}^{-1}}{\text{———}}\!\overset{q_{11}}{\circ}$ is infinite both if $q_{11}^mq_{12}q_{21}=1$ for some $m\geq 2$ (Cartan type) and if $q_{11}\in R_{m+1}$, $m\geq 3$ (Proposition 3.1.6(i)). On the other hand, if $q_{11}q_{12}^2q_{21}^2q_{22}=-1$ then $\boldsymbol{\Delta}(\chi;\mathbf{e}_3,2\mathbf{e}_1+\mathbf{e}_2)$ $\overset{q_{11}^2}{\circ}\!\overset{q_{11}^{-4}}{\text{———}}\!\overset{-q_{11}^3}{\circ}$ is infinite by Proposition 3.1.6(i). ■

Lemma 3.2.8. *If $(\boldsymbol{\Delta},\chi,E)$ is a connected arithmetic root system such that $q_{12}q_{21}\neq 1$, $q_{13}q_{31}\neq 1$, $q_{23}q_{32}=1$, $(q_{13}q_{31}q_{33}-1)(q_{33}+1)\neq 0$, and $(q_{11}+1)(q_{11}q_{13}q_{31}-1)=0$, then $(q_{11}q_{12}q_{21}q_{13}^2q_{31}^2q_{33}-1)(q_{11}^2q_{12}q_{21}q_{13}q_{31}-1)=0$.*

Proof. Consider first the case $q_{11}=-1$. Since $\mathbf{e}_1+2\mathbf{e}_3\in\boldsymbol{\Delta}^+$, one obtains the relation $\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_3\in\boldsymbol{\Delta}(\chi;\mathbf{e}_1+2\mathbf{e}_3,\mathbf{e}_2)\subset\boldsymbol{\Delta}$. Thus $q_{11}=-1$ implies that $3\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_3\notin\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_3,\mathbf{e}_1)$. Considering $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2)$ the latter gives $(q_{12}q_{21}q_{13}q_{31}-1)(-q_{13}q_{31}q_{33}+1)(-q_{13}q_{31}q_{33}q_{12}q_{21}q_{13}q_{31}-1)=0$.

Assume now that $q_{11}q_{13}q_{31}=1$. If $q_{22}=-1$ then because of Proposition 3.1.6(i) the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3)$ implies that

$$(-q_{11}q_{12}q_{21}+1)(-q_{11}q_{12}q_{21}q_{13}q_{31}-1)(-q_{11}q_{12}q_{21}q_{13}^2q_{31}^2q_{33}+1)=0.$$

Therefore by Lemma 3.2.5 it remains to consider the case $q_{12}q_{21}q_{22}=1$, and one can assume additionally that $(q_{11}q_{12}q_{21}-1)(q_{11}^2q_{12}q_{21}-1)(q_{11}^2+q_{11}+1)=0$.

If $q_{11}q_{12}q_{21}=1$ then the claim of the lemma is obviously true. If $q_{11}\in R_3$ then $3(\mathbf{e}_1+\mathbf{e}_2)+(\mathbf{e}_1+2\mathbf{e}_3)\notin\boldsymbol{\Delta}^+$, and hence $4\mathbf{e}_1+3\mathbf{e}_2+2\mathbf{e}_3\notin\boldsymbol{\Delta}^+(\chi;\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,2\mathbf{e}_1+\mathbf{e}_2)$ $\overset{q_{33}}{\circ}\!\overset{q_{11}^2q_{22}^{-1}}{\text{———}}\!\overset{q_{11}q_{22}^{-1}}{\circ}$. Therefore Lemma 2.2.1(i) gives the relation $(q_{11}^2q_{12}q_{21}-1)(q_{33}+1)(q_{11}^2q_{12}q_{21}q_{33}-1)=0$.

Finally assume that $q_{11}q_{13}q_{31}=q_{11}^2q_{12}q_{21}=q_{12}q_{21}q_{22}=1$. Then (χ,E) is not of Cartan type by Theorem 2.10.2. Thus by the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_3)$ and by Table A.1 one has $q_{33}\in R_{a+1}$ with $a\in\{2,3,4\}$. Note that $\mathbf{e}_1+\mathbf{e}_2+$

$2\mathbf{e}_3 \in \mathbf{\Delta}(\chi; \mathbf{e}_1+2\mathbf{e}_3, \mathbf{e}_2)$. Moreover, relations $(2\mathbf{e}_1+\mathbf{e}_2)+b\mathbf{e}_3 \notin \mathbf{\Delta}$ hold for all $b > a$, which in turn imply that $\chi\chi^{\mathrm{op}}(\mathbf{e}_1+\mathbf{e}_2+m\mathbf{e}_3, \mathbf{e}_1+a\mathbf{e}_3) = 1$ whenever $0 < m < a$. Insert $m = 1$ and $m = 2$. Since $(q_{13}q_{31})^{a+1} \neq 1$, one gets $a = 2$ and $q_{11}^3 = q_{33}$, and hence $q_{11}q_{12}q_{21}q_{13}^2q_{31}^2q_{33} = q_{11}^{-3}q_{33} = 1$. ∎

Lemma 3.2.9. *If $(\mathbf{\Delta}, \chi, E)$ is a connected arithmetic root system with generalized Dynkin diagram*

$$\overset{qr^{-1}}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}\overset{r}{\text{——}}\overset{s}{\circ}$$

such that $q, r \neq 1$, $q^2 \neq r^2$, then $q = -1$ or $qr = 1$ or $qr^{-1} \in R_3$.

Proof. One can assume that $q_{11} = q$, $q_{22} = s$, $q_{33} = qr^{-1}$. Thus by Lemma 3.2.5 one has $(rs-1)(s+1) = 0$. Further, Proposition 3.1.6(i) and the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2)$ $\overset{qr^{-1}}{\circ}\overset{q^{-1}}{\text{——}}\overset{qrs}{\circ}$ imply that $qrs = -1$ or $rs = 1$ or $s = -1, qrs \in R_3$, or $s = -1, qr^{-1} \in R_3$.

If $rs \neq 1$ then $s = -1$ and it remains to consider the case $-qr \in R_3$. However Proposition 3.1.6(i) and the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_1)$ $\overset{-q}{\circ}\overset{qr}{\text{——}}\overset{q}{\circ}$ give that $qr = 1$ or $q^4r^2 = 1$. The latter equation is a contradiction to $(-qr)^3 = 1$ and $qr^{-1} \neq -1$.

If $rs = 1$ then the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_1)$ $\overset{qr^{-1}}{\circ}\overset{qr}{\text{——}}\overset{q}{\circ}$ implies that $(qr-1)(q+1)(q^2r-1)(q^2-1) = 0$ or $q^4r = -1$, $\{q, qr^{-1}\} \cap R_3 \neq \{\}$. If $q^2r = 1$ then relation $(qr^{-1})^2 \neq 1$ gives that $q^6 \neq 1$. By Theorem 2.10.2 (χ, E) is not of Cartan type. Thus the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_3)$ yields $q^3 \in R_3 \cup R_4 \cup R_5$. Using this the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1+2\mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2)$ $\overset{q^{11}}{\circ}\overset{q^{-2}}{\text{——}}\overset{q}{\circ}$ implies that $qr^{-1} = q^3 \in R_3$.

Finally, if relations $rs = 1$, $q^4r = -1$, and $q \in R_3$ hold, then the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_1)$ $\overset{-q^{-1}}{\circ}\overset{-1}{\text{——}}\overset{q}{\circ}$ gives a contradiction to Lemma 3.1.7(i). ∎

Lemma 3.2.10. *If $(\mathbf{\Delta}, \chi, E)$ is a connected arithmetic root system with generalized Dynkin diagram*

$$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}\overset{r}{\text{——}}\overset{s}{\circ}$$

such that $r, s \neq 1$, $q^2 \neq 1$, then $(qr-1)(q^2r-1)(q^3r-1)(r^4s^2-1) = 0$.

Proof. Assume that the vertices of $\mathcal{D}_{\chi,E}$ correspond from left to right to $\mathbf{e}_3, \mathbf{e}_1$ and $\mathbf{e}_2$, respectively. If $qr \neq 1$ then $2\mathbf{e}_1+\mathbf{e}_2 \in \mathbf{\Delta}$ by Lemma 2.2.1(i). Further, $2\mathbf{e}_1+\mathbf{e}_3 \notin \mathbf{\Delta}$. Since $4\mathbf{e}_1+2\mathbf{e}_2+2\mathbf{e}_3 \notin \mathbf{\Delta}$ by Corollary 2.5.4, one has either $4\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_3 \notin \mathbf{\Delta}$ or $\chi\chi^{\mathrm{op}}(4\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_3, \mathbf{e}_2) = 1$. In the first case Lemma 2.2.1(i) applied to $\mathbf{\Delta}(\chi; \mathbf{e}_1+\mathbf{e}_3, 2\mathbf{e}_1+\mathbf{e}_2)$ $\overset{q}{\circ}\overset{q^2r}{\text{——}}\overset{q^4r^2s}{\circ}$ implies that $q^2r = 1$ or $q^3r = 1$, and in the second one gets $r^4s^2 = 1$. ∎

Lemma 3.2.11. *If $(\mathbf{\Delta}, \chi, E)$ is a connected arithmetic root system such that $(q_{12}q_{21}-1)(q_{13}q_{31}-1)(q_{23}q_{32}-1) \neq 0$ then it is Weyl equivalent to an arithmetic root system $(\mathbf{\Delta}', \chi', E)$ such that $q'_{11} = -1$ and $q'_{23}q'_{32} = 1$, where $q'_{ij} = \chi'(\mathbf{e}_i, \mathbf{e}_j)$.*

Proof. By Lemma 3.2.6 one can assume that $q_{11} = -1$. Set $\chi'(\mathbf{e}_i, \mathbf{e}_j) := \chi(s_{\mathbf{e}_1,E}(\mathbf{e}_i), s_{\mathbf{e}_1,E}(\mathbf{e}_j))$ for all $i, j \in \{1, 2, 3\}$ and apply Example 2.6.1 with $n = 3$ and $i = 1$. One obtains $p_{12} = -q_{12}q_{21}$, $p_{13} = -q_{13}q_{31}$, $q'_{11} = -1$, and $q'_{23}q'_{32} = p_{12}p_{13}q_{23}q_{32} = 1$ by Lemma 3.2.3. ∎

Lemma 3.2.12. *If $q_{12}q_{21} \neq 1$, $q_{13}q_{31} \neq 1$, and $q_{23}q_{32} = 1$ then $(\mathbf{\Delta}, \chi, E)$ is Weyl equivalent to an arithmetic root system $(\mathbf{\Delta}', \chi', E)$ such that equations $(q'_{11}+1)(q'_{11}q'_{13}q'_{31}-1) = 0$, $q'_{23}q'_{32} = 1$ hold, where $q'_{ij} = \chi'(\mathbf{e}_i, \mathbf{e}_j)$.*

Proof. By Lemma 3.2.7 and twist equivalence it is sufficient to prove the claim under the condition that all of the relations $q_{12}q_{21} \neq 1$, $q_{13}q_{31} \neq 1$, $q_{23}q_{32} = 1$, $q_{11}q_{12}q_{21}q_{13}q_{31} = -1$, $q_{11}^2 \neq 1$, $q_{11}q_{12}q_{21} \neq 1$, and $q_{11}q_{13}q_{31} \neq 1$ hold.

Step 1. *If $q_{22} = -1$ or $q_{33} = -1$ then the claim holds.* By twist equivalence one can assume that $q_{22} = -1$. Then Example 2.6.1 with $n = 3$ and $i = 2$ gives that $q'_{22} = -1$, $q'_{12}q'_{21} = q_{12}^{-1}q_{21}^{-1}$, $q'_{11} = -q_{11}q_{12}q_{21}$, $q'_{13}q'_{31} = q_{13}q_{31}$, $q'_{33} = q_{33}$, $q'_{23}q'_{32} = 1$, and hence $q'_{11}q'_{13}q'_{31} = 1$.

Step 2. *If $q_{13}q_{31}q_{33} \neq 1$ then the lemma is true.* By Step 1 one can assume that $q_{33} \neq -1$. Since $q_{11} \neq -1$ and $q_{11}q_{13}q_{31} \neq 1$, Proposition 3.1.6(i) gives that $q_{11}q_{13}^2q_{31}^2q_{33} = -1$ and one of the relations $q_{11} \in R_3$, $q_{33} \in R_3$ is valid.

Suppose first that $q_{11} \in R_3$. Then Example 2.6.1 with $n = 3$ and $i = 1$ together with equation $q_{11}q_{12}q_{21}q_{13}q_{31} = -1$ gives that $p_{13} = q_{11}^{-1}q_{13}q_{31}$, $q'_{33} = p_{13}^2q_{33} = -1$, and $q'_{23}q'_{32} = 1$, and hence the lemma follows from Step 1. On the other hand, if $q_{33} \in R_3$ and $q_{11}q_{13}^2q_{31}^2q_{33} = -1$ then Example 2.6.1 with $n = 3$ and $i = 3$ gives $p_{31} = q_{33}^{-1}q_{13}q_{31}$ and $q'_{11} = p_{31}^2q_{11} = -1$.

