FREDHOLM THEORY IN POLYFOLDS: WORKSHOP IN LEIPZIG AN OVERVIEW

May 3rd , 2005

1. Monday

1.1. **Lecture 1 (Hofer).** Gromov-Witten theory, Floer-Theory, Contact-Homology, or more generally Symplectic Field Theory are theories build on the study of certain compactified moduli spaces, or even infinite families of such spaces. These moduli spaces are measured and the data is encoded in convenient ways, quite as often as a so-called generating function. Common features include:

1) The moduli spaces are solutions of elliptic PDE's quite often exhibiting compactness problems, at least as seen from a more classical analytical viewpoint.
2) Very often these moduli spaces, when they are not compact admit nontrivial compactifications usually based on surviving analytic phenomena carrying names like "Bubbling-off", "Stretching the Neck", "Blow-up", "Breaking of Trajectories" hinting to violent analytic behavior.
3) In problems like Floer-Theory, Contact-Homology or Symplectic Field Theory precisely the algebraic structures of interest are those created by this "violent analytic behavior" and its "taming" by finding a workable compactification. In fact the algebra is created by the fact that many different moduli spaces interact with each other in a complicated way.

We begin with the shortcomings of classical Fredholm theory. The classical Fredholm theory can be viewed as the study of Fredholm sections of some Banach bundle $Y \to X$. For definiteness we assume that Y is Banach space bundle over the Banach manifold X. Let us denote the fiber over $x \in X$ by Y_x. If $f(x) = 0$ we can build the linearisation $f'(x) : T_xX \to Y_x$ and if $f'(x)$ is surjective we have a solution manifold near x in fact inheriting its manifold structure as a submanifold of the (big) ambient space. It is worthwhile to ask the question if it is not unnecessary luxury that the ambient space has a lot of "hard structure"

whereas we only seem to use little of it in order to obtain a smooth structure on the solution set $f^{-1}(0)$ (assuming transversality)? This question is very much justified, since in many cases, once the solution spaces are constructed, the ambient spaces are discarded and considered irrelevant. Hence a generalization of the Fredholm theory has to address the following question, which we formulate on a level which is relevant for a conceptual formulation of SFT as a "Theory of infinitely many interacting Fredholm operators":
What (perhaps new) structures do we need on the ambient space and bundle (with a preferred section called 0) to talk about transversality and an abstract perturbation theory for a section f so that at points of transversality the solution set $f^{-1}(0)$ carries in a natural way the structure of a smooth orbifold with boundary with corners? In addition we require the theory to be so general that in applications the compactified moduli spaces in Gromov-Witten theory, Floer theory or SFT would be the solution sets of the generalized Fredholm operators. Analyzing the before-mentioned theories it becomes immediately clear that one has to address a certain number of very serious issues. For example if one of the "violent analytical phenomena" occur any natural candidate for an ambient space seem to have locally varying dimensions. Hence, if we think of a manifold-type theory, the local models cannot be open sets in some Banach or Fréchet space. They need to be more general. If we still want to talk about a linearisation of a problem, which, as every analyst knows, has its century-tested benefits, we should look for some class of local models which in some way admit tangent spaces. Moreover there are some other unpleasant phenomena to deal with like dividing out by families of diffeomorphisms acting on the domain of maps where we all know that such actions (for example on any Banach space set-up will always be only continuous, but never smooth). In addition, the applications like SFT, will require the theory to have certain features of a theory of (infinite-dimensional) orbifolds with boundaries with corners. Summarizing there is a whole basket of issues which call for a more general theory. If we have a look at our list of requirements it seems that the problem of finding an adequate theory is "over-determined". Surprisingly, however, there is such a general Fredholm theory and even more surprisingly it is not much more difficult than the classical one. In this new theory we can formalize new structures which could not be formalized before. This gives a unified perspective on a variety of theories in symplectic geometry. It also seems that the theory should have applications in other fields as well, since the addressed analytical issues arise in geometric pde's of Riemannian geometry as well as the theory of nonlinear pde in general.

All the new concepts will be illustrated in the problem classes by the special case of Morse Theory. In this case everything is very transparent, but yet all difficulties already arise here.

I will describe three models which one should keep in in mind

Gromov-Witten Theory (Illustrates the analytical difficulties, and the role of symmetries and the occurrence of groupoids)

Morse-Theory (Illustrates some of the analytical difficulties and the ideas around operations)

Symplectic field theory to be described later will contain all the difficulties.

The material is taken from a series of lecture notes which are being written jointly with K. Wysocki and E. Zehnder. The first Volume is available in some preliminary form for this workshop.

Volume I. Preprint In the first part of the current volume, we introduce the new calculus and develop the functional analysis and differential geometry needed in order to construct our new spaces. Let us remind the reader that most of the constructions in differential geometry are functorial. Hence if we introduce new spaces as local models which have some kind of tangent spaces, and if we can define smooth maps between these spaces and their tangent maps, then the validity of the chain rule allows us to carry out most differential geometric constructions provided we have smooth (in the new sense) partitions of unity. Using the new local models for spaces we construct the M polyfolds. On these new spaces, we develop a nonlinear Fredholm theory and prove several variants of the implicit function theorem. All the concepts are illustrated by an application to classical Morse theory. Let us, however, note the following. Having the notion of an M-polyfold we can generalize the notion of a Lie-groupoid and define polyfolds by copying the groupoid approach to orbifolds. This is straightforward and allows us, with some of the results in volume II, to develop the Gromov-Witten theory, where in case of transversality the moduli spaces have natural smooth (in the classical sense) structures.

Volume II. In preparation In this volume, we construct a polyfold set-up for Symplectic Field theory. As a by-product, we obtain a polyfold set-up for Gromov-Witten theory. We show that there are natural polyfolds and polyfold bundles over them so that the Cauchy-Riemann operator is a Fredholm section in our new sense. The zero set is the union of all compactified moduli spaces. Polyfolds are the orbifold generalisation of M-polyfolds and have descriptions in terms of a theory of polyfold-groupoids, where the notion of manifold is replaced by that of an M-polyfold.

Volume III. In preparation Here we develop the Fredholm theory in polyfolds with operations. We illustrate it by several applications. The easiest and very instructive application is again Morse theory. A second application is the Contact homology.

Volume IV. In preparation This volume is entirely devoted to Symplectic Field theory, which is obtained as an application of a theory which one might call Fredholm theory in polyfold groupoids.

