Polyfolds and Fredholm Theory
Part 1

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Preliminary Version *

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Introduction

This book is the first in a series of four books devoted to a general nonlinear Fredholm theory on spaces of varying dimensions. The usefulness of this theory will be illustrated by a variety of applications including Morse homology, Gromov-Witten theory, Floer theory and Symplectic Field theory. We believe that there are many other possible applications to nonlinear PDE-problems beyond what we describe here. From a very abstract point of view the above mentioned problems are very similar. To explain this point we recall some facts. Gromov-Witten theory, Floer theory, Contact homology, or more generally Symplectic Field theory are all based on the study of compactified moduli spaces, or even infinite families of such spaces interacting with each other. The data of these moduli spaces are encoded in convenient ways leading for example to so-called generating functions in Gromov-Witten theory or to Floer-Homology in the Floer-Theory. Common features include the following.

- The moduli spaces are solutions of elliptic PDE's showing dramatic non-compactness phenomena having well-known names like "bubbling-off", "stretching the neck", "blow-up", "breaking of trajectories". These descriptions are a manifestation of the fact that from classical analytical viewpoint one is confronted with limiting phenomena, where classical analytical descriptions break down.
- When the moduli spaces are not compact, they admit nontrivial compactifications like the Gromov compactification of the space of pseudoholomorphic curves or the space of broken trajectories in Morse theory.
- In many problems like in Floer theory, Contact homology or Symplectic Field theory the algebraic structures of interest are precisely those created by the "violent analytical behavior" and its "taming" by suitable compactifications. In fact, the algebra is created by the complicated interactions of many different moduli spaces.

In our books, we will propose a general functional analytic approach which allows us, in particular, to understand the elliptic problems arising in symplectic geometry. Also here the lack of compactness does not permit a satisfactory classical description. On the other hand, the lack of compactness is precisely the source of interesting invariants. This is our motivation for introducing the new framework. The framework should fit into the following general scheme for producing invariants for geometric problems. Starting with a concrete problem, we have a distinguished set $\mathcal{G}$ of geometric data. The choice of $\tau \in \mathcal{G}$ leads to a nonlinear elliptic differential operator L_τ. We are interested in the solution set $\mathcal{M}$ of $L_\tau = 0$. We want to extract invariants from this solution set which are independent of the actual choice τ. In interesting cases the solution sets are not compact. The first task is then to

1) find a good compactification $\overline{\mathcal{M}}$ of the solution set $\mathcal{M}$.

This task can be very difficult, as the compactification of the space of pseudoholomorphic curves in symplectic cobordisms in [**3**] shows. However, our abstract theory will give quite a number of ideas how to construct compactifications in concrete case.

2) Next we construct a bundle $Y \to X$ so that L becomes a section whose zero-set is $\overline{\mathcal{M}}$ and not only $\mathcal{M}$.

Clearly, if the bundle $Y \to X$ would not have some additional "good properties", this formulation would be quite useless. What are these good properties? Taking classical (nonlinear) Fredholm theory as a guide, we would like our spaces to have tangents and we would like to be able to linearize problems. Information about the linearization should then allow us to design implicit function theorems. In addition, we would like to have a notion of transversality and an abstract perturbation theory which would permit to perturb a section into a general position. Finally, in case of transversality, **the solution set $\{L = 0\}$ should be a smooth manifold** or a **smooth orbifold in a natural way.** How does classical Fredholm theory achieve this? In the classical theory the ambient space X is a smooth Banach manifold and Y is a smooth Banach space bundle over it. Then L is a smooth section and its linearization at zero is Fredholm. The usual implicit function theorem then describes the solution set provided the linearization is surjective. The smooth structure on the solution set is induced from the smooth structure of the ambient space. In fact, in case of transversality the solutions set is a smooth submanifold. It should be emphasized that the classical theory has a lot of luxury build in. If one ultimately is only interested in obtaining a smooth structure on the solution space (in case of transversality), then one expects to be able to give up quite a lot of structure on the ambient space X. In our case we want the compactified solution set to be contained in the ambient space X, and an analysis of concrete examples reveals that there cannot be a structure for which X is locally homeomorphic to an open set in a Banach space. In fact, one needs models admitting locally varying dimensions in order to deal with phenomena like bubbling-off. At first sight, it seems rather doubtful whether such objects could ever have tangent spaces. Moreover, for any useful concept of smoothness in our new theory, the concrete examples listed above indicate that the shift-map