Step 3. *If $q_{12}q_{21}q_{22} = 1$, $q_{13}q_{31}q_{33} = 1$, and $(q_{11}^2+q_{11}+1)(q_{11}^2q_{12}q_{21}-1) = 0$ then the assertion of the lemma holds.* Assume first that $q_{11} \in R_3$. Apply

Example 2.6.1 with $i=1$ and note that $q_{11}q_{12}q_{21}q_{13}q_{31}=-1$, $q_{11}q_{12}q_{21}\neq 1$, $q_{11}q_{13}q_{31}\neq 1$. Thus $q'_{11}=q_{11}$, $q'_{12}q'_{21}={q'}^{-1}_{22}=q_{11}^{-1}q_{12}^{-1}q_{21}^{-1}$, $q'_{13}q'_{31}={q'}^{-1}_{33}=q_{11}^{-1}q_{13}^{-1}q_{31}^{-1}$, and $q'_{23}q'_{32}=1$. Thus $W_{\chi,E}=W_{\chi'',E}$, where the structure constants q''_{ij} of (χ'',E) satisfy the relations ${q''}^2_{11}q''_{12}q''_{21}={q''}^2_{11}q''_{13}q''_{31}=q''_{12}q''_{21}q''_{22}=q''_{13}q''_{31}q''_{33}=q''_{23}q''_{32}=1$ and ${q''}^2_{11}\neq 1$, that is (χ'',E) is of infinite Cartan type. By Theorem 2.10.2 this is a contradiction to the finiteness of $\mathbf{\Delta}$.

Suppose now that $q_{11}\notin R_3$, and hence $q_{11}^2q_{12}q_{21}=1$. If $q_{11}^aq_{13}q_{31}=1$ for some $a\geq 2$ then (χ,E) is of infinite Cartan type, which is a contradiction. On the other hand, if $q_{11}\in R_{m+1}$ with $m\geq 3$, and $(q_{13}q_{31})^{m+1}\neq 1$, then in Example 2.6.1 with $i=1$ one gets $q'_{23}q'_{32}=(q_{11}^{-1}q_{13}q_{31})^2$. If $q'_{23}q'_{32}\neq 1$ then the Lemma holds by Lemma 3.2.11. If $q'_{23}q'_{32}=1$ then $q_{13}q_{31}=-q_{11}$ which contradicts the finiteness of $\mathbf{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_3)$, Lemma 3.1.7(iii), and relation $(q_{13}q_{31})^{m+1}\neq 1$.

Now turn to the proof of the lemma. By Lemma 3.2.5 one can assume that $(2a+1)\mathbf{e}_1+2\mathbf{e}_2\notin\mathbf{\Delta}$ for all $a\in\mathbb{N}_0$. If $q_{22}=-1$ then the claim of the lemma holds by Step 1. Otherwise $q_{12}q_{21}q_{22}=1$ and either $q_{11}\in R_3$ or $q_{11}^2q_{12}q_{21}=1$. Hence Steps 2 and 3 prove the lemma. ■

Lemma 3.2.13. *If $(\mathbf{\Delta},\chi,E)$ is a connected arithmetic root system then either $\mathcal{D}_{\chi,E}$ appears in rows 9–11 of Table A.2, or it is Weyl equivalent to an arithmetic root system $(\mathbf{\Delta}',\chi',E)$ such that $q'_{23}q'_{32}=1$ and $q'_{11}q'_{13}q'_{31}=1$, where $q'_{ij}=\chi'(\mathbf{e}_i,\mathbf{e}_j)$.*

Proof. By Lemmata 3.2.11 and 3.2.12 it is sufficient to prove the claim under the additional assumption that $q_{11}=-1$ and $q_{23}q_{32}=1$, where $q_{ij}=\chi(\mathbf{e}_i,\mathbf{e}_j)$. By Lemma 3.2.5 one can assume that $q_{22}=-1$ or $q_{12}q_{21}q_{22}=1$. In the first case one can apply Example 2.6.1 with $n=3$ and $i=2$. One gets $p_{21}=-q_{12}q_{21}$, $m_{21}=1$, $p_{23}=1$, $m_{23}=0$, and hence $q'_{11}q'_{12}q'_{21}=-q_{12}q_{21}q_{11}q_{12}^{-2}q_{21}^{-2}q_{12}q_{21}=1$ and $q'_{23}q'_{32}=1$, and therefore the assertion follows using twist equivalence.

Assume now that $q_{11}=-1$, $q_{22}\neq -1$, $q_{33}\neq -1$, and $q_{12}q_{21}q_{22}=1$. If $q_{13}q_{31}q_{33}=1$ then it can happen that $q_{12}q_{21}q_{13}q_{31}=1$, and hence one has an example in row 8 of Table A.2, and the claim of the lemma holds. Otherwise $\mathcal{D}_{\chi,E}$ appears in rows 9–11 of Table A.2. Therefore one can assume additionally that $q_{13}q_{31}q_{33}\neq 1$. Then one can apply Lemma 3.2.8 to conclude that either $q_{12}q_{21}q_{13}q_{31}=1$ or $q_{12}q_{21}q_{13}^2q_{31}^2q_{33}=-1$. If $q_{12}q_{21}q_{13}q_{31}=1$ then using Example 2.6.1 with $n=3$ and $i=1$ one gets $q'_{23}q'_{32}=1$, $q'_{11}=q'_{22}=-1$, and hence the claim of the lemma holds by the first part of the proof. On the other hand, if $q_{12}q_{21}q_{13}^2q_{31}^2q_{33}=-1$ and $q_{12}q_{21}q_{13}q_{31}\neq 1$ then one can use twice Example 2.6.1, first with $i=1$ and then with $i=2$, to obtain a Weyl equivalent arithmetic root system $(\mathbf{\Delta}',\chi',E)$ such that $q'_{22}=q'_{33}=-1$, $q'_{23}q'_{32}\neq 1$, and $q'_{13}q'_{31}=1$, and hence the assertion of the lemma holds again by the first paragraph of the proof. ■

In what follows Theorem 3.2.1 will be reformulated in two pieces and the proofs will be given.

Theorem 3.2.14. *If $\mathcal{D}_{\chi,E}$ appears in Table A.2 then $(\mathbf{\Delta},\chi,E)$ is an arithmetic root system, where $\mathbf{\Delta}=\bigcup\{F\,|\,(\mathrm{id},F)\in W_{\chi,E}\}$. Further, two such arithmetic root systems $(\mathbf{\Delta},\chi,E)$, $(\mathbf{\Delta}',\chi',E)$ are Weyl equivalent if and only if $\mathcal{D}_{\chi,E}$ and $\mathcal{D}_{\chi',E}$ appear in the same row of Table A.2 and can be presented with the same set of fixed parameters.*

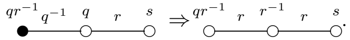
Proof. The second assertion can be checked case by case using Example 2.6.1. Therefore $\mathfrak{D}^{\chi,E}$ is finite for all pairs (χ,E) such that $\mathcal{D}_{\chi,E}$ appears in Table A.2. Now the first assertion follows by direct calculations from Lemmata 2.8.2 and 2.8.3. The structure of $\mathcal{G}_{\chi,E}$ is given in Table A.2. Here $[m]$ denotes the Coxeter group generated by two elements s_1,s_2 and the relations $s_1^2=s_2^2=(s_1s_2)^m=1$, and $[m_1,m_2]$ is the Coxeter group generated by three elements s_1,s_2,s_3 and the relations $s_1^2=s_2^2=s_3^2=(s_1s_2)^{m_1}=(s_1s_3)^2=(s_2s_3)^{m_2}=1$. ■

Theorem 3.2.15. *If $(\mathbf{\Delta},\chi,E)$ is a connected arithmetic root system then $\mathcal{D}_{\chi,E}$ appears in Table A.2.*

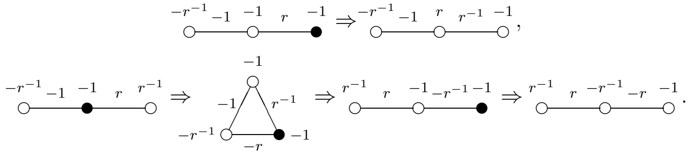
Proof. By Theorem 3.2.14 it suffices to consider $(\mathbf{\Delta},\chi,E)$ up to Weyl equivalence. By Lemma 3.2.13 one can assume that $q_{23}q_{32}=1$ and $q_{11}q_{13}q_{31}=1$. Further, by Lemma 3.2.8 one can suppose that equation $(q_{33}+1)(q_{33}-q_{11})(q_{11}-q_{12}q_{21}q_{33})(q_{11}q_{12}q_{21}-1)=0$ holds.

Step 1. *Up to Weyl equivalence one can assume that $q_{11}q_{13}q_{31}=1$ and $(q_{33}+1)(q_{33}-q_{11})=0$.* Assume that $(q_{33}+1)(q_{11}-q_{33})\neq 0$. If $q_{11}q_{12}q_{21}=1$ then by Lemma 3.2.5 one has $(q_{22}+1)(q_{12}q_{21}q_{22}-1)=0$ and hence the claim of Step 1 holds by exchanging $\mathbf{e}_2$ and $\mathbf{e}_3$. Otherwise the last equation above Step 1 implies that $q_{11}=q_{12}q_{21}q_{33}$. Further, by Lemma 3.2.9 one obtains that either $q_{11}=-1$ or $q_{33}\in R_3$. In the second case ($q_{33}\in R_3$) one can apply Example 2.6.1 with $i=3$ ($q:=q_{11}$, $r:=q_{12}q_{21}$, $s:=q_{22}$). Here and in the following the node corresponding to the basis vector $\mathbf{e}_i$ and the reflection

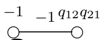
$s_{\mathbf{e}_i,E}$, respectively, will be denoted by a black circle.

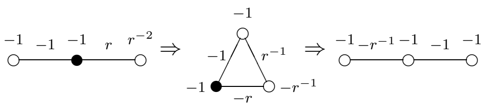


Using additionally equation $(q_{22}+1)(q_{12}q_{21}q_{22}-1)=0$, which follows from Lemma 3.2.5, one obtains an arithmetic root system which satisfies the conditions of Step 1 up to twist equivalence. In the final case $q_{11}=-1$ one has again the equation $(q_{22}+1)(q_{12}q_{21}q_{22}-1)=0$ from Lemma 3.2.5. Thus the claim follows from the following transformations ($r=q_{12}q_{21}$).



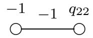
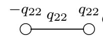
Using Step 1 and Proposition 3.1.6(i) for $\mathbf{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2)$, 9 special cases will be considered which together prove the claim of the theorem. In all of them the vertices of $\mathcal{D}_{\chi,E}$ correspond from left to right to the basis vectors $\mathbf{e}_3,\mathbf{e}_1$, and $\mathbf{e}_2$ of E, respectively.

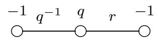
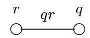
Step 2. *If $q_{11}=q_{13}q_{31}=q_{33}=-1$ then the claim of the theorem holds.* Exchanging $\mathbf{e}_2$ and $\mathbf{e}_3$, Lemma 3.2.8 implies that $q_{22}=-1$ or $q_{12}q_{21}q_{22}=1$ or $q_{12}^2q_{21}^2q_{22}=1$ or $q_{12}q_{21}=-1$. If $q_{12}q_{21}q_{22}=1$ then $\mathcal{D}_{\chi,E}$ appears in one of rows 1,9, or 10 of Table A.2. If $q_{22}=-1$ then the finiteness of $\mathbf{\Delta}(\chi;\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2)$  and Lemma 3.1.7(i) imply that $q_{12}q_{21}\in R_2\cup R_3\cup R_4\cup R_6$. Then $\mathcal{D}_{\chi,E}$ appears in row 1,17,6 and 7 of Table A.2, respectively. If $q_{12}^2q_{21}^2q_{22}=1$ then by the transformations ($r=q_{12}q_{21}$)



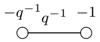
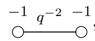
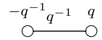
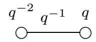
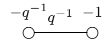
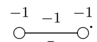
all such arithmetic root systems are Weyl equivalent to one of the previous case.

Finally, if $q_{12}q_{21}=-1$ then the finiteness of $\mathbf{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2)$ and Lemma 3.1.7(i) give that $q_{22}\in R_2\cup R_3\cup R_4\cup R_6$. If $q_{22}\in R_2\cup R_4$ then $\mathcal{D}_{\chi,E}$

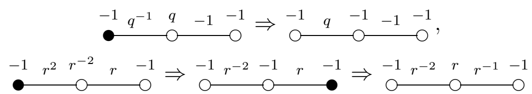
appears in rows 1 and 2 of Table A.2, respectively. If $q_{22}\in R_6$ then (χ,E) is of infinite Cartan type which contradicts Theorem 2.10.2. If $q_{22}\in R_3$ then $\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3\in\mathbf{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2)$  and hence the finiteness of $\mathbf{\Delta}(\chi;\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2)$  contradicts Proposition 3.1.6(i).