1.2. Lecture 2: New Concepts of Smooth Structures (Hofer).

(Volume I, Chap.1) Definition of a sc-structure on a Banach Space

Definition 1.1. *Let E be a Banach space. A sc-smooth structure or sc-structure on E is given by a nested sequence of Banach spaces E_m, $m \in \mathbb{N}$, satisfying*

1) *For $m \leq n$ the space E_n is a linear subspace of E_m and $E_0 = E$.*
2) *The inclusion $E_n \to E_m$ for $m < n$ is a compact operator with dense image.*
3) *The vector space E_∞ defined by*
$$E_\infty = \bigcap_{m \in \mathbb{N}} E_m$$
is dense in every E_m.

For $i \geq 0$ we denote by E^i the space E_i with the sc-structure $(E^i)_k = E_{i+k}$.

One can define sc-subspaces, linear sc-operators, direct sums, sc-splittings $E = F \oplus_{sc} G$ etc.
One can define the induced structure on an open subset U by $U_m = E_m \cap U$. Also we can define U^i.

The Tangent Space of the sc-space U is
$$TU = U^1 \oplus E.$$

sc^0-maps: $\varphi : U \to F$ is sc^0 if for all m $\varphi(U_m) \subset F_m$ and all these maps are continuous.

sc^1-maps: There are two equivalent definitions. Here is the first one

Definition 1.2. *Let E and F be sc-smooth Banach spaces and $U \subset E$ an open subset. A sc^0-map $f : U \to F$ is said to be sc^1 provided the following holds:*

1) *For every* $m \geq 1$ *the induced map*
$$f : U_m \to F_{m-1}$$
is of class C^1. *In particular the derivative gives the continuous map*
$$U_m \to L(E_m, F_{m-1}) : x \to Df(x).$$
2) *For* $x \in U_m$ *and* $m \geq 1$ *the map* $Df(x)$ *induces (see below for an explanation) a continuous linear operator* $Df(x) : E_{m-1} \to F_{m-1}$ *and the resulting map*
$$U_m \times E_{m-1} \to F_{m-1} : (x, h) \to Df(x)h$$
is continuous.

Here is the second

Definition 1.3. *Let* E *and* F *be sc-smooth Banach spaces and let* $U \subset E$ *be an open subset. An* sc^0*-map* $f : U \to F$ *is said to be* $\mathbf{sc^1}$ *or of* **class** $\mathbf{sc^1}$ *if the following conditions hold true.*

(1) *For every* $x \in U_1$ *there exists a linear map* $Df(x) \in \mathcal{L}(E_0, F_0)$ *satisfying for* $h \in E_1$
$$\frac{1}{\|h\|_1}\|f(x+h) - f(x) - Df(x)h\|_0 \to 0 \quad \text{as } \|h\|_1 \to 0.$$
(2) *The* **tangent map** $Tf : TU \to TF$ *defined by*
$$Tf(x, h) = (f(x), Df(x)h)$$
is an sc^0*-map.*

If f is sc^1 can define the tangent map
$$Tf : TU \to TF : (Tf)(x, h) = (f(x), Df(x)h).$$
The important observation is the validity of a chain rule.

Theorem 1.4 (Chain Rule)**.** *If* $f : U \to F$ *and* $g : V \to G$*, with* $f(U) \subset V$ *are* sc^1*, so is* $g \circ f$ *and*
$$T(g \circ f) = (Tg) \circ (Tf).$$

If one recalls that differential geometry is functorially build from having the notion of a smooth map, the chain rule, the existence of smooth partitions of unity and functorial constructions on vector spaces, it becomes apparent that one build everything in the same way around sc-smooth maps (with some modifications). Within this frame-work the surprising fact is that we can build in a quite natural way (the splicing construction, see later) new smooth spaces with varying dimensions. These will serve as the local models for new types of smooth spaces. Again one has functoriality so that one can develop a differential geometry on these new spaces. All ambient spaces occurring in Morse-Homology, Floer-Theory, Gromov-Witten, Contact Homology and more generally SFT can be build on these local models.

- Mention example of the shift and rotation on a cylinder which is important for SFT (Abbas describes the special case in "Problem 1")
- Introduce the notion of sc-manifold
- Mention corner recognition which is stated for polyfolds later.

1.3. Problem 1: Smoothness of the Shift and some related maps (Abbas). (Vol. I, Chap 1)

- Example of a sc-Structure and the $\mathbb{R}$-shift operator. This example shows that a map which barely is continuous in the usual sense is indeed sc-smooth. It is shown that all results would fail for alternative concepts of smoothness.

- Mention the technical theorem

Theorem 1.5. *Let $\varphi : (0, 1] \to [0, \infty)$ be the exponential gluing profile and define for a non-zero complex number a with $|a| < 1$ the associated data (R, ϑ) in the usual way. Then the following three maps are sc-smooth*

$$\Gamma_i : B \oplus E \to E$$

by

(a) *Let $f_1 : \mathbb{R} \to \mathbb{R}$ be smooth and constant outside of a compact interval so that $f_1(+\infty) = 0$. Define $\Gamma_1(0, h)(s, t) = f_1(-\infty)h(s, t)$ and $\Gamma_1(a, h)(s, t) = f_1(s - \frac{R}{2})h(s, t)$.*
(b) *Let $f_2 : \mathbb{R} \to \mathbb{R}$ be a smooth map which is constant outside of a compact interval so that $f_2(-\infty) = 0$. Define $\Gamma_2(0, h) = 0$ and $\Gamma_2(a, h)(s, t) = f_2(s - \frac{R}{2})h(s, t)$.*
(c) *Let $f_3 : \mathbb{R} \to \mathbb{R}$ be a compactly supported smooth map and define $\Gamma_3(0, h) = 0$ and $\Gamma(a, h)_3(s, t) = f_3(s - \frac{R}{2})h(s - R, t - \vartheta)$.*

Only indicate proofs for

Lemma 1.6. *Let $F = H^2(\mathbb{R}, \mathbb{R}^n)$ equipped with the sc-structure where level m consists of regularity $(m + 2, \delta_m)$-maps. Using the exponential* *gluing profile* φ *let* $R=\varphi(r)$. *Any of the following three maps*

$$\Gamma_i:[0,1)\oplus F\to F,\ i=1,2,3$$

is sc-smooth:

1) *Let* $f_1:\mathbb{R}\to\mathbb{R}$ *be a smooth map which is outside of a compact interval constant so that* $f_1(+\infty)=0$ *and define* $\Gamma_1(0,h)(s)=f_1(-\infty)h(s)$ *and* $\Gamma_1(r,h)(s)=f_1(s-\frac{R}{2})h(s)$.
2) *Let* $f_2:\mathbb{R}\to\mathbb{R}$ *be a smooth map which is outside of a compact interval constant so that* $f_2(-\infty)=0$ *and define* $\Gamma_2(0,h)=0$ *and* $\Gamma_2(r,h)(s)=f_2(s-\frac{R}{2})h(s)$.
3) *Let* $f_3:\mathbb{R}\to\mathbb{R}$ *be a compactly supported smooth map and define* $\Gamma_3(0,h)=0$ *and* $\Gamma_3(r,h)(s)=f_3(s-\frac{R}{2})h(s-R)$.