$$\mathbb{R} \times L^2(\mathbb{R}) \to L^2(\mathbb{R}) : (t, u) \to u(\cdot + t)$$

should be smooth (in a new sense yet do be defined). This map is just barely continuous in the classical sense. Surprisingly there is a way of overcoming this problem. One can generalize calculus by introducing a new concept of smoothness. This then allows us to define new local models for spaces which still admit tangent spaces and which replace the open sets in Banach spaces. Moreover, the classical smoothness of transition maps can be replaced by the new concept of smoothness. Our local models now have varying dimensions. The spaces obtained this way are called **polyfolds** and their bundles are called **polyfold bundles**. As it turns out, on these polyfold bundles can be developed a Fredholm theory which satisfies all our requirements. Roughly a section is a Fredholm section if in suitable local coordinates it has a suitable normal form. Our notion of a chart is very weak because the concept of smoothness is relaxed. As a consequence, we obtain more flexibility in bringing problems into a normal form. In other words, more problems than before turn out to be nonlinear Fredholm problems.

Finally, we want to make more precise what it means that many Fredholm problems interact with each other. Let us explain some background. The compactifications of the solution spaces which one introduces are usually composed of ingredients which are solutions of PDE's obtained from the original PDE's as limit cases. They are usually Fredholm problems. If we take the disjoint union of all these Fredholm problems, we obtain a new Fredholm problem having the following structure. We find a Fredholm section f of a polyfold bundle $Y \to X$. The polyfold X will have a "boundary with corners", a notion to be defined. Polyfolds are, in particular, second countable paracompact spaces. As it will turn out, there is a function $d : X \to \mathbb{N}$, called the degeneracy function, so that every point $x \in X$ has an open neighborhood $U(x)$ satisfying $d|U(x) \leq d(x)$. Now the boundary ∂X is by definition the subset of X consisting of points having degeneracy $d(x) \geq 1$. Consider the connected components in $\{x \in X \mid d(x) = 1\}$. The closure of such a connected component will be called a face. Every point $x \in \partial X$ lies in the intersection of at most $d(x)$ faces. There is a rich structure if we know a priori that every point lies in precisely $d(x)$-many faces. This will be the case in all our applications. Now we are able to formalize what it means that Fredholm problems are interacting with each other. We describe only a particular case of our more general theory. There is a countable set S (equipped with the discrete topology) and a subset D of $S \times X \times X$ which is the union of connected components in $X \times X$ and a smooth map $\circ : D \to X$ whose image is ∂X and which meets certain axioms. This map extends to a map $S \times Y \times Y \to Y$ as a fiber-wise linear isomorphism. The Fredholm section $f : X \to Y$ is compatible with the operation $\circ$, if

$$f(\circ(s, x, x')) = \circ(s, f(x), f(x')).$$

The interpretation is as follows. Given two points x and x' in X and the "recipe" s in S, we can construct a new element $\circ(s, x, x')$. If, for example, x and x' are solutions of the Fredholm problem then $\circ(s, x, x')$ is also a solution. This way we can explain the boundary ∂X in terms of X via the operation $\circ$. The above structure $(f : X \to Y, \circ)$ is called a Fredholm problem with operation. As it turns out Morse homology, Floer theory, Contact homology and Symplectic Field theory can be understood as Fredholm problems with operations. Hence the third step in a concrete problem is

3) identify the operation.

We shall develop a theory of cobordisms between Fredholm problems with operations and some interesting algebra to describe such problems.