Step 3. *If $\mathcal{D}_{\chi,E}$ is*  *with $q^2\neq 1$, $r\neq 1$, then the claim of the theorem holds.* The finiteness of $\mathbf{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1)$  and Proposition 3.1.6(i) imply that $(qr-1)(q^2r-1)(r+1)(qr^2-1)=0$ or $q^3r^3=-1$, $\{q,r\}\cap R_3\neq\{\}$. However if $(qr-1)(q^2r-1)(r+1)(qr^2-1)\neq 0$ and say $q\in R_3$ then by Table A.1 one has $r\in R_3\cup R_4\cup R_{18}\cup R_{30}$ which is a contradiction to $r^3=-1$.

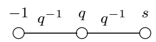
In the case $qr=1$ $\mathcal{D}_{\chi,E}$ appears in row 8 of Table A.2.

If $q^2r=1$ and $r\neq -1$ then the finiteness of $\mathbf{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3)$  and Lemma 3.1.7(ii) imply that $q\in R_3\cup R_5\cup R_6\cup R_8$. If $q\in R_3\cup R_6$ then $\mathcal{D}_{\chi,E}$ appears in row 15 and 17, respectively. On the other hand, if $q\in R_5$ then $\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3\notin\mathbf{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2)$ , but $2\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3\in\mathbf{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3)$ by Lemma 2.2.1(i). However $\mathbf{\Delta}(\chi;2\mathbf{e}_1+2\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1)$  is of infinite Cartan type which is a contradiction to Theorem 2.10.2 and Proposition 2.7.1. Finally, if $q\in R_8$ then $4\mathbf{e}_1+3\mathbf{e}_2+3\mathbf{e}_3\in\mathbf{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1)$  and $4\mathbf{e}_1+3\mathbf{e}_2+\mathbf{e}_3\in\mathbf{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_3)$  which is a contradiction to $\mathbf{\Delta}(\chi;4\mathbf{e}_1+3\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_3)=\mathbf{\Delta}\cap\mathbb{Z}\{\mathbf{e}_3,4\mathbf{e}_1+3\mathbf{e}_2+\mathbf{e}_3\}$ .

It remains to consider the cases $r=-1$ and $qr^2=1$, respectively. Here one can use the transformations



to obtain arithmetic root systems which were already considered in Step 2 and Step 3, respectively.

Step 4. *If $\mathcal{D}_{\chi,E}$ is*  *with $q^2\neq 1$, $s^2\neq 1$, then the claim* *of the theorem holds.* Since $\boldsymbol{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2)$ $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{s}{\circ}$ and $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2)$ $\overset{-1}{\circ}\overset{q^{-1}}{\text{———}}\overset{s}{\circ}$ are finite, Lemma 3.1.8 implies that either $s^m=q$ for $m\in\{1,2,3\}$ or $s\in R_3$. If $s=q$ or $s^2=q$ then $\mathcal{D}_{\chi,E}$ appears in row 4 and row 5 of Table A.2, respectively. If $s^3=q$ then Proposition 3.1.6(i) and the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1+2\mathbf{e}_2)$ $\overset{-s^{-2}}{\circ}\overset{s^{-2}}{\text{———}}\overset{s}{\circ}$ give that either $s^4=-1$ or $s^5=1$, $\{s,-s^{-2}\}\cap R_3\neq\{\}$. The second case contradicts himself. If $s\in R_8$ then one has $2\mathbf{e}_1+3\mathbf{e}_2\in\boldsymbol{\Delta}$ and $2\mathbf{e}_1+3\mathbf{e}_2+2\mathbf{e}_3\notin\boldsymbol{\Delta}(\chi;2\mathbf{e}_1+3\mathbf{e}_2,\mathbf{e}_3)$ $\overset{s^3}{\circ}\overset{s^{-6}}{\text{———}}\overset{-1}{\circ}$.This contradicts to the fact that $2(\mathbf{e}_1+\mathbf{e}_3)+3\mathbf{e}_2\in\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2)$.

Suppose now that $s\in R_3$ and $q^3\neq 1$. Then Proposition 3.1.6(i) and the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1+2\mathbf{e}_2)$ $\overset{-q^{-1}s}{\circ}\overset{q^{-2}s}{\text{———}}\overset{q^{-1}s}{\circ}$ imply that equation $(q^2-s)(q^{-3}s^2-1)(q^{-3}s^2+1)=0$ holds. If $s=q^2$ then $q^3\neq 1$ gives that $q\in R_6$, $s=-q^{-1}$. In this case $\mathcal{D}_{\chi,E}$ appears in row 14 of Table A.2. On the other hand, if $s^2=q^3$ then $s=q^{-3}$, $q\in R_9$, and $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1+2\mathbf{e}_2)$ corresponds to the generalized Dynkin diagram $\overset{-q^{-4}}{\circ}\overset{q^{-5}}{\text{———}}\overset{q^{-4}}{\circ}$. This contradicts Theorem 2.10.2. In the same way $s^2=-q^3$ leads to a contradiction.

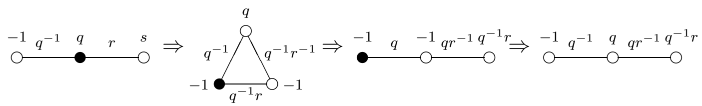
Step 5. *If $\mathcal{D}_{\chi,E}$ is* $\overset{-1}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}\overset{r}{\text{———}}\overset{r^{-1}}{\circ}$ *with $q^2\neq 1$, $r^2\neq 1$, $qr\neq 1$, then the claim of the theorem holds.* By Table A.1 and the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2)$ $\overset{q}{\circ}\overset{r}{\text{———}}\overset{r^{-1}}{\circ}$ one has either $q^mr=1$ with $m\in\{2,3\}$ or $q\in R_{m+1}$ with $m\in\{2,3,4\}$ and $r^{m+1}\neq 1$. If $q^2r=1$ or $q^3r=1$ then $\mathcal{D}_{\chi,E}$ appears in row 6 and 7 of Table A.2, respectively. If $q\in R_5$ then $r\in R_{30}$ and $q=-r^{-3}$ again by the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2)$. Then $\boldsymbol{\Delta}(\chi;2\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3)$ $\overset{-r^4}{\circ}\overset{r^6}{\text{———}}\overset{-1}{\circ}$ is not finite.

If $q\in R_4$ then by the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2)$ one gets either $r\in R_{24}$, $q=r^6$, or $r\in R_8$, $q=r^2$. Since $\chi\chi^{\mathrm{op}}(2\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_3)=-r\neq 1$, one has $3\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3\in\boldsymbol{\Delta}$. In both cases $\boldsymbol{\Delta}(\chi;3\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2)$ $\overset{r^2}{\circ}\overset{qr^2}{\text{———}}\overset{q}{\circ}$ is not finite.

Finally, if $q\in R_3$ then using the list in Appendix A the finiteness of $\boldsymbol{\Delta}(\chi;2\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3)$ $\overset{qr}{\circ}\overset{q}{\text{———}}\overset{-1}{\circ}$ implies that either $q^2r=1$ or $qr=-1$ or $(qr)^2q=1$. The case $q^2r=1$ was already considered, and $r^2=1$ was excluded. Thus $qr=-1$ and then $\mathcal{D}_{\chi,E}$ appears in row 16 of Table A.2.

Step 6. *If $\mathcal{D}_{\chi,E}$ is* $\overset{-1}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}\overset{r}{\text{———}}\overset{s}{\circ}$ *with $q^2\neq 1$, $s^2\neq 1$, $qr\neq 1$, and*

$rs\neq 1$, then the claim of the theorem holds. By Proposition 3.1.6(i) one has $qr^2s=-1$ and $\{q,s\}\cap R_3\neq\{\}$. If $q\in R_3$ then $r\neq q$. The transformations



give that $(\boldsymbol{\Delta},\chi,E)$ is Weyl equivalent to an arithmetic root system which was already considered in the previous steps.

When $s\in R_3$ and $q\notin R_3$ then by Table A.1 there are only three possibilities: either $r\in R_{24}$, $q=r^{-6}$, or $q\in R_{18}$, $r=q^{-2}$, or $q\in R_{30}$, $r=q^{-3}$. In all cases $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_3)$ $\overset{-r^{-1}}{\circ}\overset{qr}{\text{———}}\overset{-1}{\circ}$ is not finite by Table A.1.

Step 7. *If $\mathcal{D}_{\chi,E}$ is* $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}\overset{r}{\text{———}}\overset{-1}{\circ}$ *with $q^2\neq 1$, $r\neq 1$, then the claim of the theorem holds.* By Lemma 3.2.10 one has $(qr-1)(q^2r-1)(q^3r-1)(r^4-1)=0$. If $qr=1$ then $\mathcal{D}_{\chi,E}$ appears in row 4 of Table A.2. If $q^2r=1$ then the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_3,\mathbf{e}_1+\mathbf{e}_2)$ $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-q^{-1}}{\circ}$ and Lemma 3.1.7(iii) imply that $q\in R_3\cup R_4\cup R_6\cup R_8$. If $q\in R_3\cup R_6\cup R_4$ then $\mathcal{D}_{\chi,E}$ appears in row 13 and 3 of Table A.2, respectively. If $q\in R_8$ then $2\mathbf{e}_1+3\mathbf{e}_2+2\mathbf{e}_3\notin\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_3,\mathbf{e}_2)$ $\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{-1}{\circ}$ and $3\mathbf{e}_1+3\mathbf{e}_2+2\mathbf{e}_3\in\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3)$ $\overset{-q^{-1}}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$, but $\boldsymbol{\Delta}(\chi;3\mathbf{e}_1+3\mathbf{e}_2+2\mathbf{e}_3,\mathbf{e}_1)$ $\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{q}{\circ}$ is not finite.

Suppose now that $q^3r=1$. Then Proposition 3.1.6(i) and the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_3)$ $\overset{-q^{-2}}{\circ}\overset{q^{-2}}{\text{———}}\overset{q}{\circ}$ give that $q\in R_8$. In this case $\boldsymbol{\Delta}(\chi;\mathbf{e}_1+\mathbf{e}_2,\mathbf{e}_3)$ $\overset{-q^{-2}}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$ is not finite.

Finally, if $r^4=1$ and $q^mr\neq 1$ for all $m\in\{1,2,3\}$ then $r\in R_2\cup R_4$. The finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_3,2\mathbf{e}_1+\mathbf{e}_2)$ $\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{-q^4r^2}{\circ}$ and Proposition 3.1.6(i) imply that either $r=-1$, $q^4=1$, or $r\in R_4$, $q^4=-1$. Since $q^2r\neq 1$, only the case $q\in R_8$, $q^6r=1$ can appear, but this is a contradiction to the finiteness of $\boldsymbol{\Delta}(\chi;\mathbf{e}_1,\mathbf{e}_2)$.

Step 8. *If $\mathcal{D}_{\chi,E}$ is* $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}\overset{r}{\text{———}}\overset{r^{-1}}{\circ}$ *with $q^2\neq 1$, $r^2\neq 1$, then the claim of the theorem holds.* By Lemma 3.2.10 one has $(qr-1)(q^2r-1)(q^3r-1)=0$. If $qr=1$ or $q^2r=1$ then $\mathcal{D}_{\chi,E}$ appears in row 1 and row 3 of Table A.2, respectively. If $q^3r=1$ then (χ,E) is of infinite Cartan type which is a contradiction to Theorem 2.10.2.

Step 9. *If* $\mathcal{D}_{\chi,E}$ *is* $\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{s}{\circ}$ *with* $q^2 \neq 1$, $s^2 \neq 1$, *and* $q \neq s$, *then the claim of the theorem holds.* If $s^m q^{-1} = 1$ for some $m \geq 2$ then (χ, E) is of Cartan type. Thus by Theorem 2.10.2 one has $m = 2$ and then $\mathcal{D}_{\chi,E}$ appears in row 2 of Table A.2.

Assume now that $s \in R_{m+1}$ with $m \geq 2$, and $q^{m+1} \neq 1$. Then the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2)$ gives that $2 \leq m \leq 4$. Moreover, if $m > 2$ then either $q \in R_{30}$, $s = -q^3$, or $q \in R_{24}$, $s = -q^6$, or $q \in R_8$, $s = -q^2$. In all cases one obtains a contradiction to the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{e}_3)$ $\overset{s^4q^{-1}}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$.