1.4. **Lecture 3: Splicings and Polyfolds (Hofer).** (Vol I, Chap 2)

• Splicings and their tangents.

• M-polyfolds.

• The degeneracy function, faces and face-structured.

• Polyfold Groupoids, étale and proper (proper is defined different in infinite dimensions, but is equivalent to the standard definition in the finite-dimensional case manifold case.)

• Strong Bundles, sc^+-sections.

• Mention bundles over Polyfold Groupoids.

1.5. **Lecture 4: Concrete Splicings and Gluings I(Wehrheim).** (Vol. I, Chap. 2)

Example showing dimension jumps and example of finite-dimensional polyfolds. Splicings in Morse-theory. Splicings for SFT to be continued next lecture.

1.6. **Problem 2:Splicing and Polyfolds (Dupont, Chance, McDuff).** (Vol. I, Chap. 1)

• Construct M-polyfold charts and indicate what to do in Manifolds.

2. Tuesday

2.1. **Lecture 5: Concrete Splicing and Gluings II(Wehrheim).** (Vol. II, see also Vol. I, Chap. 2)

• Splicings for the base and the fiber.

Denote for $\sigma \geq 0$ and $m \geq 0$ by $H^{m,\sigma}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$ the Hilbert space of functions

$$u : \mathbb{R}^{\pm} \times S^1 \to \mathbb{R}^N$$

with distributional partial derivative up to order m weighted by $e^{\sigma|s|}$ belonging to L^2:

$$[(s,t) \to (D^{\alpha}u)(s,t)e^{\sigma|s|}] \in L^2(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n}) \text{ for } |\alpha| \leq m.$$

Further we define $H^{m,\sigma}_c(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$ to be the Hilbert space of functions $u : \mathbb{R}^{\pm} \times S^1 \to \mathbb{R}^{2n}$ so that

$$u - c_0 \in H^{m,\sigma}(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^N)$$

for a suitable constant $c_0 \in \mathbb{R}^{2n}$. We call c_0 the asymptotic constant of u. Let us define for an integer $k \geq 0$ and a real number τ

$$\| q \|_{k,\tau} = \left[\sum_{|\alpha| \leq k} \| (D^{\alpha}q)e^{\tau|s|} \|^2_{L^2} \right]^{\frac{1}{2}}.$$

Here $\| \cdot \|_{L^2}$ is the standard L^2-norm. If $u \in H^{m,\sigma}_c(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$ we can write it as $u = c + r$ where c is the asymptotic constant and define a norm by

$$\| u \|_{H^{m,\sigma}_c} = \left[\| r \|^2_{m,\sigma} + |c|^2 \right]^{\frac{1}{2}}.$$

Consider the sc-Hilbert space E consisting of all pairs (η^+, η^-) so that $\eta^{\pm} \in H^{3,\delta_0}_c(\mathbb{R}^{\pm} \times S^1, \mathbb{R}^{2n})$, where $\delta_0 \in (0, 2\pi)$, and the asymptotic constants are the same. We equip E with the sc-structure for which the level m corresponds to regularity $(m+3, \delta_m)$, where δ_m is a sequence satisfying

$$0 < \delta_0 < \delta_1 < \delta_2 < \ldots.$$

Clearly

Proposition 2.1. *The sequence of nested spaces E_m defines a sc-smooth structure on E.*

The norms $\| \cdot \|^E_m$ are defined by

$$\| (\eta^+, \eta^-) \|^E_m = \left[\| r^+ \|^2_{m+3,\delta_m} + \| r^- \|^2_{m+3,\delta_m} + |c|^2 \right]^{\frac{1}{2}}.$$

Next we will construct a suitable splicing for E. For the following construction fix a smooth cut-off function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying

$$\beta(s) = 1 \text{ for } s \leq -1$$
$$\beta'(s) < 0 \text{ for } s \in (-1, 1)$$
$$\beta(s) + \beta(-s) = 1.$$

Let us first define the complete gluing operation. Given $(\eta^+,\eta^-)\in E$ define the plus-gluing as a map

$$\oplus_a(\eta^+,\eta^-):[0,R]\times S^1\to\mathbb{R}^{2n}$$

by the following formula

$$\oplus_a(\eta^+,\eta^-)(s,t)=\beta(s-\frac{R}{2})\eta^+(s,t)+(1-\beta(s-\frac{R}{2}))\eta^-(s-R,t-\vartheta).$$

The minus-gluing is a map

$$\ominus_a(\eta^-,\eta^+):\mathbb{R}\times S^1\to\mathbb{R}^{2n}$$

where

$$\begin{aligned}&\ominus_a(\eta^-,\eta^+)(s,t)\\ =\;&-(1-\beta(s-\frac{R}{2}))(\eta^+(s,t)-\frac{1}{2}([\eta^+]_R+[\eta^-]_{-R})\\ &+\beta(s-\frac{R}{2})(\eta^-(s-R,t-\vartheta)-\frac{1}{2}([\eta^+]_R+[\eta^-]_{-R}).\end{aligned}$$

Let us observe that $\oplus_a(\eta^+,\eta^-)$ is of class $H^3([0,R]\times S^1,\mathbb{R}^{2n})$ if $(\eta^+,\eta^-)\in E$. Further $\ominus_a(\eta^+,\eta^-)$ consists of a map with antipodal asymptotic constants so that

$$(s,t)\to\ominus_a(\eta^+,\eta^-)(s,t)-(1-2\beta(s-\frac{R}{2}))c_0$$

belongs to $H^{3,\delta_0}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$. Here c_0 is the asymptotic constant. With other words $\ominus_a(\eta^+,\eta^-)$ belongs to the space $H_c^{3,\delta_0}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ consisting of maps η with $\eta\pm c\in H_c^{3,\delta_0}(\mathbb{R}^\pm\times S^1)$ for a suitable constant c. Hence the asymptotic constants of η are antipodal.