The four volumes are organized in the following way,

Volume I. In the first part of the current volume, we introduce the new calculus and develop the functional analysis and differential geometry needed in order to construct our new spaces. Let us remind the reader that most of the constructions in differential geometry are functorial. Hence if we introduce new spaces as local models which have some kind of tangent spaces, and if we can define smooth maps between these spaces and their tangent maps, then the validity of the chain rule allows us to carry out most differential geometric constructions provided we have smooth (in the new sense) partitions of unity. Using the new local models for spaces we construct the M polyfolds. On these new spaces, we develop a nonlinear Fredholm theory and prove several variants of the implicit function theorem. All the concepts are illustrated by an application to classical Morse theory. Let us, however, note the following. Having the notion of an M-polyfold we can generalize the notion of a Lie-groupoid and define polyfolds by copying the groupoid approach to orbifolds. This is straightforward and allows us, with some of the results in volume II, to develop the Gromov-Witten theory, where in case of transversality the moduli spaces have natural smooth (in the classical sense) structures.

Volume II. In this volume, we construct a polyfold set-up for Symplectic Field theory. As a by-product, we obtain a polyfold set-up for Gromov-Witten theory. We show that there are natural polyfolds and polyfold bundles over them so that the Cauchy-Riemann operator is a Fredholm section in our new sense. The zero set is the union of all compactified moduli spaces. Polyfolds are the orbifold generalisation of M-polyfolds and have descriptions in terms of a theory of polyfold-groupoids, where the notion of manifold is replaced by that of an M-polyfold.

Volume III. Here we develop the Fredholm theory in polyfolds with operations. It will be illustrated by several applications. The easiest and very instructive application is again Morse theory. Another application is the Contact homology.

Volume IV: This volume is entirely devoted to Symplectic Field theory, which is obtained as an application of a theory which one might call Fredholm theory in polyfold groupoids.

The tentative titles of the books are:

"Fredholm Theory in Polyfolds I: Functional Analytic Methods";
"Fredholm Theory in Polyfolds II: The Polyfolds in Symplectic Field Theory";
"Fredholm Theory in Polyfolds III: Operations";
"Fredholm Theory in Polyfolds IV: Applications to Symplectic Field Theory".

CHAPTER 1

SC-Calculus in Banach Spaces

In order to develop the generalized nonlinear Fredholm theory needed for the symplectic field theory we start with some calculus issues. In the following "sc" stands for "scale" as well as for "scale compact" and the meaning will become clear in the definition below.

1.1. sc-Smooth Spaces

We begin by introducing the notion of an sc-smooth structure on a Banach space and on its open subsets.

Definition 1.1. *Let E be a Banach space. An* **sc-smooth structure** *on E is given by a nested sequence*

$$E = E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots \supseteq \bigcap_{m\geq 0} E_m = E_\infty$$

of Banach spaces E_m, $m \in \mathbb{N} = \{0, 1, 2, \cdots\}$, having the following properties.

- *If $m < n$, the inclusion $E_n \hookrightarrow E_m$ is a compact operator.*
- *The vector space E_∞ defined by*

  $$E_\infty = \bigcap_{m\geq 0} E_m$$

  is dense in E_m for every $m \geq 0$.

It follows, in particular, that $E_n \subseteq E_m$ is dense and the embedding is continuous if $m < n$. We note that E_∞ has the structure of a Frechet space. If $U \subset E$ is an open subset we define the **induced sc-smooth structure** on U to be the nested sequence $U_m = U \cap E_m$. Given an sc-smooth structure on U we observe that U_m inherits the sc-smooth structure defined by $(U_m)_k = U_{m+k}$. In the following most of the time there is no possibility of confusing the Banach space E_m with the sc-Banach space E_m. In case where is an ambiguity we will write $\mathbf{E}^m$ to emphasize that we are dealing with the Banach space with sc-structure $(E_{m+i})_{i\geq 0}$. Similarly we will distinguish between U_m and $\mathbf{U}_m$.

Here is an example.

EXAMPLE 1.2. Consider $E = L^2(\mathbb{R})$. Take a strictly increasing sequence of real numbers $\delta_0 = 0 < \delta_1 < \delta_2 \cdots$ starting at 0. Then define the space $E_m = H^{m,\delta_m}$ to consist of all L^2-functions u whose weak derivatives $D^k u$ up to order m belong to L^2 if weighted by $e^{\delta_m |s|}$, i.e.,

$$e^{\delta_m |\cdot|} D^k u \in L^2(\mathbb{R}) \text{ for all } \quad k \leq m.$$

The space E_m is equipped with the inner product

$$(u, v)_m = \sum_{0 \leq k \leq m} (e^{\delta_m |s|} D^k u, e^{\delta_m |s|} D^k v).$$

Using the appropriate compact Sobolev embedding theorem on bounded domains and the strictly increasing weights at the infinities one establishes the compactness of the inclusion operators $E_n \hookrightarrow E_m$ for $n > m$. Clearly, the images are dense. Armed with this example the reader should make a similar construction for functions defined on $\mathbb{R} \times S^1$. Such maps will be important in SFT.