If $m = 2$ then the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + 2\mathbf{e}_2)$ $\overset{s}{\circ}\overset{sq^{-2}}{\text{——}}\overset{sq^{-1}}{\circ}$ and Proposition 3.1.6(i) imply that

$$(sq^{-2} - 1)(s^2q^{-2} - 1)(sq^{-1} + 1)(s^2q^{-3} - 1)(sq^{-5} + 1) = 0.$$

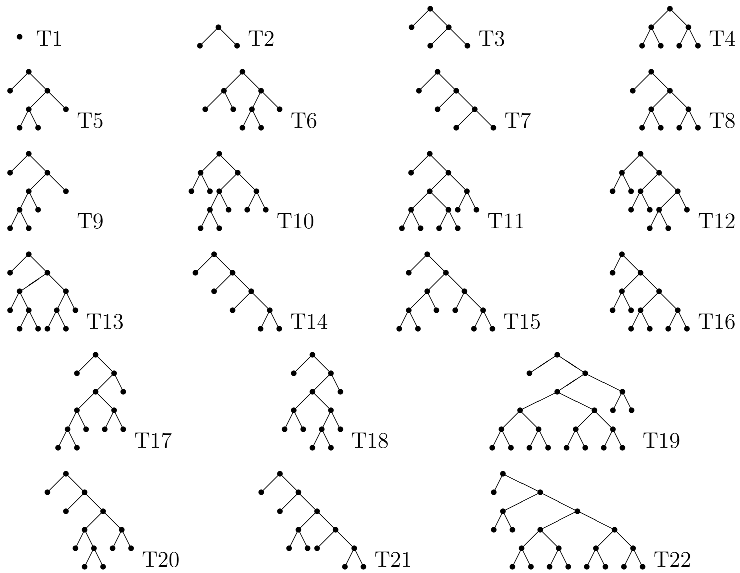
Since $q^3 \neq 1$, this yields $(sq + 1)(sq^{-1} + 1)(sq^3 - 1)(sq^{-5} + 1) = 0$. If $sq = -1$ or $s = -q$ or $sq^3 = 1$ then $\mathcal{D}_{\chi,E}$ appears in row 12, 16 and 18 of Table A.2, respectively. If $s = -q^5$ then either $q \in R_6$, $sq = -1$, or $q \in R_{30}$. In the second case $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + 2\mathbf{e}_2)$ is not finite by Table A.1.

Step 10. *If* $\mathcal{D}_{\chi,E}$ *is* $\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}\overset{r}{\text{——}}\overset{s}{\circ}$ *with* $q^2 \neq 1$, $s^2 \neq 1$, $qr \neq 1$, *and* $rs \neq 1$, *then the claim of the theorem holds.* By Proposition 3.1.6(i) one has $qr^2s = -1$ and $\{q, s\} \cap R_3 \neq \{\}$. The finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3)$ $\overset{-r^{-1}}{\circ}\overset{qr}{\text{——}}\overset{q}{\circ}$ and Proposition 3.1.6(i) imply that $(q^2r - 1)(q^3r - 1) = 0$. By the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1, \mathbf{e}_2)$ if $q^2r = 1$ then $q \in R_{18}$ and if $q^3r = 1$ then $q \in R_{30}$. Both cases contradict to the finiteness of $\mathbf{\Delta}(\chi; \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3)$. ■

Appendix A

Types of Nichols Algebras

On this page we collect all full binary trees which appear as the type of some Nichols algebra of diagonal type studied in the present thesis.



	generalized Dynkin diagrams	fixed parameters	tree
1	$\overset{q}{\circ}\quad\overset{r}{\circ}$	$q, r \in k^*$	T1
2	$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$	$q \in k^* \setminus \{1\}$	T2
3	$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{-1}{\circ}\quad\overset{-1}{\circ}\overset{q}{\text{——}}\overset{-1}{\circ}$	$q \in k^* \setminus \{-1, 1\}$	T2 T2
4	$\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{q^2}{\circ}$	$q \in k^* \setminus \{-1, 1\}$	T3
5	$\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{-1}{\circ}\quad\overset{-q^{-1}}{\circ}\overset{q^2}{\text{——}}\overset{-1}{\circ}$	$q \in k^* \setminus \{-1, 1\}$, $q \notin R_4$	T3 T3
6	$\overset{\zeta}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}\quad\overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\text{——}}\overset{\zeta q^{-1}}{\circ}$	$\zeta \in R_3$, $q \in k^* \setminus \{1, \zeta, \zeta^2\}$	T3 T3
7	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}\quad\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in R_3$	T3 T3
8	$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-\zeta^{2}}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in R_{12}$	T4 T5 T5 T7 T7
9	$\overset{-\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{-\zeta^{2}}{\circ}\quad\overset{-\zeta^{2}}{\circ}\overset{\zeta^{3}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{-1}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in R_{12}$	T4 T5 T7
10	$\overset{-\zeta}{\circ}\overset{\zeta^{-2}}{\text{——}}\overset{\zeta^{3}}{\circ}\quad\overset{\zeta^{3}}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}$	$\zeta \in R_9$	T6 T9 T14
11	$\overset{q}{\circ}\overset{q^{-3}}{\text{——}}\overset{q^3}{\circ}$	$q \in k^* \setminus \{-1, 1\}$, $q \notin R_3$	T8
12	$\overset{\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{\zeta^{-1}}{\circ}\quad\overset{\zeta^{2}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}\quad\overset{\zeta}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}$	$\zeta \in R_8$	T8 T8 T8
13	$\overset{\zeta^{6}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-\zeta^{-4}}{\circ}\quad\overset{\zeta^{6}}{\circ}\overset{\zeta}{\text{——}}\overset{\zeta^{-1}}{\circ}$ $\overset{-\zeta^{-4}}{\circ}\overset{\zeta^{5}}{\text{——}}\overset{-1}{\circ}\quad\overset{\zeta}{\circ}\overset{\zeta^{-5}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in R_{24}$	T10 T13 T17 T21
14	$\overset{\zeta}{\circ}\overset{\zeta^{2}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in R_5$	T11 T16
15	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{3}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in R_{20}$	T11 T11 T16 T16
16	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{\zeta^{5}}{\circ}\quad\overset{\zeta^{3}}{\circ}\overset{-\zeta^{4}}{\text{——}}\overset{-\zeta^{-4}}{\circ}$ $\overset{\zeta^{5}}{\circ}\overset{-\zeta^{-2}}{\text{——}}\overset{-1}{\circ}\quad\overset{\zeta^{3}}{\circ}\overset{-\zeta^{2}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in R_{15}$	T12 T15 T18 T20
17	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in R_7$	T19 T22

Table A.1: Weyl equivalence for rank 2 arithmetic root systems

Using another ordering of Table A.1 Theorem 3.1.1 becomes equivalent to the following.

Let $E = \{\mathbf{e}_1, \mathbf{e}_2\}$ be a basis of $\mathbb{Z}^2$ and χ a bicharacter on $\mathbb{Z}^2$. Set $q_{ij} := \chi(\mathbf{e}_i, \mathbf{e}_j)$ for $i, j \in \{1, 2\}$. Then $W_{\chi,E}$ is full and finite if and only if up to permutation of $\mathbf{e}_1$ and $\mathbf{e}_2$ one of the following is true.

1. $q_{12}q_{21} = 1$.
2. $q_{12}q_{21} \neq 1$, $q_{12}q_{21}q_{22} = 1$, and
   - $q_{11}q_{12}q_{21} = 1$ or
   - $q_{11} = -1$, $q_{12}^2 q_{21}^2 \neq 1$ or
   - $q_{11}^2 q_{12}q_{21} = 1$ or
   - $q_{11}^3 q_{12}q_{21} = 1$, $q_{11}^2 \neq 1$ or
   - $q_{11} \in R_3$, $q_{12}^3 q_{21}^3 \neq 1$ or
   - $q_{12}q_{21} \in R_8$, $q_{11} = (q_{12}q_{21})^2$ or
   - $q_{12}q_{21} \in R_{24}$, $q_{11} = (q_{12}q_{21})^6$ or
   - $q_{12}q_{21} \in R_{30}$, $q_{11} = (q_{12}q_{21})^{12}$.
3. $q_{12}q_{21} \neq 1$, $q_{11}q_{12}q_{21} \neq 1$, $q_{12}q_{21}q_{22} \neq 1$, $q_{22} = -1$, $q_{11} \in R_2 \cup R_3$, and
   - $q_{11} = -1$, $q_{12}^2 q_{21}^2 \neq 1$ or
   - $q_{11} \in R_3$, $q_{12}q_{21} \in \{q_{11}, -q_{11}\}$ or
   - $q_0 := q_{11}q_{12}q_{21} \in R_{12}$, $q_{11} = q_0^4$ or
   - $q_{12}q_{21} \in R_{12}$, $q_{11} = -(q_{12}q_{21})^2$ or
   - $q_{12}q_{21} \in R_9$, $q_{11} = (q_{12}q_{21})^{-3}$ or
   - $q_{12}q_{21} \in R_{24}$, $q_{11} = -(q_{12}q_{21})^4$ or
   - $q_{12}q_{21} \in R_{30}$, $q_{11} = -(q_{12}q_{21})^5$.
4. $q_{12}q_{21} \neq 1$, $q_{11}q_{12}q_{21} \neq 1$, $q_{12}q_{21}q_{22} \neq 1$, $q_{22} = -1$, $q_{11} \notin R_2 \cup R_3$, and
   - $q_{12}q_{21} = q_{11}^{-2}$ or
   - $q_{11} \in R_5 \cup R_8 \cup R_{12} \cup R_{14} \cup R_{20}$, $q_{12}q_{21} = q_{11}^{-3}$ or
   - $q_{11} \in R_{10} \cup R_{18}$, $q_{12}q_{21} = q_{11}^{-4}$ or
   - $q_{11} \in R_{14} \cup R_{24}$, $q_{12}q_{21} = q_{11}^{-5}$ or

   - $q_{12}q_{21} \in R_8$, $q_{11} = (q_{12}q_{21})^{-2}$ or
   - $q_{12}q_{21} \in R_{12}$, $q_{11} = (q_{12}q_{21})^{-3}$ or
   - $q_{12}q_{21} \in R_{20}$, $q_{11} = (q_{12}q_{21})^{-4}$ or
   - $q_{12}q_{21} \in R_{30}$, $q_{11} = (q_{12}q_{21})^{-6}$.
5. $q_{12}q_{21} \neq 1$, $q_{11}q_{12}q_{21} \neq 1$, $q_{12}q_{21}q_{22} \neq 1$, $q_{11} \neq -1$, $q_{22} \in R_3$ and
   - $q_0 := q_{11}q_{12}q_{21} \in R_{12}$, $q_{11} = q_0^4$, $q_{22} = -q_0^2$ or
   - $q_{12}q_{21} \in R_{12}$, $q_{11} = q_{22} = -(q_{12}q_{21})^2$ or
   - $q_{12}q_{21} \in R_{24}$, $q_{11} = (q_{12}q_{21})^{-6}$, $q_{22} = (q_{12}q_{21})^{-8}$ or
   - $q_{11} \in R_{18}$, $q_{12}q_{21} = q_{11}^{-2}$, $q_{22} = -q_{11}^3$ or
   - $q_{11} \in R_{30}$, $q_{12}q_{21} = q_{11}^{-3}$, $q_{22} = -q_{11}^5$.