Let us define the shifted plus-gluing by

$$\boxplus_a(\eta^+,\eta^-)(s,t)=\oplus_a(\eta^+,\eta^-)(s+\frac{R}{2},t+\frac{\vartheta}{2}),$$

where $(s,t)\in[-\frac{R}{2},\frac{R}{2}]\times S^1$. Similarly the shifted minus-gluing

$$\boxminus_a(\eta^+,\eta^-)(s,t)=\ominus_a(\eta^+,\eta^-)(s+\frac{R}{2},t+\frac{\vartheta}{2}).$$

For most of our later construction the "gluing" rather than the "shifted gluing" is more convenient. However, for the following estimates the shifted gluing is more useful. Let us also remark the following. The subspaces Q_a and P_a of E defined by

$$Q_a=\{(\eta^+,\eta^-)\,|\,\boxminus_a(\eta^+,\eta^-)=0\}\text{ and }P_a=\{(\eta^+,\eta^-)\,|\,\boxplus_a(\eta^+,\eta^-)=0\}$$

or the same ones, say $\hat{Q}_a$ and $\hat{P}_a$, defined by using $\oplus_a$ and $\ominus_a$. Also note that these are complementary subspaces preserving the sc-structure

$$E = Q_a \oplus P_a.$$

For $a \in B$, the open unit ball in $\mathbb{C}$, define the sc-Hilbert space G^a as follows. We put $G^0 = E$ with its sc-structure. For $a \in B \setminus \{0\}$ let (R, ϑ) be the associated pair. We define

$$G^a = H^3([-\frac{R}{2}, \frac{R}{2}] \times S^1, \mathbb{R}^{2n}) \times H_c^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^{2n}).$$

As sc-structure we take

$$G_m^a = H^{3+m}([-\frac{R}{2}, \frac{R}{2}] \times S^1, \mathbb{R}^{2n}) \times H_c^{3+m,\delta_m}(\mathbb{R} \times S^1, \mathbb{R}^{2n}).$$

We define the Hilbert space norm $\| (q,p) \|_m^a$ on G^a by

$$\begin{aligned} & \| (q,p) \|_m^a \\ = \; & \left[e^{\delta_m R} \left(\| q - [q]_0 + p(+\infty) \|_{3+m,-\delta_m}^2 \right.\right. \\ & \left.\left. + \| p - (1-2\beta) p(+\infty) \|_{m+3,\delta_m}^2 \right) + |[q]_0 - p(+\infty)|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

We define

$$\| (\eta^+, \eta^-) \|_m^0 = \left[\| r^+ \|_{m+3,\delta_m}^2 + \| r^- \|_{m+3,\delta_m}^2 + |c|^2 \right]^{\frac{1}{2}}.$$

This is a Hilbert space norm. Note that the precise values of the norms matter in our later estimates. Here $p(+\infty)$ is the positive asymptotic limit of p and $[q]_0$ is the meanvalue

$$[q]_0 = \int_{S^1} q(0,t)dt.$$

Observe that the above defines a Banach space norm on G_m^a. Define $Y \to B$ by putting

$$Y = \bigcup_{a \in B} \{a\} \times G^a$$

with the obvious projection $Y \to B$. We define for every fiber and level m the norm $\| \cdot \|_m^a$ just introduced. In order to avoid confusion with other norms we will write it sometimes as $\| \cdot \|_m^{Y_a}$. Define $Z = B \times E$ with norm on level m given by $\| \cdot \|_m^Z$, which is defined as

$$\| (\eta^+, \eta^-) \|_m^Z = \left[\| r^+ \|_{m+3,\delta_m}^2 + \| r^- \|_{m+3,\delta_m}^2 + |c|^2 \right]^{\frac{1}{2}},$$

where c is the common asymptotic limit. Sometimes we suppress the index Z if there is no danger of confusion. Consider now the fiber preserving map

$$\boxdot : Z \to Y : (a, (\eta^+, \eta^-)) \to (a, (\boxplus_a(\eta^+, \eta^-), \boxminus_a(\eta^+, \eta^-))).$$

We have the following result

Theorem 2.2. *The map $\boxdot$ is a bijection which is fiber-wise linear. Moreover there exists a constant $C_m > 0$ independent of a so that*

$$C_m^{-1} \cdot \| (\eta^+, \eta^-) \|_m^E \leq \| \boxdot_a(\eta^+, \eta^-) \|_m^{Y_a} \leq C_m \cdot \| (\eta^+, \eta^-) \|_m^E .$$

Proof. Clearly the map is fiber-wise linear. Assume first that for some a we have $\boxdot_a(\eta^+, \eta^-) = 0$. If we write $\eta^\pm = c + r^\pm$, where c is the asymptotic constant, then it follows

$$0 = \boxminus_a(\eta^+, \eta^-) = \boxminus_a(r^+, r^-)$$

which implies that $[r^+]_R + [r^-]_R = 0$. This implies using that $\boxplus_a(\eta^+, \eta^-) = 0$ via an integration over $\{0\} \times S^1$ that the asymptotic constant c vanishes. Hence we deduce that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \beta & 1-\beta \\ \beta - 1 & \beta \end{bmatrix} \cdot \begin{bmatrix} r^+(s + \frac{R}{2}, t + \frac{\vartheta}{2}) \\ r^-(s - \frac{R}{2}, t - \frac{\vartheta}{2}) \end{bmatrix}.$$

This implies that $r^\pm$ vanishes. Since we already know that the asymptotic constant vanishes we see that $\eta^\pm = 0$. Assume next that $(q, p) \in G^a$ are given. If we can solve

$$\boxdot_a(\eta^+, \eta^-) = (q, p)$$

it follows that necessarily

$$[q]_0 = c + \frac{1}{2}([r^+]_R + [r^-]_{-R}) \text{ and } p(+\infty) = \frac{1}{2}([r^+]_R + [r^-]_{-R}).$$

Given q and p we can compute $[q]_0$ and $p(+\infty)$ and determine c by

$$c = [q]_0 - p(+\infty).$$

Consider now the equation

$$\begin{bmatrix} q - c \\ p - (1 - 2\beta)p(+\infty) \end{bmatrix} = \begin{bmatrix} \beta & 1-\beta \\ \beta - 1 & \beta \end{bmatrix} \cdot \begin{bmatrix} r^+(s + \frac{R}{2}, t + \frac{\vartheta}{2}) \\ r^-(s - \frac{R}{2}, t - \frac{\vartheta}{2}) \end{bmatrix}.$$

This can be solved by inverting the "β-matrix", resulting in a solution (r^+, r^-) with vanishing asymptotic limits. Now define

$$\eta^\pm = [q]_0 - p(+\infty) + r^\pm.$$

This is the desired solution.