Given E and F with sc-smooth structures then the Banach space $E \oplus F$ carries the sc-smooth structure defined by $(E \oplus F)_m = E_m \oplus F_m$.

DEFINITION 1.3. *Let U and V be open subsets of sc-smooth Banach spaces. A continuous map $\varphi : U \to V$ is said to be of* **class sc^0** *or simply* **sc^0** *if it induces continuous maps on every level, i.e., the induced maps*

$$\varphi : U_m \to V_m$$

are all continuous.

We illustrate the concept by the following example.

EXAMPLE 1.4. Take the space E from Example 1.2 equipped with the sc-structure given there. Define the $\mathbb{R}$-action of translation

$$\Phi : \mathbb{R} \oplus E \to E, \quad (t, u) \mapsto t * u \quad \text{by} \tag{1.1}$$

$$(t * u)(s) = u(s + t).$$

It is not difficult to prove that the map $(t, u) \mapsto t * u$ is of class sc^0, see Lemma 1.39 below.

Next we define the tangent bundle.

DEFINITION 1.5. *Let U be an open subset in a sc-smooth Banach space E equipped with the induced sc-smooth structure. Then the* **tangent bundle of U** *is defined by $TU = U^1 \oplus E$ with the induced sc-smooth structure defined by the nested sequence*

$$(TU)_m = (U^1 \oplus E)_m = U_{m+1} \oplus E_m$$

together with the sc^0-projection

$$p : TU \to U^1.$$

Note that the tangent bundle is not defined on U but merely on the smaller subset U_1 of level 1. For instance, in Example 1.2 the tangent bundle of E is given by

$$TE = H^{1,\delta_1} \oplus L^2$$

with the sc-smooth structure $(TE)_m = H^{1+m,\delta_{m+1}} \oplus H^{m,\delta_m}$.

1.2. sc-Smooth Maps

Next we introduce the notion of a sc^1-map. Let us first recall that a map $f : U \to V$ between open subsets of Banach spaces E and F is differentiable at the point $x \in U$ if there exists a bounded linear operator $L : E \to F$ satisfying

$$\frac{1}{\|h\|_E}\|f(x+h) - f(x) - Lh\|_F \ \to 0 \ \text{ as } \ \|h\|_E \to 0.$$

The operator L is then called the derivative of f at x and is denoted by $df(x)$. The map is of class C^1 if it is differentiable at every $x \in U$ and if the map $U \to \mathcal{L}(E, F)$, $x \to df(x)$ is continuous. Here $\mathcal{L}(E, F)$ is the space of bounded linear operators equipped with the norm

$$\|L\|_{\mathcal{L}(E,F)} \ = \sup_{\{h\in E|\ \|h\|_E\leq 1\}} \|Lh\|_F.$$

If E is infinite-dimensional, then the requirement that the map

$$U \to \mathcal{L}(E, F),\ x \to df(x)$$

is continuous is much stronger than the requirement that the map

$$U \oplus E \to F,\ (x, h) \to df(x)h$$

is continuous. The latter merely implies that

$$U \to \mathcal{L}_{\text{co}}(E, F)$$

is continuous, where $\mathcal{L}_{\text{co}}(E, F)$ stands for $\mathcal{L}(E, F)$ equipped, however, with the compact open topology which is not a normable topology if $\dim(E) = \infty$ but merely a locally convex topology.