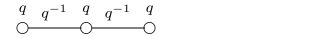
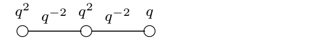
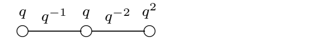
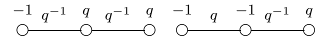
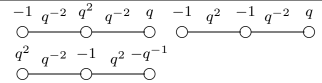
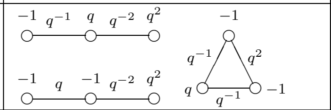
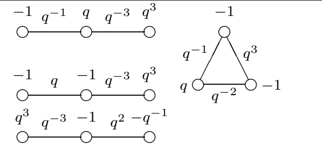
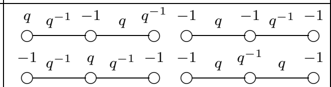
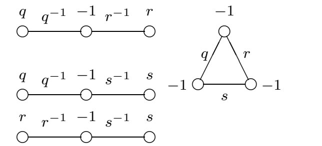
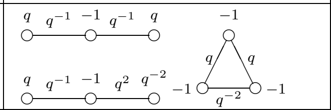
row	gener. Dynkin diagrams	fixed parameters	$\mathcal{G}_{\chi,E}$
1		$q \in k^* \setminus \{1\}$	$[3,4]$
2		$q \in k^* \setminus \{-1,1\}$	$[3,4]$
3		$q \in k^* \setminus \{-1,1\}$	$[3,4]$
4		$q \in k^* \setminus \{-1,1\}$	$\mathbb{Z}_2 \times [3]$
5		$q \in k^* \setminus \{-1,1\}$ $q \notin R_4$	$\mathbb{Z}_2 \times [4]$
6		$q \in k^* \setminus \{-1,1\}$	$\mathbb{Z}_2 \times [4]$
7		$q \in k^* \setminus \{-1,1\}$ $q \notin R_3$	$\mathbb{Z}_2 \times [6]$
8		$q \in k^* \setminus \{-1,1\}$	$\mathbb{Z}_2^3$
9		$q, r, s \in k^* \setminus \{1\}$ $qrs = 1$, $q \neq r$, $q \neq s$, $r \neq s$	$\mathbb{Z}_2^3$
10		$q \in k^* \setminus \{-1,1\}$ $q \notin R_3$	$\mathbb{Z}_2 \times [4]$

Table A.2: Weyl equivalence for connected rank 3 arithmetic root systems

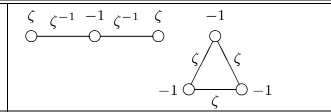
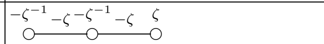
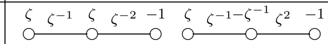
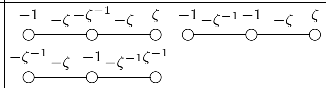
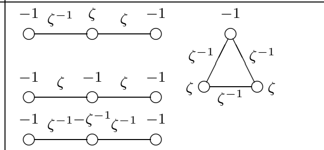
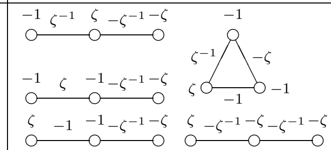
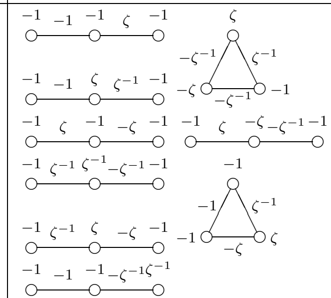
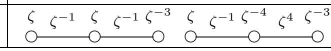
row	gener. Dynkin diagrams	fixed param.	$\mathcal{G}_{\chi,E}$
11		$\zeta \in R_3$	$[3,4]$
12		$\zeta \in R_3$	$[3,4]$
13		$\zeta \in R_3 \cup R_6$	$[3,4]$
14		$\zeta \in R_3$	$\mathbb{Z}_2 \times [4]$
15		$\zeta \in R_3$	$\mathbb{Z}_2 \times [6]$
16		$\zeta \in R_3$	$[6]$
17		$\zeta \in R_3$	$\mathbb{Z}_2^3$
18		$\zeta \in R_9$	$[3,4]$

Table A.2: Weyl equivalence for connected rank 3 arithmetic root systems

Nichols-Algebren von diagonalem Typ und arithmetische Wurzelsysteme

deutsche Zusammenfassung

1. Einleitung

Der Ursprung der Theorie der Hopfalgebren liegt in den Arbeiten von Hopf um 1940 über topologische Eigenschaften von Lie-Gruppen. Er beobachtete, dass für eine zusammenhängende Lie-Gruppe G die Kohomologiegruppe $H^*(G)$ mit Koeffizienten in einem Körper nicht nur die Struktur einer graduierten Algebra hat, die durch die diagonale Abbildung $G \to G \times G$, $g \mapsto (g, g)$, induziert wird, sondern auch eine Komultiplikation $\Delta : H^*(G) \to H^*(G) \otimes H^*(G)$ besitzt, die von der Multiplikationsabbildung $\mathrm{m} : G \times G \to G$ kommt. Hopfalgebren wurden als rein algebraische Objekte seit etwa 1960 studiert. Eine gute Einführung in das Thema kann etwa in [36], [1], und [31] gefunden werden.

Bis in die 80er Jahre waren die meisten bekannten Hopfalgebren entweder kommutativ oder kokommutativ. Das änderte sich schlagartig durch die Entdeckung der quantisierten Kac-Moody-Algebren durch Drinfel'd [13] und Jimbo [25]. Danach wurden neue Verbindungen der Theorie der Hopfalgebren zu verschiedenen Bereichen der Mathematik wie Knotentheorie, Deformationstheorie, Lie-Theorie, Darstellungstheorie, Kombinatorik, und andere, gefunden, und viele Mathematiker wurden auf das Gebiet aufmerksam.

Eine erste Fragestellung in vielen mathematischen Gebieten ist die vollständige Klassifikation der Objekte, die von Interesse sind. In der Theorie der Hopfalgebren ist das ein sehr kompliziertes Problem und nur unter starken Zusatzbedingungen sind vollständige Lösungen bekannt. Bislang sind im Wesentlichen halbeinfache Hopfalgebren, punktierte Hopfalgebren, und Hopfalgebren einer gegebenen Dimension n, wobei n eine kleine Zahl ist oder nur wenige Primfaktoren hat, untersucht worden. Ein sehr guter Übersichtsartikel ist zum Beispiel [2]. Nichols-Algebren, die die zentralen Objekte in dieser Habilitationsschrift sind, treten in natürlicher Art und Weise bei der Klassifikation von punktierten Hopfalgebren mit der Methode von Andruskiewitsch und Schneider [5] auf. Eine Hopfalgebra H über einem Körper k heißt *punktiert*, wenn alle einfachen Unterkoalgebren von H eindimensional sind. Gruppenalgebren, universelle einhüllende Algebren von Lie-Algebren, und quantisierte Kac-Moody-Algebren sind punktiert. Das Koradikal einer punktierten Hopfalgebra H ist eine Gruppenalgebra kG. Die zur Koradikalfiltrierung assoziierte graduierte Koalgebra $\mathrm{gr}\, H$ ist eine graduierte Hopfalgebra und die Unteralgebra der Elemente vom Grad 0 ist kG. Man erhält eine kanonische Projektion $\pi : \mathrm{gr}\, H \to kG$ von Hopfalgebren. Die Menge $R = \{x \in \mathrm{gr}\, H \mid (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\}$, wobei Δ das Koprodukt von $\mathrm{gr}\, H$

bezeichnet, nennt man das Diagramm von H. Es ist eine graduierte, gezopfte Hopfalgebra und $\mathrm{gr}\, H$ ist das Biprodukt (Bosonisierung) $R\#kG$ von R und kG im Sinne von Radford (beziehungsweise Majid). Der Unterraum der Elemente von R vom Grad 0 und vom Grad 1 ist $k1$ beziehungsweise $P(R)$, wobei $P(R)$ die Menge der primitiven Elemente von R bezeichnet. Die von $k1$ und $P(R)$ erzeugte Unteralgebra von R ist eine graduierte, gezopfte Hopf-Unteralgebra und sie ist eine wichtige Invariante von H — sie wird *Nichols-Algebra* genannt. Die Methode von Andruskiewitsch und Schneider funktioniert wie folgt. Bestimme zuerst alle möglichen Nichols-Algebren und die zugehörigen Diagramme R, und finde die dazu passenden Gruppen G derart, dass $R\#kG$ eine graduierte Hopfalgebra ist. Im letzten Schritt muss man alle Hopfalgebren H finden, die der Gleichung $\mathrm{gr}\, H = R\#kG$ genügen.

Die symmetrische und die äußere Algebra eines Vektorraumes, die in vielen Bereichen der Mathematik vorkommen, sind die bekanntesten Beispiele für Nichols-Algebren. Die systematische Untersuchung von Nichols-Algebren begann mit der Arbeit von Nichols [32] über Bialgebren vom Typ Eins. In seiner Arbeit wurden Schiefderivationen definiert und graduierte, punktierte Hopfalgebren mit sehr kleinem Koradikal betrachtet. Unabhängig davon wurde von Woronowicz [38] eine Differentialrechnung auf Quantengruppen initiiert, in der die äußere Algebra zu einem Hopf-Bimodul eine zentrale Rolle spielt. Dieses Objekt ist im Wesentlichen wie in Definition 2.1 durch Erzeuger und Relationen gegeben. Lusztig [29] definierte die quantisierte einhüllende Lie-Algebra $U_q(\mathfrak{n}_+)$, wobei $\mathfrak{n}_+$ der nilpotente Teil einer halbeinfachen Lie-Algebra bezüglich einer festen Wahl von positiven Wurzeln ist, als den Quotienten einer Tensoralgebra nach dem Kern einer symmetrischen Bilinearform. In diesem Zugang wird die Cartan-Matrix zur Definition der Bilinearform benutzt und die positiven Wurzeln werden mit den Erzeugern der Poincaré-Birkhoff-Witt-Basis von $U_q(\mathfrak{n}_+)$ identifiziert. Schauenburg [34] entdeckte, dass Nichols-Algebren und die äußeren Algebren von Woronowicz die selben Objekte beschreiben, und dass Lusztigs Konstruktion einen Spezialfall beschreibt. Ist R das Diagramm einer punktierten Hopfalgebra H, dann hat $P(R)$ die Struktur eines gezopften Vektorraumes (siehe Abschnitt 2). Schauenburg zeigte, dass die durch R gegebene Nichols-Algebra allein durch die Kenntnis des gezopften Vektorraums $P(R)$ mit Woronowicz's Definition beschrieben werden kann (siehe Definition 1). Im Folgenden wird die durch einen gezopften Vektorraum (V, σ) gegebene Nichols-Algebra durch $\mathcal{B}(V)$ bezeichnet. In der Arbeit [5] schlugen Andruskiewitsch und Schneider die weiter oben beschriebene Methode zur Klassifikation von punktierten Hopfalgebren vor, die sich mittlerweile als Erfolg versprechend erwiesen hat. Sie führten die Klassifikation unter den Zusatzbedingungen durch, dass der Grundkörper k algebraisch abgeschlossen ist, $\operatorname{char} k = 0$ gilt, und das Diagramm von H ein quantisierter linearer Raum ist. Als Folgerung konnten sie sowohl alle punktierten Hopfalgebren der Dimension p^3 bestimmen, wobei p eine ungerade Primzahl ist, als auch die, deren Koradikal abelsch ist und Kodimension p oder p^2 hat. Man sagt, dass ein d-dimensionaler gezopfter Vektorraum (V, σ), wobei $\sigma \in \operatorname{Aut}_k(V \otimes V)$, von *diagonalem Typ* ist, wenn V eine Basis $\{x_i \mid 1 \le i \le d\}$ mit folgender Eigenschaft besitzt: für jedes $i, j \in \{1, 2, \ldots, d\}$ gibt es ein $q_{ij} \in k^* = k \setminus \{0\}$ derart, dass $\sigma(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ gilt. Rosso [33] zeigte eine beachtenswerte Klassifikationsaussage: Ist V ein gezopfter Vektorraum über $\mathbb{C}$ von diagonalem Typ ist und sind die Strukturkonstanten q_{ii} von V alle positiv, dann hat $\mathcal{B}(V)$ genau dann endliche Gel'fand-Kirillov-Dimension, wenn V von endlichem Cartan-Typ ist (siehe dazu Abschnitt 4). Man beachte, dass diese Nichols-Algebren alle unendlich-dimensional sind. Rosso hat auch eine Methode zur Konstruktion von $U_q(\mathfrak{n}_+)$ durch Induktion über den Rang der Cartan-Matrix angegeben. Andruskiewitsch und Graña [3] zeigten die Äquivalenz verschiedener Definitionen einer Nichols-Algebra. Graña [17] klassifizierte alle Nichols-Algebren, deren Dimension kleiner als 32 ist. Er [16] sowie Milinski und Schneider [30] zeigten, dass Nichols-Algebren freie Moduln über ihren Nichols-Unteralgebren sind. In den Arbeiten [3], [17], [30] und [4] findet man einige Beispiele für endlich-dimensionale Nichols-Algebren mit nicht-abelscher Strukturgruppe. Kharchenko [27] zeigte, dass jede Hopfalgebra, die von einer Gruppenalgebra einer abelschen Gruppe und von einem gezopften Vektorraum von diagonalem Typ erzeugt wird, eine eingeschränkte Poincaré-Birkhoff-Witt-Basis besitzt. Um so eine Basis zu konstruieren, benutzte Kharchenko Lyndon-Wörter. Er gab auch Bedingungen an die Höhe der Erzeuger der Poincaré-Birkhoff-Witt-Basis. Andruskiewitsch und Schneider [6] bewiesen unter einer schwachen Bedingung an die Strukturkonstanten q_{ij}, dass endlich-dimensionale Nichols-Algebren vom Cartan-Typ sind. Darüberhinaus bestimmten sie alle koradikal-graduierten Hopfalgebren der Dimension p^4 mit ungerader Primzahl p. Im Übersichtsartikel [7] klassifizierten Andruskiewitsch und Schneider punktierte Hopfalgebren H unter der Voraussetzung, dass das Diagramm von H vom Cartan-Typ A_n ist. Sie gaben auch eine Liste der damals bekannten, auf Nichols [32] und Graña [17] zurückgehenden endlich-dimensionalen Nichols-Algebren von Rang 2 an, die von diagonalem aber nicht vom Cartan-Typ sind. Durch Kharchenkos Artikel [27] inspiriert zeigte Ufer [37], dass gezopfte Hopfalgebren, die von einem gezopften Vektorraum mit triangulärer Zopfung erzeugt werden, eine Poincaré-Birkhoff-Witt-Basis besitzen. Diese Aussage ist wesentlich für die Bestimmung von punktierten Hopfalgebren mit endlicher Gel'fand-Kirillov-Dimension und abelschem Koradikal. In [4] untersuchten Andruskiewitsch und Graña die Beziehung zwischen Nichols-Algebren von nichtabelschem Gruppentyp und dem topologischen Begriff eines Racks. In einer Reihe von Arbeiten [18, 19, 22, 21, 20, 23] wurde dann die Klassifikation von endlich-dimensionalen Nichols-Algebren von diagonalem Typ und Rang 2 und 3, und die der Nichols-Algebren vom Cartan-Typ über Körper der Charakteristik 0 durchgeführt, sowie für die Nichols-Algebren von Rang 2 explizit ein System von definierenden Relationen bestimmt. Die ersten sehr technischen Beweise benutzten Kharchenkos Poincaré-Birkhoff-Witt-Basis und Schiefderivationen. Später wurden Weyl-Gruppoide und arithmetische Wurzelsysteme eingeführt, was zu einer Vereinfachung der Beweise führte, und die Klassifikation in Rang 3 ermöglichte. Unlängst bestimmten Andruskiewitsch und Schneider [8] alle punktierten endlich-dimensionalen Hopfalgebren H unter den folgenden Bedingungen: der Grundkörper ist algebraisch abgeschlossen und hat Charakteristik 0, das Koradikal kG von H ist abelsch, und die Primfaktoren von $|G|$ sind größer als 7. In diesem Artikel wurden Resultate aus [18, 19, 22] benutzt. Die Kenntnis der Nichols-Algebren von diagonalem Typ ist auch hilfreich, wenn die Strukturgruppe nicht abelsch ist, wie das in [17] und [9] demonstriert wurde.