Next we want to derive our estimates for suitable constants C_m independent of a. If $(q, p) = \boxdot_a(\eta^+, \eta^-)$ then we have with the previous notation the relationship

$$\begin{bmatrix} q - [q]_0 + p(+\infty) \\ p - (1 - 2\beta)p(+\infty) \end{bmatrix} = \begin{bmatrix} \beta & 1-\beta \\ \beta - 1 & \beta \end{bmatrix} \cdot \begin{bmatrix} r^+(s + \frac{R}{2}, t + \frac{\vartheta}{2}) \\ r^-(s - \frac{R}{2}, t - \frac{\vartheta}{2}) \end{bmatrix}.$$

It follows from a straight forward calculation observing that r^+ and r^- as well as $p-(1-2\beta)p(+\infty)$ have vanishing asymptotic limits that for a suitable constant $C>0$ (depending on m) which is independent of a and with the abbreviation $c=[q]_0-p(+\infty)$

$$\begin{aligned} & C^{-1}\left[\| q-c\|^2_{m+3,-\delta_m}+\| p-(1-2\beta)p(+\infty)\|^2_{m+3,\delta_m}\right] \\ \leq\ & e^{-\delta_m R}\left[\| r^+\|^2_{m+3,\delta_m}+\| r^-\|^2_{m+3,\delta_m}\right] \\ \leq\ & C\cdot\left[\| q-c\|^2_{m+3,-\delta_m}+\| p-(1-2\beta)p(+\infty)\|^2_{m+3,\delta_m}\right]. \end{aligned}$$

Here all the norms are the standard weighted Sobolev norms. Multiplying this inequality by $e^{\delta_m R}$, adding $|c|^2$ and taking square roots our assertion follows immediately. □

Let us note the following result which emphasizes that the choice of norms $\|\cdot\|_m^{Y_a}$ is well-adapted to the gluing procedure.

Proposition 2.3. *Assume that $(a,\eta^+,\eta^-)\rightarrow(a_0,\eta_0^+,\eta_0^-)$ on level m in $B\times E$. Then*

$$\| \boxdot_a(\eta^+,\eta^-)\|^a_m \rightarrow \| \boxdot_{a_0}(\eta_0^+,\eta_0^-)\|^{a_0}_m .$$

Proof. We estimate using the previous theorem

$$\begin{aligned} & |\,\| \boxdot_a(\eta^+,\eta^-)\|^a_m - \| \boxdot_{a_0}(\eta_0^+,\eta_0^-)\|^{a_0}_m\,| \\ \leq\ & C\cdot \| (\eta^+-\eta_0^+,\eta^- -\eta_0^-)\|_m \\ & + C\cdot |\,\| \boxdot_a(\eta_0^+,\eta_0^-)\|^{Y_a}_m - \| \boxdot_{a_0}(\eta_0^+,\eta_0^-)\|^{Y_{a_0}}_m\,|. \end{aligned}$$

In order to draw our conclusion we only have to show that the second term converges to 0 since this is true for the first one. If $a_0\neq 0$ the result follows from an easy calculation. Assume next that $a_0=0$. Assume that $a\neq 0$. Note that by a density argument and using the previous theorem we may assume that $\eta_0^\pm$ are constant outside of a compact domain. This constant is, of course, the common asymptotic constant c. We compute for a small enough that

$$p_a := \boxminus_a(\eta_0^+,\eta_0^-)=0.$$

Further with $q_a=\boxplus_a(\eta_0^+,\eta_0^-)$ we see that $[q_a]_0=c$. Hence

$$\begin{aligned} & \| (q_a,p_a)\|^{Y_a}_m \\ =\ & \left[|c|^2+e^{\delta_m R}\| q-c\|^2_{3+m,-\delta_m}\right]^{\frac{1}{2}} \\ =\ & \left[|c|^2+e^{\delta_m R}\| \boxplus_a(r^+,r^-)\|^2_{3+m,-\delta_m}\right]^{\frac{1}{2}}. \end{aligned}$$

Now we observe, using that $r^+(s,t)=0$ for $s\geq K$ and $r^-(s,t)=0$ for $s\leq -K$ for a suitable constant K for $|a|$ small enough

$$\begin{aligned} & \| D^\alpha(\boxplus_a(r^+,r^-))e^{-\delta_m|s|}\|^2_{L^2} \\ = \; & \| ((D^\alpha r^+)(s+\frac{R}{2},t+\frac{\vartheta}{2})+(D^\alpha r^-)(s-\frac{R}{2},t-\frac{\vartheta}{2}))e^{-\delta_m|s|}\|^2_{L^2} \\ = \; & e^{-\delta_m R}\left[\| (D^\alpha r^+)e^{\delta_m|s|}\|^2_{L^2}+\| (D^\alpha r^-)e^{\delta_m|s|}\|^2_{L^2}\right]. \end{aligned}$$

Summing up all these terms with respect to $|\alpha|\leq m+3$, adding $|c|^2$ and taking the square root shows that

$$\| (q_a,p_a)\|^{Y_a}_m=\| (\eta_0^+,\eta_0^-)\|^Z_m$$

for $|a|$ small enough, assuming that $\eta_0^\pm$ are constant outside of a compact set. This proves the desired result. □

Using the map $\boxdot_a : Z\to Y$ define for $a\in B$ two subspaces of Z_a. Namely K_a and K_a^c as follows:

$$\begin{aligned} K_a &= \{(\eta^+,\eta^-)\mid \boxminus_a(\eta^+,\eta^-)=0\} \\ K_a^c &= \{(\eta^+,\eta^-)\mid \boxplus_a(\eta^+,\eta^-)=0\}. \end{aligned}$$

Clearly

$$E=K_a\oplus K_a^c.$$

We observe that $K_0=E$ and $K_0^c=0$. Denote by

$$\pi_a : E\to E$$

the projection onto K_a along K_a^c. Clearly $\pi_0=Id$. For $a\neq 0$ let (R,ϑ) be the associated pair. We have

Lemma 2.4. *For $a\neq 0$ the projection π_a is given by the explicit formula*

$$(\hat{\eta}^+,\hat{\eta}^-)=\pi_a(\eta^+,\eta^-)$$

where

$$\begin{aligned} \hat{\eta}^+(s,t) &= \tfrac{\tau^2}{\alpha}\eta^+ + \tfrac{\tau(1-\tau)}{\alpha}\bar{\eta}^- + \tfrac{1}{2}(1-\tfrac{\tau}{\alpha})([\eta^+]_R+[\eta^-]_{-R}) \\ \hat{\eta}^-(s',t') &= \tfrac{\bar{\tau}(1-\bar{\tau})}{\bar{\alpha}}\bar{\eta}^+ + \tfrac{\bar{\tau}^2}{\bar{\alpha}}\eta^- + \tfrac{1}{2}(1-\tfrac{\bar{\tau}}{\bar{\alpha}})([\eta^+]_R+[\eta^-]_{-R}). \end{aligned}$$