Definition 1.6. *Let E and F be sc-smooth Banach spaces and let $U \subset E$ be an open subset. An sc^0-map $f : U \to F$ is said to be* $\mathbf{sc^1}$ *or of* **class** $\mathbf{sc^1}$ *if the following conditions hold true.*

(1) *For every $x \in U_1$ there exists a linear map $Df(x) \in \mathcal{L}(E_0, F_0)$ satisfying for $h \in E_1$*

$$\frac{1}{\|h\|_1}\|f(x+h) - f(x) - Df(x)h\|_0 \to 0 \quad \text{as } \|h\|_1 \to 0.$$

(2) *The* **tangent map** *$Tf : TU \to TF$ defined by*

$$Tf(x,h) = (f(x), Df(x)h)$$

is an sc^0-map.

The linear map $Df(x)$ will in the following often be called the **linearization** of f at the point x.

The second condition requires that $Tf : (TU)_m \to (TF)_m$ is continuous for every $m \geq 0$. In detail, the map

$$\begin{aligned} U_{m+1} \oplus E_m &\to F_{m+1} \oplus F_m \\ (x,h) &\mapsto (f(x), Df(x)h) \end{aligned}$$

is continuous for every $m \geq 0$. It follows, in particular, that

$$Df(x) \in \mathcal{L}(E_m, F_m) \tag{1.2}$$

if $x \in U_{m+1}$, for every $m \geq 0$.

$$\begin{array}{ccccc} F_{m+1} & \longrightarrow & F_m & \longrightarrow & F_{m-1} \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f} \\ U_{m+1} & \longrightarrow & U_m & \longrightarrow & U_{m-1} \end{array} \qquad \begin{array}{ccccc} F_{m+1} & \longrightarrow & F_m & \longrightarrow & F_{m-1} \\ \uparrow{\scriptstyle Df(x)} & & \uparrow{\scriptstyle Df(x)} & & \uparrow{\scriptstyle Df(x)} \\ E_{m+1} & \longrightarrow & E_m & \longrightarrow & E_{m-1} \end{array}$$

From the first condition in Definition 1.6 it follows that $f : U_1 \to F_0$ is differentiable and the derivative $df(x)$ at $x \in U_1$ is given by

$$df(x) = Df(x) \in \mathcal{L}(E_1, F_0).$$

Actually, we shall show that $f \in C^1(U_1, F)$. This is a consequence of the following alternative definition of a class sc^1-map.

Proposition 1.7 (**Alternative definition**). *Let E and F be sc-smooth Banach spaces and let $U \subset E$ be an open subset. An sc^0-map*

$f : U \to F$ is of **class sc^1 if and only if** *the following conditions hold true.*

(1) *For every $m \geq 1$, the induced map*

$$f : U_m \to F_{m-1}$$

is of class C^1. In particular, the derivative df is the continuous map

$$U_m \to \mathcal{L}(E_m, F_{m-1}),\ x \to df(x).$$

(2) *For every $m \geq 1$ and every $x \in U_m$ the continuous linear operator $df(x) : E_m \to F_{m-1}$ has an extension to a continuous linear operator $Df(x) : E_{m-1} \to F_{m-1}$. In addition, the map*

$$\begin{aligned} U_m \oplus E_{m-1} &\to F_{m-1} \\ (x, h) &\to Df(x)h \end{aligned}$$

is continuous.

Proof. Assume that $f : U \to F$ is of class sc^1 according to Definition 1.6. Then $f : U_1 \to F$ is differentiable at every point x with the derivative $df(x) = Df(x)|E_1 \in \mathcal{L}(E_1, F)$, so that the extension of $df(x) : E_1 \to F$ to a continuous linear map $E \to F$ is the postulated map $Df(x)$. We claim that the derivative $x \mapsto df(x)$ from U_1 into $\mathcal{L}(E_1, F)$ is continuous. Arguing indirectly we find $\varepsilon > 0$ and sequences $x_n \to x$ in U_1 and h_n of unit norm in E_1 satisfying

$$\|df(x_n)h_n - df(x)h_n\|_0 \geq \varepsilon. \tag{1.3}$$

Taking a subsequence we may assume, in view of the compactness of the embedding $E_1 \hookrightarrow E_0$, that $h_n \to h$ in E_0. Hence, by the continuity property (2) in Definition 1.6, $df(x_n)h_n = Df(x_n)h_n \to Df(x)h$ in F_0. Consequently,

$$df(x_n)h_n - df(x)h_n = Df(x_n)h_n - Df(x)h_n \to Df(x)h - Df(x)h = 0$$

in F_0, in contradiction to (1.3).