Die Untersuchungen in der vorliegenden Habilitationsarbeit wurden hauptsächlich durch folgende Frage von Andruskiewitsch [2], [5] motiviert.

Frage 5.9. Entscheide für einen beliebigen gezopften Vektorraum V von diagonalem Typ, ob $\mathcal{B}(V)$ endlich-dimensional ist. Wenn ja, bestimme $\dim_k \mathcal{B}(V)$, und gebe eine „schöne" Darstellung von $\mathcal{B}(V)$ mit Hilfe von Erzeugern und Relationen an.

Das Ziel dieser Habilitationsschrift ist, neue Strukturen zu entwickeln, die eine vollständige Beantwortung der obigen Frage ermöglichen. Die Komplexität des Problems wird deutlich, wenn man beachtet, welcher Art die bislang bekannten partiellen Antworten sind. Wenn die Dimension von V eins ist, so ist $\mathcal{B}(V)$ genau dann endlich-dimensional, wenn q_{11} eine primitive m-te Einheitswurzel ist, wobei $2 \le m < \infty$ [32]. In diesem Fall gilt $\mathcal{B}(V) = k[x]/(x^m)$. Wenn V vom Cartan-Typ ist, so hat man die Klassifikationen von Rosso, Andruskiewitsch und Schneider. Darüberhinaus waren einige einfache Beispiele vom Rang 2 bekannt, die nicht vom Cartan-Typ sind. Es war auch bekannt, dass Nichols-Algebren von diagonalem Typ viele natürliche Nichols-Unteralgebren besitzen, und damit bietet sich eine Klassifikation durch Induktion nach der Dimension von V (wie für halbeinfache Lie-Algebren) an. Dafür war natürlich entscheidend, dass man die obige Frage im Fall $\dim_k V = 2$ beantworten kann, siehe [2, Question 5.40]. Eines der wichtigsten Probleme war, eine Methode zu finden, mit deren Hilfe man schnell sehen kann, wann eine Nichols-Algebra unendlich-dimensional ist. Die Hauptidee in [19] war, dass man in $\mathcal{B}(V)$ nach solchen Nichols-Unteralgebren $\mathcal{B}(W)$ sucht, für die $W \subset V$ nicht zwingend gilt. Glücklicherweise konnten für alle unendlich-dimensionalen Nichols-Algebren vom Rang 2 solche unendlich-dimensionalen Nichols-Unteralgebren gefunden werden. Die Endlichkeit der restlichen Nichols-Algebren wurde in [18], siehe auch Theorem 2, mit Hilfe von Kharchenkos Resultaten [27] durch Bestimmung einer hinreichend großen Menge konkreter Relationen bewiesen. Es stellte sich heraus, dass unter diesen endlich-dimensionalen Algebren viele die selbe Dimension haben. Daher lag die Vermutung nahe, dass eine zusätzliche Struktur existiert, die dieses Phänomen erklärt. Andererseits existierte eine überraschende Verbindung zwischen Lie-Theorie und der Theorie der Nichols-Algebren von diagonalem Typ, und die Weyl-Gruppe spielt eine zentrale Rolle in der Theorie der halbeinfachen Lie-Algebren. Es lag also nahe, nach einer Struktur zu suchen, die die Rolle der Weyl-Gruppe übernehmen könnte. Im Hinblick auf die Tatsache, dass Kharchenkos eingeschränkte Poincaré-Birkhoff-Witt-Basis einer Nichols-Algebra $\mathcal{B}(V)$ von diagonalem Typ $\mathbb{Z}^d$-graduiert ist, wobei $d = \dim_k V$, ist es naheliegend, die $\mathbb{Z}^d$-Grade der Erzeuger der Poincaré-Birkhoff-Witt-Basis als Verallgemeinerungen von positiven Wurzeln zu betrachten. In [22] wurde gezeigt, dass der Vektorraum $\mathcal{B}(V) \otimes k[y_i]$, wobei $y_i \in V^*$ homogen bezüglich der $\mathbb{Z}^d$-Graduierung ist, eine Algebrenstruktur besitzt, die eine Nichols-Algebra $\mathcal{B}(W)$ von diagonalem Typ und Rang d enthält, die verschieden von $\mathcal{B}(V)$ ist. Zusätzlich konnte das Wurzelsystem von $\mathcal{B}(W)$ mit Hilfe jenes von $\mathcal{B}(V)$ beschrieben und die Nichols-Algebra $\mathcal{B}(V)$ aus der Kenntnis von W als gezopfter Vektorraum zurück gewonnen werden. Diese Tatsache führte zur Definition des Weyl-Gruppoids, der Weyl-Äquivalenz, und des arithmetischen Wurzelsystems. Im Gegensatz zu den Elementen der Weyl-Gruppe einer halbeinfachen Lie-Algebra sind die eines Weyl-Gruppoids keine Permutationen von Wurzeln, sondern bilden eine einzelne Menge von einfachen Wurzeln auf eine andere derartige Menge ab. Aus diesem Grund ist die Komposition von Elementen eines Weyl-Gruppoids nur

partiell möglich. Die durch die Gruppoid-Wirkung induzierte Äquivalenz von gezopften Vektorräumen ermöglicht die drastische Vereinfachung der Klassifikation endlich-dimensionaler Nichols-Algebren von Rang 2 und diagonalem Typ. Bei der Betrachtung von Nichols-Algebren höheren Ranges hat es sich in [23] als vorteilhaft erwiesen, Untersysteme von arithmetischen Wurzelsystemen zu studieren. Die in [23] verwendeten Methoden scheinen geeignet zu sein, eine unendlich-dimensionale Nichols-Algebra immer als solche zu erkennen.

Durch Lusztigs Konstruktion [29] der quantisierten einhüllenden Algebren von halbeinfachen Lie-Algebren ist klar, dass die quantisierten Serre-Relationen ausreichen, eine Nichols-Algebra vom Cartan-Typ mit Erzeugern und Relationen zu definieren. Für allgemeinere Nichols-Algebren von diagonalem Typ benötigt man jedoch kompliziertere Relationen. Um die vielfältigen endlich-dimensionalen Nichols-Algebren von Rang 2 und diagonalem Typ besser handhaben zu können, wurde in [18] zunächst angenommen, dass die Grade der Erzeuger von Kharchenkos eingeschränkter Poincaré-Birkhoff-Witt-Basis durch einen vollen Binärbaum kodiert werden können. Diese Annahme wurde durch die Klassifikation in [19] gerechtfertigt, und ihre Notwendigkeit durch die Untersuchungen über arithmetische Wurzelsysteme theoretisch begründet. Durch die explizite Bestimmung einer hinreichenden Menge von Relationen, die man am Binärbaum ablesen kann, konnte gezeigt werden, dass gewisse Nichols-Algebren von diagonalem Typ endlich-dimensional sind. Die Dimensionen der graduierten Komponenten kann man ebenfalls dem Binärbaum entnehmen. Das gab zusammen mit den Resultaten in [19] eine vollständige Antwort auf die obige Frage von Andruskiewitsch für den Fall, dass $\dim_k V = 2$ gilt. Das analoge Problem für den Fall $\dim_k V > 2$ ist immer noch offen.

In der vorliegenden Habilitationsschrift werden folgende Bezeichnungen verwendet. Die Symbole $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, und $\mathbb{R}$ stehen für die natürlichen Zahlen ohne 0, dieselben mit der 0, die ganzen Zahlen, beziehungsweise die reellen Zahlen. Wenn nicht anders erwähnt, ist k ein Körper der Charakteristik 0. Für $m \in \mathbb{N}_0$ und $q \in k^* = k \setminus \{0\}$ werden die Definitionen $(m)_q := \sum_{i=0}^{m-1} q^i$ und $(m)_q^! := \prod_{i=1}^{m} (i)_q$ benutzt. Tensorproduktzeichen $\otimes$ ohne unteren Index bedeuten immer Tensorprodukt über k. Das m-fache Tensorprodukt von V mit sich selbst wird mit $V^{\otimes m}$ bezeichnet, und für die Tensoralgebra $\bigoplus_{m=0}^{\infty} V^{\otimes m}$ wird das Symbol $V^{\otimes}$ verwendet.

2. Definitionen und Notation

Sei k ein Körper der Charakteristik 0, $d \in \mathbb{N}$, und $\chi : \mathbb{Z}^d \times \mathbb{Z}^d \to k^* = k \setminus \{0\}$ ein Bicharakter auf $\mathbb{Z}^d$, das heißt, die Gleichungen

$$\begin{gathered} \chi(0,a) = \chi(a,0) = 1, \\ \chi(a+b,c) = \chi(a,c)\chi(b,c), \qquad \chi(a,b+c) = \chi(a,b)\chi(a,c) \end{gathered} \tag{1}$$

gelten für alle $a,b,c \in \mathbb{Z}^d$. Sei $E = \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ eine Basis von $\mathbb{Z}^d$. Das *verallgemeinerte Dynkin-Diagramm* des Paares (χ, E) ist ein Graph $\mathcal{D}_{\chi,E}$ mit folgenden Eigenschaften:
(i) es gibt eine bijektive Abbildung ϕ von $\{1,2,\dots,d\}$ auf die Menge der Ecken von $\mathcal{D}_{\chi,E}$,
(ii) für alle $i \in \{1,2,\dots,d\}$ ist die Ecke $\phi(i)$ mit q_{ii} markiert,
(iii) für alle $i,j \in \{1,2,\dots,d\}$ ist die Zahl n_{ij} der Kanten zwischen $\phi(i)$ und $\phi(j)$ entweder 0 oder 1. Wenn $i = j$ oder $\chi(\mathbf{e}_i,\mathbf{e}_j)\chi(\mathbf{e}_j,\mathbf{e}_i) = 1$ dann gilt $n_{ij} = 0$, ansonsten $n_{ij} = 1$ und die Kante ist mit $\chi(\mathbf{e}_i,\mathbf{e}_j)\chi(\mathbf{e}_j,\mathbf{e}_i)$ markiert.

Ein gezopfter Vektorraum ist ein Paar (V,σ), wobei V ein Vektorraum über k ist und die Abbildung $\sigma \in \mathrm{Aut}_k(V \otimes V)$ der Zopf-Gleichung

$$(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)$$

auf $V \otimes V \otimes V$ genügt.

Sei (V,σ) ein d-dimensionaler gezopfter Vektorraum, wobei $d \in \mathbb{N}$.