From our technical results it follows that (π,B,E) is a sc-smooth splicing. More precisely

Theorem 2.5. *For given and $0<\delta_0<\delta_1<..$ in the definition of E the triple $\mathcal{S}=(\pi,B,E)$ is a sc-smooth splicing.*

We will call $\mathcal{S}$ the nodal splicing. The result follows as an application of the previously described technical results. Of particular interest will be $\mathcal{S}$ when E is equipped with the sc-structure associated to a sequence $0 < \delta_0, \delta_1 < ... \leq 2\pi - \sigma$ for a small $\sigma \in (0, 2\pi)$. The bound on δ_m by 2π will be important in sc-smoothness considerations for the polyfold set-up in symplectic field theory.

Define $K = \bigcup_{a \in B} \{a\} \times K_a$ and $\bar{Y}$ by

$$\bar{Y} = [\{0\} \times E] \bigcup \left[\bigcup_{a \in B \setminus \{0\}} \{a\} \times H^3([-\frac{R}{2}, \frac{R}{2}] \times S^1, \mathbb{R}^{2n}) \right].$$

The norm on the fiber over a is the induced norm from Y. Then by construction

Theorem 2.6. *The map*

$$\Phi : K \to \bar{Y} : (a, (\eta^+, \eta^-)) \to (a, \boxplus_a(\eta^+, \eta^-))$$

is a bijection which is fiber-wise linear. Moreover we have the estimate with $q = \boxplus_a(\eta^+, \eta^-)$

$$\begin{aligned} & C_m^{-1} \left[e^{\delta_m R} \parallel q - [q]_0 \parallel^2_{m+3,-\delta_m} + |[q]_0|^2 \right]^{\frac{1}{2}} \\ \leq \ & \parallel (\eta^+, \eta^-) \parallel_m \\ \leq \ & C_m \left[e^{\delta_m R} \parallel q - [q]_0 \parallel^2_{m+3,-\delta_m} + |[q]_0|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

2.2. **Lecture 6: Concrete Polyfold Charts I (Hofer).** (Vol. II)

Gromov-Witten case

- State the results from DM-Theory (Small disk structure good family etc)
- Transversal constraints.

2.3. **Problem 3: Polyfold Charts in Morse Theory (Dupont, Chance, McDuff).**

- Show that the space of curves connecting two points is a sc-manifold. Transversal constraint, etc..

2.4. **Lecture 7: Concrete Polyfold Charts II (Hofer).** (Vol. II)

- Use of splicings.
- Technical result for transition maps
- Smoothness of transition maps

2.5. **Lecture 8: Fredholm Operators I: Contraction Germs (Wehrheim).** (Vol. I, Chap. 5)

- Contractions germs
- Smoothness

2.6. **Problem 4: Polyfold Charts in Morse Theory (Dupont, Chance, McDuff).** (Vol. I, Chap. 3)

- Continuation. One should have shown at this point that the space of possibly broken trajectories is a M-polyfold.
- If possible indicate how to to it on the manifold.

3. Wednesday

3.1. **Lecture 9: Fredholm Operators II: Global Theory (Hofer).** (Vol. I, Chap. 6)

- Prove certain results within the global theory, perhaps some perturbation results.

3.2. **Lecture 10: Cauchy Riemann Operator I as a Polyfold Fredholm section I (Abbas).** (Vol. II)

- The main point of this talk is the relationship between Cauchy-Riemann operator and full gluing. The apriori estimate for the Cauchy-Riemann operator with full gluing.

Let $K \triangleleft_B \hat{K} \to K$ be as described above. For (a, η^+, η^-) consider the following map

$$L_a(\eta^+, \eta^-) = \hat{\boxplus}_a^{-1} \circ \bar{\partial} \circ \boxplus_a(\eta^+, \eta^-).$$

To be more precise, $L_a(\eta^+, \eta^-)$ is defined by

$$\hat{\boxplus}_a(L_a(\eta^+, \eta^-)) = \bar{\partial} \circ \boxplus_a(\eta^+, \eta^-) \text{ and } \hat{\boxminus}_a(L_a(\eta^+, \eta^-)) = 0.$$

The assignment $a \to L_a(\eta^+, \eta^-)$ is a family of linear first order differential operators if we view it as a map

$$K_a \to \hat{K}_a.$$

Of course, this map has a natural extension as section of $(B \oplus E) \triangleleft_B \hat{K}$.

Taking the complementary splicings we can define a second operator L_a^c as a section of $(B \oplus E) \triangleleft_B \hat{K}^c \to B \oplus E$ by

$$L_a^c(\eta^+, \eta^-) = \hat{\boxminus}_a^{-1} \circ \bar{\partial} \circ \boxminus_a(\eta^+, \eta^-).$$

More precisely

$$\hat{\boxplus}_a(L_a^c(\eta^+,\eta^-)) = 0 \quad \text{and} \quad \hat{\boxminus}_a(L_a(\eta^+,\eta^-)) = \bar{\partial} \circ \boxminus_a(\eta^+,\eta^-).$$

Define a section $\bar{L}$ of $(B \oplus E) \triangleleft F$ by

$$\bar{L}_a(\eta^+,\eta^-) = L_a(\eta^+,\eta^-) + L_a^c(\eta^+,\eta^-).$$

We will now discuss the above ingredients in detail.

Proposition 3.1. *The sections L of $(B \oplus E) \triangleleft_B \hat{K} \to B \oplus E$ and L^c of $(B \oplus E) \triangleleft_B \hat{K}^c \to B \oplus E$ are sc-smooth.*

Proof. This follows by deriving explicit formulas for the maps and applying the technical results. □

Next consider L_a^c and restrict it to K_a^c. Recall that

Proposition 3.2. *The Cauchy Riemann operator $\bar{\partial}$ induces a linear isomorphism*

$$H_c^{m+3,\delta_m}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to H^{m+2,\delta_m}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

for all m. Here the condition $\delta_m \in (0, 2\pi)$ is crucial.

Then we have

Proposition 3.3. *For every $a \in B$ the map*

$$L_a^c : K_a^c \to \hat{K}_a^c$$

is a linear isomorphism.