Next we prove that $f : U_{m+1} \to F_m$ is differentiable at $x \in U_{m+1}$ with derivative

$$df(x) = Df(x)|E_{m+1} \in \mathcal{L}(E_{m+1}, F_m)$$

so that the required extension of $df(x)$ is the operator $Df(x) \in \mathcal{L}(E_m, F_m)$. The map $f : U_1 \to F_0$ is of class C^1 and $df(x) = Df(x)$. Since, by continuity property (2) in Definition 1.6, the map $(x, h) \mapsto Df(x)h$

from $U_{m+1}\oplus E_m\to F_m$ is continuous, we can estimate for $x\in U_{m+1}$ and $h\in E_{m+1}$,

$$
\begin{aligned}
&\frac{1}{\|h\|_{m+1}}\cdot\|f(x+h)-f(x)-Df(x)h\|_m\\
&\qquad=\frac{1}{\|h\|_{m+1}}\cdot\Big\|\int_0^1\big[Df(x+\tau h)\cdot h-Df(x)\cdot h\big]\,d\tau\Big\|_m\\
&\qquad\leq\int_0^1\Big\|\Big[Df(x+\tau h)\cdot\frac{h}{\|h\|_{m+1}}-Df(x)\cdot\frac{h}{\|h\|_{m+1}}\Big]\Big\|_m\,d\tau.
\end{aligned}
$$

Take a sequence $h\to 0$ in E_{m+1}. By the compactness of the embedding $E_m\hookrightarrow E_{m+1}$ we may assume that $\frac{h}{\|h\|_{m+1}}\to h_0$ in E_m. By the continuity property in Definition 1.6 we now conclude that the intengrand converges uniformly in τ to $\|Df(x)h_0-Df(x)h_0\|_m=0$ as $h\to 0$ in E_{m+1}. This shows that $f:U_{m+1}\to F_m$ is indeed differentiable at x with derivative $df(x)$ being the bounded linear operator

$$df(x)=Df(x)\in\mathcal{L}(E_{m+1},F_m).$$

The continuity of $x\mapsto df(x)\in\mathcal{L}(E_{m+1},F_m)$ follows by the argument already used above, so that $f:U_{m+1}\to F_m$ is of class C^1. This finishes the proof of Proposition 1.7. ■

The situation in Proposition 1.7 is illustrated by the two diagrams

$$
\begin{tikzcd}
F_m \arrow[r] & F_{m-1} \\
U_m \arrow[u, "f"] \arrow[r] & U_{m-1} \arrow[u, "f"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & F_{m-1} \\
E_m \arrow[ur, "df(x)"] \arrow[r] & E_{m-1} \arrow[u, "Df(x)"']
\end{tikzcd}
$$

where $x\in U_m\subset E_m$.

Remark 1.8. The extension $Df(x)$ in Proposition 1.7 is unique because $E_m\subset E_{m-1}$ is dense. In general, if $A:G\to F$ is a continuous linear operator between Banach spaces and if $G\subset E$ is a dense linear subspace of another Banach space E, the question of extending A to a continuous linear operator $\widehat{A}:E\to F$ is immediately answered. By the density of G in E an extension is unique, if it exists. A necessary and sufficient condition for the existence of a continuous extension is the existence of a positive constant C such that

$$\|Ah\|_F \leq C \cdot \|h\|_E \quad \text{for all } h \in G.$$

It is important to note that the second condition in (1.7) says that the map $U_m \to \mathcal{L}_{\text{co}}(E_{m-1}, F_{m-1})$ is continuous where $\mathcal{L}_{\text{co}}$ is the space of continuous linear operators equipped with the compact open topology (rather than the usual operator topology).