Definition 1. Für $m \geq 2$ und $j \in \mathbb{N}$ seien $S_m \in \mathrm{End}_k(V^{\otimes m})$ und $S_{1,j} \in \mathrm{End}_k(V^{\otimes j+1})$ die Abbildungen

$$\begin{aligned} S_m &= \prod_{j=1}^{m-1} (\mathrm{id}^{\otimes m-j-1} \otimes S_{1,j}), \\ S_{1,j} &= \mathrm{id} + \sigma_{12}^{-1} + \sigma_{12}^{-1}\sigma_{23}^{-1} + \cdots + \sigma_{12}^{-1}\sigma_{23}^{-1}\cdots\sigma_{j,j+1}^{-1}. \end{aligned}$$

Dann ist der Unterraum $S = \bigoplus_{m=2}^{\infty} \ker S_m$ der Tensoralgebra $V^{\otimes}$ ein zweiseitiges Ideal, und die Algebra $\mathcal{B}(V) = V^{\otimes}/S$ heißt die zu (V,σ) assoziierte *Nichols-Algebra*. Das eindeutige maximale Ideal von $\mathcal{B}(V)$ wird durch $\mathcal{B}(V)^+$ bezeichnet.

In der vorliegenden Habilitationsschrift werden nur gezopfte Vektorräume (V,σ) von diagonalem Typ betrachtet. In diesem Fall besitzt V eine Basis $\{x_1,\dots,x_d\}$ derart, dass $\sigma(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ für jedes $i,j \in \{1,2,\dots,d\}$

und für gewisse Zahlen $q_{ij} \in k^*$ gilt. Die Zahlen q_{ij} heißen *Strukturkonstanten* von V. Sei $E = \{\mathbf{e}_1,\dots,\mathbf{e}_d\}$ eine Basis des $\mathbb{Z}$-Moduls $\mathbb{Z}^d$. Dann sind $V^{\otimes}$ und $\mathcal{B}(V)$ $\mathbb{Z}^d$-graduierte Algebren, wobei die Erzeuger x_i Grad $\deg x_i = \mathbf{e}_i$ haben. Sei χ der zu V und E assoziierte eindeutige Bicharakter, für den $\chi(\mathbf{e}_i,\mathbf{e}_j) = q_{ij}$ gilt für alle i,j. Der Einfachheit halber wird auch $\chi(x,y)$ anstelle von $\chi(\deg x, \deg y)$ geschrieben, wenn x und y homogene Elemente von $V^{\otimes}$ oder $\mathcal{B}(V)$ sind. Das verallgemeinerte Dynkin-Diagramm von (χ,E) wird auch als das verallgemeinerte Dynkin-Diagramm von V bezeichnet.

Betrachte die wohlgeordnete Menge $X := \{\alpha_1,\dots,\alpha_d\}$ mit d Elementen, wobei $\alpha_i < \alpha_j$ für $i < j$, als Alphabet. Seien $\mathbb{X}$ und $\mathbb{X}^+$ die Mengen der Wörter beziehungsweise nichtleeren Wörter, die mit Hilfe des Alphabets X gebildet werden können. Dann induziert die Relation $<$ auf X die lexikographische Ordnung auf $\mathbb{X}$, die ebenfalls durch das Symbol $<$ bezeichnet wird. Die Länge $|u|$ eines Wortes u ist die Anzahl der Zeichen in u. Ein Wort $u \in \mathbb{X}^+$ heißt *Lyndon-Wort*, wenn für jede Zerlegung $u = vw$ mit $v,w \in \mathbb{X}^+$ die Relation $vw < wv$ gilt. Jedes Lyndon-Wort u mit $|u| > 1$ besitzt eine Zerlegung in ein Produkt $u = vw$ von zwei Lyndon-Wörtern v und w. Diejenige unter diesen, für die $|v|$ minimal ist, heißt *Shirshow-Zerlegung* of u.

Assoziiere induktiv zu jedem Lyndon-Wort $u \in \mathbb{X}$ das Element $[u] \in \mathcal{B}(V)$ wie folgt. Definiere $[\alpha_i] := x_{d+1-i}$ für $1 \leq i \leq d$, und

$$[u] := [w][v] - \chi([w],[v])[v][w],$$

falls $u = vw$ die Shirshow-Zerlegung von u ist. Außerdem sei h_u folgendermaßen gewählt. Wenn m die kleinste Zahl ist derart, dass $[u]^m$ sich als Linearkombination von Produkten $[u_1][u_2]\cdots[u_i]$ schreiben läßt, wobei $i \in \mathbb{N}$ und alle u_j sind Lyndon-Wörter mit $u < u_j$, und wenn außerdem entweder $m = 1$ gilt oder $\chi([u],[u])$ ist eine primitive m-te Einheitswurzel, dann sei $h_u = m$. Ansonsten definiere $h_u := \infty$. Sei $B := \{u \in \mathbb{X} \,|\, u \text{ ist Lyndon-Wort}, h_u > 1\}$.

Theorem 1. *[27, Theorem 2] Die Menge* $\{[u_1]^{n_1}[u_2]^{n_2}\cdots[u_i]^{n_i} \,|\, i \in \mathbb{N}_0, u_j \in B, u_i < \cdots < u_2 < u_1, n_j < h_{u_j} \forall j\}$ *ist eine Basis des Vektorraumes* $\mathcal{B}(V)$.

Die Endlichkeit der Mengen $\mathbf{\Delta}^+(\mathcal{B}(V)) := \{\deg[u] \,|\, u \in B\} \subset \mathbb{Z}^d$ und $\mathbf{\Delta}(\mathcal{B}(V)) := \mathbf{\Delta}^+(\mathcal{B}(V)) \cup -\mathbf{\Delta}^+(\mathcal{B}(V))$ wird später mit der Hilfe der arithmetischen Wurzelsysteme wichtige Rückschlüsse auf die Strukturkonstanten q_{ij} ermöglichen.

3. Nichols-Algebren und Binärbäume

Um Nichols-Algebren mit Hilfe einer relativ einfachen Menge von Relationen beschreiben zu können, müssen einige technischen Definitionen eingeführt werden.

Sei T ein voller Binärbaum mit Wurzel $\mathbf{r}$, das heißt ein nichtleerer Baum, in dem jeder Knoten genau Null oder zwei Kinder hat. Bezeichne $N_0(T)$ und $N_2(T)$ die Menge aller Knoten von T mit 0 beziehungsweise 2 Kindern. Seien $\bar{N}_2(T)$ und $\bar{N}(T)$ die disjunkte Vereinigung von $N_2(T)$ beziehungsweise $N_0(T) \cup N_2(T)$ mit einer zweielementigen Menge $\{\text{‘}L\text{’}, \text{‘}R\text{’}\}$. Für ein $a \in N_2(T)$ bezeichne a_{L} das linke und a_{R} das rechte Kind von a. Definiere induktiv über den Abstand von $\mathbf{r}$ die Abbildungen $(\cdot)^{\mathrm{L}}, (\cdot)^{\mathrm{R}} : N_0(T) \cup N_2(T) \to \bar{N}_2(T)$ durch die Vorschrift $\mathbf{r}^{\mathrm{L}} := \text{‘}L\text{’}$, $\mathbf{r}^{\mathrm{R}} := \text{‘}R\text{’}$, ${a_{\mathrm{R}}}^{\mathrm{L}} := a$, ${a_{\mathrm{L}}}^{\mathrm{R}} := a$, ${a_{\mathrm{L}}}^{\mathrm{L}} := a^{\mathrm{L}}$, ${a_{\mathrm{R}}}^{\mathrm{R}} := a^{\mathrm{R}}$ für alle $a \in N_2(T)$. Der Binärbaum T wird dazu benutzt, Elemente der Tensoralgebra $V^{\otimes}$ zu kodieren, wobei V ein gezopfter Vektorraum von diagonalem Typ ist. Für $a \in \bar{N}(T)$ sei $\tau_0(a) \in V^{\otimes}$ das folgende Element:

- $\tau_0(\text{‘}L\text{’}) := x_2$, $\tau_0(\text{‘}R\text{’}) := x_1$.
- Wenn $a \in N_0(T) \cup N_2(T)$, dann sei
  $\tau_0(a) := \tau_0(a^{\mathrm{R}})\tau_0(a^{\mathrm{L}}) - \chi(\tau_0(a^{\mathrm{R}}), \tau_0(a^{\mathrm{L}}))\tau_0(a^{\mathrm{L}})\tau_0(a^{\mathrm{R}})$.

Im folgenden Theorem werden die Abkürzungen $\chi(a,b) := \chi(\tau_0(a), \tau_0(b))$ und $p_a := \chi(a,a)^{-1}$ für $a, b \in \bar{N}(T)$ verwendet. Die Zahl $\ell^{\mathrm{R}}(b)$ für ein $b \in N_2(T)$ sei 1, wenn ${b^{\mathrm{R}}}_{\mathrm{L}} \neq b$, ansonsten definiere $\ell^{\mathrm{R}}(b) := \ell^{\mathrm{R}}(b^{\mathrm{R}}) + 1$. Für $a \in \bar{N}(T)$ sei

$$\lambda(a) := \begin{cases} 0 & \text{wenn } a \in \{\text{‘}L\text{’}, \text{‘}R\text{’}\}, \\ \chi(\text{‘}L\text{’}, \text{‘}R\text{’})^{-1} - \chi(\text{‘}R\text{’}, \text{‘}L\text{’}) & \text{wenn } a = \mathbf{r}, \\ \chi(a^{\mathrm{L}}, a^{\mathrm{R}})^{-1} - \chi(a^{\mathrm{R}}, a^{\mathrm{L}}) + \lambda(b) & \text{sonst}, \end{cases}$$

wobei b der Vater von a ist. Weiterhin sei

$$\mu(b) := \begin{cases} \lambda(b) & \text{wenn } \ell^{\mathrm{R}}(b) = 1, \\ \lambda(b)\mu(b^{\mathrm{R}}) & \text{sonst} \end{cases}$$

für $b \in N_2(T)$.

Das Hauptresultat des ersten Kapitels dieser Habilitationsschrift ist die folgende Aussage.

Theorem 2. *Sei V ein gezopfter Vektorraum von diagonalem Typ mit $\dim_k V = 2$. Angenommen, das verallgemeinerte Dynkin-Diagramm von V kommt in Tabelle A.1 vor und ist mit dem vollen Binärbaum Tn markiert, wobei n aus $\{1, 2, \dots, 22\}$ und Tn wie im Anhang A ist. Dann ist $\mathcal{B}(V)$ vom Typ Tn, das heißt, die Menge*

$$\Big\{ \prod_{a \in \bar{N}_2(T)} \tau(a)^{i_a} \,\Big|\, 0 \le i_a < \operatorname{ord} \chi(a,a) \quad \forall a \in \bar{N}_2(T) \Big\} \tag{2}$$

bildet eine Basis von $\mathcal{B}(V)$. Die Algebra $\mathcal{B}(V)$ ist der Quotient von $V^{\otimes}$ nach dem durch die Menge

$$\begin{aligned} &\{\tau_0(a) \mid a \in N_0(T)\} \cup \{\tau_0(a)^{\operatorname{ord} p_a} \mid a \in \bar{N}_2(T), 2 \le \operatorname{ord} p_a < \infty\} \cup \\ &\quad \{\tau_0(b)\tau_0(c^{\mathrm{L}}) - \chi(b, c^{\mathrm{L}})\tau_0(c^{\mathrm{L}})\tau_0(b) - \mu(b)/(\ell^{\mathrm{R}}(b)+1)^!_{p_c}\tau_0(c)^{\ell^{\mathrm{R}}(b)+1} \mid \\ &\qquad b \in N_2(T), c := b^{\mathrm{L}} \in N_2(T)\} \end{aligned} \tag{3}$$

erzeugten Ideal. Insbesondere ist die Nichols-Algebra $\mathcal{B}(V)$ endlich-dimensional, wenn q_{11}, q_{22} und $q_{12}q_{21}$ Einheitswurzeln sind, und q_{11} und q_{22} beide nicht 1 sind.

Im obigen Produkt in (2) werden die Elemente von $\bar{N}_2(T)$ derart angeordnet, dass vorne ‘L’ und hinten ‘R’ steht, und dazwischen die Knoten von $N_2(T)$ aufsteigend nach den Zahlen $m_1(a)/m_2(a)$ sortiert sind, wobei $\deg \tau_0(a) = m_1(a)\mathbf{e}_1 + m_2(a)\mathbf{e}_2$. Die Zahl $\operatorname{ord} \chi(a,a)$ ist die Ordnung von $\chi(a,a)$ in der multiplikativen Gruppe k^* wenn $\chi(a,a) \neq 1$ eine Einheitswurzel ist, und ∞ sonst.

4. Arithmetische Wurzelsysteme

Sei $d \in \mathbb{N}$. Bezeichne $\widetilde{W}$ die Menge aller Paare (T, F), wobei F eine Basis von $\mathbb{Z}^d$ ist, und $T \in \operatorname{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$. Die partielle Verknüpfung $\circ : \widetilde{W} \times \widetilde{W} \to \widetilde{W}$, wobei

$$(T_1, F_1) \circ (T_2, F_2) := \begin{cases} (T_1T_2, F_2) & \text{wenn } T_2(F_2) = F_1, \\ \text{nicht definiert} & \text{sonst}, \end{cases}$$

definiert die Struktur eines Gruppoids auf $\widetilde{W}$.