Proof. The maps

$$\boxminus_a : K_a^c \to H_c^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

and

$$\hat{\boxminus}_a : \hat{K}_a^c \to H^{2,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

are linear sc-isomorphism. Further

$$\bar{\partial} : H_c^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to H^{2,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

is a linear sc-isomorphism. □

Observe that a solution of the problem

$$(\xi^+,\xi^-) := \bar{L}_a(\eta^+,\eta^-) \in \hat{K}_a$$

solves in particular

$$L_a^c(\eta^+,\eta^-) = 0$$

which implies that $\boxminus_a(\eta^+,\eta^-) = 0$. With other words $(\eta^+,\eta^-) \in K_a$. The section L^c is with other words what we call a filler, see [?].

Denote by $\dot{E}$ the subspace of E obtained by taking the closure of smooth maps in E whose support is compact and contained in $(0,+\infty)\times S^1$ or $(-\infty,0)\times S^1$, respectively. Then $\dot{E}$ inherits from E a sc-structure. We need the following result:

Proposition 3.4. *For every m the Cauchy-Riemann operator defines a surjective Fredholm operator of real index $2n$*

$$H^{m+3,-\delta_m}(\mathbb{R}\times S^1,\mathbb{R}^{2n})\to H^{m+2,-\delta_m}(\mathbb{R}\times S^1,\mathbb{R}^{2n}).$$

The condition $\delta_m\in(0,2\pi)$ is crucial.

Now we can give a proof the following important result.

Theorem 3.5. *Given $m\geq 0$ there exists a constant C_m independent of $a\in B$ so that for $(\eta^+,\eta^-)\in\dot{E}$ we have the estimate*

$$\| \bar{L}_a(\eta^+,\eta^-) \|^F_m\geq C_m\cdot \| (\eta^+,\eta^-) \|^E_m .$$

Proof. Given $(\eta^+,\eta^-)\in\dot{E}$ we may view $\boxplus_a(\eta^+,\eta^-)$ as an element in $H^{3,-\delta_0}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ with support in $[-\frac{R}{2},\frac{R}{2}]\times S^1$. Let us introduce the following abbreviations we put

$$q=\bar{\partial}\boxplus_a(\eta^+,\eta^-)\quad\text{and}\quad p=\bar{\partial}\boxminus_a(\eta^+,\eta^-),$$

and consequently

$$(\bar{\partial}q,\bar{\partial}p)=\hat{\boxdot}_a\bar{L}_a(\eta^+,\eta^-).$$

Since we assume that $(\eta^+,\eta^-)\in\dot{E}$ the asymptotic constant c is vanishing. We recall the following consequence from the gluing formulae

$$[q]_0=c+\frac{1}{2}([r^+]_R+[r^-]_{-R})=c+p(+\infty).$$

Consequently, assuming that $c=0$ which is the case for data in $\dot{E}$, we deduce

$$[q]_0=p(+\infty).$$

Let us further note that a complement of the constant functions in $H^{m+3,-\delta_m}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ are the functions q with $[q]_0=0$. We compute for suitable constants C^i only depending on m but not on a

$$
\begin{aligned}
& \| L_a(\eta^+,\eta^-) \|_m^F \\
\geq\ & C^1 \cdot \| \hat{\boxdot}_a(L_a(\eta^+,\eta^-)) \|_m^{\hat{Y}_a} \\
=\ & C^1 \cdot \| (\bar{\partial} q, \bar{\partial} p) \|_m^{\hat{Y}_a} \\
=\ & C^1 \cdot e^{\delta_m \frac{R}{2}} \left[\| \bar{\partial} q \|^2_{m+2,-\delta_m} + \| \bar{\partial} p \|^2_{m+2,\delta_m} \right]^{\frac{1}{2}} \\
=\ & C^1 \cdot e^{\delta_m \frac{R}{2}} \left[\| \bar{\partial}(q-[q]_0) \|^2_{m+2,-\delta_m} + \| \bar{\partial} p \|^2_{m+2,\delta_m} \right]^{\frac{1}{2}} \\
\geq\ & C^2 e^{\delta_m \frac{R}{2}} \left[\| q-[q]_0 \|^2_{m+3,-\delta_m} + \| p \|^2_{m+3,\delta_m} \right]^{\frac{1}{2}} \\
=\ & C^2 \left[e^{\delta_m R} \left[\| q-[q]_0 \|^2_{m+3,-\delta_m} \right.\right. \\
& \left.\left. + \| p-(1-2\beta)p(+\infty) \|^2_{m+3,\delta_m} + |p(+\infty)|^2 \right] \right]^{\frac{1}{2}} \\
\geq\ & C^3 e^{\delta_m \frac{R}{2}} \left[\| q-[q]_0 + p(+\infty) \|^2_{m+3,-\delta_m} \right. \\
& \left. + \| p-(1-2\beta)p(+\infty) \|^2_{m+3,\delta_m} \right]^{\frac{1}{2}} \\
\geq\ & C^3 \left[e^{\delta_m R} \left[\| q-[q]_0 + p(+\infty) \|^2_{m+3,-\delta_m} \right.\right. \\
& \left.\left. + \| p-(1-2\beta)p(+\infty) \|^2_{m+3,\delta_m} \right] + |[q]_0 - p(+\infty)|^2 \right]^{\frac{1}{2}} \\
=\ & C^3 \cdot \| (q,p) \|_m^{Y_a} \\
=\ & C^3 \cdot \| \boxdot_a(\eta^+,\eta^-) \|_m^{Y_a} \\
\geq\ & C^4 \cdot \| (\eta^+,\eta^-) \|_m^E \, .
\end{aligned}
$$

□

State the following since of relevance for the Contact Homology and SFT.

Assume that $t \to J(t)$, associates to $t \in S^1$ a complex multiplication on $\mathbb{R}^{2n}$ compatible with the standard symplectic form ω. Moreover we are given a loop of real linear maps $t \to A(t)$, $A(t) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, so that the map

$$x \to -J(t)\dot{x} - A(t)x$$

for $x \in H^1(S^1, \mathbb{R}^{2n})$ defines a self-adjoint operator in $L^2(S^1, \mathbb{R}^{2n})$ where the latter is equipped with the L^2-inner product

$$(u,v) = \int_{S^1} \omega(u(t), J(t)v(t))dt.$$

Let us assume that the problem $-J(t)\dot{x}(t) - A(t)x(t) = 0$ does not have a 1-periodic solution other than the trivial one. Then there exists a number $a_0 > 0$ so that the interval $[-a_0, a_0]$ does not contain any eigenvalue of the self-adjoint operator $x \to Tx := -J(t)\dot{x} - A(t)x$.