Sei F eine Basis von $\mathbb{Z}^d$, χ ein Bicharakter auf $\mathbb{Z}^d$, und wähle $\mathbf{f}' \in F$. Angenommen, für alle $\mathbf{f}'' \in F \setminus \{\mathbf{f}'\}$ existieren die Zahlen

$$m(\mathbf{f}', \mathbf{f}'') := \min\{m \in \mathbb{N}_0 \,|\, \chi(\mathbf{f}', \mathbf{f}')^m \chi(\mathbf{f}', \mathbf{f}'')\chi(\mathbf{f}'', \mathbf{f}') = 1 \text{ oder } \chi(\mathbf{f}', \mathbf{f}')^{m+1} = 1,\ \chi(\mathbf{f}', \mathbf{f}') \neq 1\}. \tag{4}$$

Wähle $m(\mathbf{f}', \mathbf{f}') = -2$ und definiere die Abbildung $s_{\mathbf{f}',F} \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$ durch

$$s_{\mathbf{f}',F}(\mathbf{f}'') := \mathbf{f}'' + m(\mathbf{f}', \mathbf{f}'')\mathbf{f}'.$$

Definition 2. Sei E eine Basis von $\mathbb{Z}^d$ und χ ein Bicharakter auf $\mathbb{Z}^d$. Sei $W_{\chi,E}$ das kleinste Untergruppoid von $\widetilde{W}$, das (id, E) enthält, und folgender Eigenschaft genügt: wenn $(\mathrm{id}, F) \in W_{\chi,E}$ für eine Basis F von $\mathbb{Z}^d$ und $s_{\mathbf{f},F}$ ist definiert für ein $\mathbf{f} \in F$, so liegen auch $(s_{\mathbf{f},F}, F)$ und $(\mathrm{id}, s_{\mathbf{f},F}(F))$ in $W_{\chi,E}$. Das Gruppoid $W_{\chi,E}$ wird als das zum Paar (χ, E) assoziierte *Weyl-Gruppoid* bezeichnet.

Weyl-Gruppoide bilden die wichtigste Grundlage für die Untersuchungen in den Kapiteln 2 und 3 der vorliegenden Arbeit. Das Weyl-Gruppoid $W_{\chi,E}$ heißt *voll*, wenn $s_{\mathbf{f},F}$ wohldefiniert ist für jede Basis F von $\mathbb{Z}^d$ mit $(\mathrm{id}, F) \in W_{\chi,E}$ und für jedes $\mathbf{f} \in F$.

Definition 3. Ein Tripel $(\boldsymbol{\Delta}, \chi, E)$, wobei $\boldsymbol{\Delta} \subset \mathbb{Z}^d$, heißt *arithmetisches Wurzelsystem*, wenn $W_{\chi,E}$ voll und endlich ist und die Gleichung $\boldsymbol{\Delta} = \bigcup\{F \,|\, (\mathrm{id}, F) \in W_{\chi,E}\}$ gilt. Die Kardinalität von E wird auch als Rang von $(\boldsymbol{\Delta}, \chi, E)$ bezeichnet.

Arithmetische Wurzelsysteme sind in ihren Eigenschaften ähnlich zu Wurzelsystemen, die in der Theorie der halbeinfachen Lie-Algebren auftreten.
Sei $(\boldsymbol{\Delta}, \chi, E)$ ein arithmetisches Wurzelsystem. Für eine Basis F von $\mathbb{Z}^d$ mit $(\mathrm{id}, F) \in W_{\chi,E}$ schreibe $\boldsymbol{\Delta}_F^+ := \boldsymbol{\Delta} \cap \mathbb{N}_0 F$.

Proposition 3. *Sei $(\boldsymbol{\Delta}, \chi, E)$ ein arithmetisches Wurzelsystem und F eine Basis von $\mathbb{Z}^d$ derart, dass $(\mathrm{id}, F) \in W_{\chi,E}$. Dann gilt $\boldsymbol{\Delta} = \boldsymbol{\Delta}_F^+ \cup -\boldsymbol{\Delta}_F^+$.*

Proposition 4. *Sei $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ eine Basis von $\mathbb{Z}^d$ und bezeichne $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ das Skalarprodukt auf $\mathbb{R}^d$ mit $\langle \mathbf{e}_i, \mathbf{e}_j \rangle_{\mathbb{R}^d} = \delta_{ij}$ (Kroneckers δ). Wenn $(\boldsymbol{\Delta}, \chi, E)$ ein arithmetisches Wurzelsystem ist, dann gibt es für jedes $\beta \in \mathbb{Z}^d$, das der Relation $\langle \beta, \mathbf{e} \rangle_{\mathbb{R}^d} \neq 0$ für jedes $\mathbf{e} \in \boldsymbol{\Delta}$ genügt, eine eindeutig bestimmte Basis $F = \{\mathbf{f}_1, \ldots, \mathbf{f}_d\}$ von $\mathbb{Z}^d$ derart, dass $(\mathrm{id}, F) \in W_{\chi,E}$ und $\langle \mathbf{f}_i, \beta \rangle_{\mathbb{R}^d} > 0$ für jedes i.*

Die Klassifikation endlich-dimensionaler Nichols-Algebren lässt sich mit Hilfe des folgenden Theorems auf die der arithmetischen Wurzelsysteme zurückführen.

Theorem 5. *Sei $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ eine Basis von $\mathbb{Z}^d$ und $(\boldsymbol{\Delta}, \chi, E)$ ein arithmetisches Wurzelsystem. Für den gezopften Vektorraum V von diagonalem Typ, dessen Dimension d und Strukturkonstanten $q_{ij} := \chi(\mathbf{e}_i, \mathbf{e}_j)$ sind, gilt $\boldsymbol{\Delta}(\mathcal{B}(V)) = \boldsymbol{\Delta}$. Genauer, in Kharchenkos Poincaré-Birkhoff-Witt-Basis aus Theorem 1 gibt es zu jedem $\mathbf{e} \in \boldsymbol{\Delta}$ genau ein $u \in B$ mit $\deg[u] = \mathbf{e}$ und es gilt $h_u = \mathrm{ord}\,\chi(\mathbf{e}, \mathbf{e})$.*
Umgekehrt, wenn V ein d-dimensionaler gezopfter Vektorraum von diagonalem Typ ist und $\boldsymbol{\Delta}^+(\mathcal{B}(V))$ ist endlich, dann ist $(\boldsymbol{\Delta}(\mathcal{B}(V)), \chi, E)$ ein arithmetisches Wurzelsystem, wobei $\chi : \mathbb{Z}^d \times \mathbb{Z}^d \to k^*$ der zu V und E assoziierte Bicharakter ist (siehe Abschnitt 2). Diese Korrespondenzen sind invers zueinander.

Man sagt, dass ein d-dimensionaler gezopfter Vektorraum V vom *Cartan-Typ* ist, wenn es für jedes $i \neq j$ eine nichtpositive ganze Zahl a_{ij} derart gibt, dass die Strukturkonstanten von V den Gleichungen $q_{ii}^{a_{ij}} = q_{ij}q_{ji}$ genügen. In diesem Fall wählt man $a_{ii} := 2$ und man nimmt an, dass der Betrag von a_{ij} minimal ist für jedes $i \neq j$. Die Matrix $(a_{ij})_{i,j \in \{1,\ldots,d\}}$ wird als die Cartan-Matrix von V bezeichnet. Die Klassifikation endlich-dimensionaler Nichols-Algebren vom Cartan-Typ ist vollständig lösbar mit Hilfe arithmetischer Wurzelsysteme.

Theorem 6. *Sei $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ eine Basis von $\mathbb{Z}^d$ und χ ein Bicharakter auf $\mathbb{Z}^d$. Sei (V, σ) der gezopfte Vektorraum von diagonalem Typ mit Strukturkonstanten $q_{ij} = \chi(\mathbf{e}_i, \mathbf{e}_j)$. Angenommen, V ist vom Cartan-Typ und $C := (a_{ij})_{i,j \in \{1,2,\ldots,d\}}$ sei die Cartan-Matrix von V.*
(i) Wenn C nicht von endlichem Typ ist, dann ist $\boldsymbol{\Delta}(\mathcal{B}(V))$ unendlich.
(ii) Wenn C von endlichem Typ ist, dann kann $\boldsymbol{\Delta}(\mathcal{B}(V))$ mit der Menge der Wurzeln der zu C assoziierten halbeinfachen Lie-Algebra identifiziert werden. Die Zahlen h_u, die in Theorem 1 vorkommen, hängen in jeder zusammenhängenden Komponente von C nur von der Länge der zugehörigen positiven Wurzel ab.

Wenn die Menge $\{\chi(\mathbf{e}, \mathbf{e}) \,|\, \mathbf{e} \in E\}$ die -1 oder eine primitive dritte Einheitswurzel enthält, dann ist ein arithmetisches Wurzelsystem $(\boldsymbol{\Delta}, \chi, E)$ nicht notwendig vom Cartan-Typ, und eine vollständige Klassifikation ist wesentlich komplizierter. Hier ist die Betrachtung von Untersystemen hilfreich.

Sei $(\mathbf{\Delta}, \chi, E)$ ein arithmetisches Wurzelsystem und $H \subset \mathbb{R}^d$ ein Unterraum von $\mathbb{R}^d$. Definiere $\Gamma := \mathbb{R}(\mathbf{\Delta} \cap H) \cap \mathbb{Z}^d$.

Proposition 7. *Es gibt eine eindeutig bestimmte Basis E_H des Unterraums $\mathbb{R}(\mathbf{\Delta} \cap H)$ von $\mathbb{R}^d$ derart, dass die Relationen $E_H \subset \mathbf{\Delta}^+ \cap H \subset \mathbb{R}_+ E_H$ gelten, wobei $\mathbb{R}_+ = \{r \in \mathbb{R} \,|\, r > 0\}$. Das Tripel $(\mathbf{\Delta} \cap H, \chi|_{\Gamma \times \Gamma}, E_H)$ ist ein arithmetisches Wurzelsystem.*

Darüberhinaus ist es hilfreich, Äquivalenzrelationen von arithmetischen Wurzelsystemen zu definieren. Man sagt, dass zwei arithmetische Wurzelsysteme $(\mathbf{\Delta}_1, \chi_1, E_1)$ und $(\mathbf{\Delta}_2, \chi_2, E_2)$ vom Rang d *Weyl-äquivalent* sind, wenn es ein Element $(T, E_1) \in W_{\chi_1, E_1}$ und eine Bijektion $\varphi \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^d)$ mit $\varphi(T(E_1)) = E_2$ derart gibt, dass für jedes $\mathbf{e} \in \mathbb{Z}^d$ die Gleichung

$$\chi_2(\varphi(T(\mathbf{e})), \varphi(T(\mathbf{e}))) = \chi_1(T(\mathbf{e}), T(\mathbf{e}))$$

gilt. Die arithmetischen Wurzelsysteme $(\mathbf{\Delta}_1, \chi_1, E_1)$ und $(\mathbf{\Delta}_2, \chi_2, E_2)$ heißen *twist-äquivalent* wenn sie Weyl-äquivalent sind mit $T = \mathrm{id}$.

Im Kapitel 3 wird eine vollständige Klassifikation von arithmetischen Wurzelsystemen der Ränge $d = 2$ und $d = 3$ angegeben. Es ist einfach zu zeigen, dass man nur solche Paare (χ, E) studieren muss, für die das verallgemeinerte Dynkin-Diagramm ein zusammenhängender Graph ist.

Theorem 8. *Twist-Äquivalenzklassen arithmetischer Wurzelsysteme vom Rang 2 stehen in eineindeutiger Beziehung zu verallgemeinerten Dynkin-Diagrammen, die in Tabelle A.1 vorkommen. Zwei solche arithmetische Wurzelsysteme sind genau dann Weyl-äquivalent, wenn die entsprechenden verallgemeinerten Dynkin-Diagramme in derselben Zeile von Tabelle A.1 erscheinen und mittels der selben Parameter dargestellt werden können.*

Theorem 9. *Twist-Äquivalenzklassen zusammenhängender arithmetischer Wurzelsysteme vom Rang 3 stehen in eineindeutiger Beziehung zu verallgemeinerten Dynkin-Diagrammen, die in Tabelle A.2 vorkommen. Zwei solche arithmetische Wurzelsysteme sind genau dann Weyl-äquivalent, wenn die entsprechenden verallgemeinerten Dynkin-Diagramme in derselben Zeile von Tabelle A.2 erscheinen und mittels der selben Parameter dargestellt werden können.*

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Erklärung zu Kenntnis und Anerkennung der gültigen Habilitationsordnung

Hiermit erkläre ich, die zum Zeitpunkt der Stellung dieses Habilitationsantrages gültige Habilitationsordnung der Fakultät für Mathematik und Informatik der Universität Leipzig zu kennen und anzuerkennen.

Leipzig, den 2. Dezember 2005

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Dr. István Heckenberger