Fix a number $a_1 > 0$ so that also $[-a_0 - a_1, a_0 + a_1]$ does not contain any eigenvalue. Let $\sigma_0 =< \sigma_1 < \sigma_2 < ... < 1$ be a strictly increasing sequence of numbers. Define an associated δ-sequence by

$$\delta_k = a_0 + a_1\sigma_k.$$

Define the space $\hat{E}$ by

$$\hat{E} = H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n}) \times H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n}),$$

with the sc-structure where the m-th level corresponds to regularity $(m+3, \delta_m)$. The following result is well-known

Theorem 3.6. *Assume that $(-a, a)$ is the maximal interval not containing an element of the spectrum of T. Then for every $\delta \in (-a, a)$ and $m \geq 1$ the linear operator*

$$\frac{\partial}{\partial s} - T : H^{m,\delta}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to H^{m-1,\delta}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

is a linear isomorphism.

Let us also define $\hat{F}$ by

$$\hat{F} = H^{2,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n}) \times H^{2,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n}),$$

with sc-structure where level m corresponds to regularity $(m+2, \delta_m)$. We can use for $\hat{E}$ and $\hat{F}$ the same $\hat{\boxdot}$-gluing. Denote by $\dot{E}$ the closure of the pairs (η^+, η^-) which have compact support in the interior of the half-cylinders. We consider the map

$$\bar{T}_a : \dot{E} \to \hat{F} : \hat{\boxdot}_a^{-1}(\bar{T}, \bar{T})\hat{\boxdot}_a.$$

Here $\bar{T} = \frac{\partial}{\partial s} - T$. We can carry through all arguments from the previous section with the obvious modification. We obtain the following result

Theorem 3.7. *For m there exists a constant $C_m > 0$ independent of $a \in B$ so that for $(\eta^+, \eta^-) \in \dot{E}$ we have*

$$\| \bar{T}_a(\eta^+, \eta^-) \|_M^{\hat{F}} \geq C_m \cdot \| (\eta^+, \eta^-) \|_m^E .$$

For SFT the following estimate will be important which is a mixture of the previous results. Let $\dot{E}$ be as before and similarly $\hat{F}$. We assume that $J(t), A(t) : \mathbb{R}^{2n-2} \to \mathbb{R}^{2n-2}$ are as previously described. Then let J_0 be the standard structure on $\mathbb{R}^2$. We consider for $a \in B$

$$G_a : \dot{E} \to \hat{F}$$

defined as follows. Write $\eta^{\pm} = (\eta_1^{\pm}, \eta_2^{\pm})$, where $\eta_1^{\pm} \in \mathbb{R}^2$ and $\eta_2^{\pm} \in \mathbb{R}^{2n-2}$. We define the total gluing $\bar{\boxdot}_a$ by

$$\bar{\boxdot}_a(\eta^+, \eta^-) = (\boxdot_a(\eta_1^+, \eta_1^-), \hat{\boxdot}_a(\eta_2^+, \eta_2^-)).$$

Observe that for the $\mathbb{R}^2$-component the total gluing involves the averages. We define $\hat{\boxdot}a(\xi^+,\xi^-)$ by

$$\hat{\boxdot}_a(\xi^+,\xi^-) = (\hat{\boxdot}_a(\xi_1^+,\xi_1^-),\hat{\boxdot}_a(\xi_2^+,\xi_2^-)).$$

Then we put

$$G_a(\eta^+,\eta^-) = \hat{\boxdot}_a^{-1}((\bar{\partial}_{J_0},\bar{\partial}_{J_0}),(\bar{T},\bar{T}))\circ\bar{\boxdot}_a.$$

As a combination of the previous result and the results in the previous section we obtain

Theorem 3.8. *For every level m there exists a constant $C_m > 0$ so that for ever $(\eta^+,\eta^-)\in\dot{E}$ we have*

$$\| G_a(\eta^+,\eta^-) \|_M^{\hat{F}} \geq C_m\cdot \| (\eta^+,\eta^-) \|_m^E .$$

3.3. **Problem 5: Questions and Answers (Hofer et al).**

3.4. **Problem 6: Gradient Flow in Morse Homology I (Wysocki).** (Vol. I, Chap. 6)

• Work in the case of Dupont/Chance/McDuff, i.e 3 points and the curves connecting them. State the result for the linear operator $\partial_t + A$ similar to the CR-operator in Abbas' talk then show how you can obtain the the contraction germ characterization (carried out in the second lecture note for the CR-operator)

3.5. **Problem 7: Gradient Flow in Morse Homology I (Wysocki).** (Vol. I, Chap. 6)

• Continuation. May be one gives a formulation at the end of the talk that for a closed manifold with a Morse-function the space of connecting (perhaps) broken curves defines a M-polyfold and the vector fields along a strong polyfold bundle and $x\to\dot{x}-f'(x)$ is a sc-Fredholm section.

4. Thursday

4.1. **Lecture 11: Cauchy Riemann Operator II as a Polyfold Fredholm section II (Hofer).** (Vol. II)

• Rely on the linear results by Abbas and the special case described by Wysocki to describe the situation of the CR-operator.

4.2. **Lecture 12: Transversality and Perturbations (Hofer).** (Vol. I, Chap. 6)

• Abstract background material about perturbations and transversality.

4.3. **Lecture 13: Polyfold Groupoids and Multi-Sections (McDuff).** (Vol. I, Cap. 7)

• The use of multi-sections in Lie groupoids and Polyfold groupoids.

4.4. **Lecture 14: Fredholm Theory and Operations I (Hofer).** (Vol. III)

• Degeneration structures and associated Algebra, Operation.

4.5. **Problem 8: A finite-dimensional example for operations (Fish, Siefring).** (Vol. III)

• Explain on the simple example in the lecture indexing and some of the algebra occurring.

4.6. **Problem 9: Operations in Morse-Theory/Floer Theory (Fish, Siefring).** (Vol. III)

• Continuation + some more real life examples.

5. Friday

5.1. **Lecture 15: Fredholm Theory with Operations II (Hofer).** (Vol. III)

• Perturbation Theory.

5.2. **Lecture 16: Fredholm Theory with Operations III (Hofer).** (Vol. III)

• Sketch of some proofs.

5.3. **Problem 10: Questions and Answers (Hofer et al).**

5.4. **Lecture 17: Applications to SFT I (Hofer).** (Vol. III + Vol. IV)

• Operations in SFT.

5.5. **Lecture 18: Applications to SFT II (Hofer).** (Vol. III + Vol. IV)

- Operations in SFT and some representations..

5.6. **Problem 11: Questions and Answers (Hofer et al).**

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