If the sc-continuous map $f : U \subset E \to F$ is of class sc^1, then its tangent map

$$Tf : TU \to TF$$

is an sc-continuous map. If now Tf is of class sc^1, then $f : U \to F$ is called of **class $\mathbf{sc^2}$** . In this case the tangent map of Tf,

$$T(Tf) : T(TU) \to T(TF)$$

is sc-continuous. We shall use the notation $T^2f = T(Tf)$ and $T^2U = T(TU)$ and $T^2F = T(TF)$. Then $T^2U = T(TU) = T(U^1 \oplus E) = (U^2 \oplus E^1) \oplus (E^1 \oplus E)$ has the sc-smooth structure

$$(T^2U)_m = U_{m+2} \oplus E_{m+1} \oplus E_{m+1} \oplus E_m, \quad m \geq 0.$$

Proceeding this way inductively, the map $f : U \to F$ is called of **class $\mathbf{sc^k}$** if the sc-continuous map $T^{k-1}f : T^{k-1}U \to T^{k-1}F$ is of class sc^1. Its tangent map $T(T^{k-1}f)$ is then denoted by T^kf. It is a sc-continuous map $T^kU \to T^kF$. A map which is of class sc^k for every k is called **sc-smooth** or of **class $\mathbf{sc^\infty}$** . To illustrate these concepts we shall prove the following consequences of the definitions.

Proposition 1.9. *If $f : U \subset E \to F$ is of class sc^k, then*

$$f : U_{m+k} \to F_m$$

is of class C^k for every $m \geq 0$.

Proof. For $k = 1$ the proposition follows from our alternative definition (Proposition 1.7). In this case the tangent map $Tf : TU \to TF$ is of the form

$$Tf(x_1, x_2) = (f(x_1), Df(x_1)[x_2]).$$

In particular, if $(x_1, x_2) \in U_{m+1} \oplus E_{m+1} \subset E_{m+1} \oplus E_m$ and $df(x_1) : E_{m+1} \to F_m$ is the derivative of f at x_1, then $Df(x_1)[x_2] = df(x_1)[x_2]$. The general case follows from the following claim which we prove by induction.

Let $f : U \to F$ be of class sc^k. Then $f : U_{m+k} \to F_m$ is of class C^k for all $m \geq 0$. In addition, if $m \geq 0$ and π is the projection of $(T^kF)_m$ onto the last factor F_m, then, at every point $x = (x_1, \dots x_{2^k}) \in E_{m+k} \oplus E_{m+k} \oplus \cdots \oplus E_{m+k} \subset (T^kU)_m$, the composition $\pi \circ T^k f(x)$ is a linear combination of terms of the form

$$d^j f(x_1)[x_{i_1}, \dots, x_{i_j}], \tag{1.4}$$

where $1 \leq j \leq k$.

We already know that the claim is true when $k = 1$. Assuming that our claim holds for $k \geq 1$ we show that it is also true for a map $f : U \to F$ of class sc^{k+1}. Given such a map f, then its tangent map $T^kf : T^kU \to T^kF$ is of class sc^1. Thus, in view of Proposition 1.7, the map $T^kf : (T^kU)_{m+1} \to (T^kF)_m$ is of class C^1 for every $m \geq 0$. In particular, it is also of class C^1 when considered as a map from $E_{m+k+1} \oplus \cdots \oplus E_{m+k+1} \subset (T^kU)_{m+1}$ into $(T^kF)_m$. Taking the projection π from $(T^kF)_m$ onto the last factor F_m the composition $\pi \circ T^kf : E_{m+k+1} \oplus E_{m+k+1} \oplus \cdots \oplus E_{m+k+1} \to F_m$ is continuously differentiable. By our inductive assumption, at points $x = (x_1, \dots, x_{2^k}) \in E_{m+k} \oplus \cdots \oplus E_{m+k}$, the map $\pi \circ T^kf(x)$ is a linear combinations of maps of the form (1.4). Because f is C^k, every term $d^jf(x_1)[x_{i_1}, \dots, x_{i_j}]$ with $j \leq k-1$ defines a C^1-map whose derivative is equal to

$$d^{j+1}f(x_1)[\widehat{x}_1, x_{i_1}, \dots, x_{i_j}] + \sum_{1 \leq l \leq j} d^jf(x_1)[x_{i_1}, \dots, \widehat{x}_{i_l}, \dots x_{i_j}] \tag{1.5}$$

where $(\widehat{x}_1, \widehat{x}_{i_1}, \dots, \widehat{x}_{i_j}) \in E_{m+k+1} \oplus \cdots \oplus E_{m+k+1}$.
Hence $d^kf(x_1)[x_{i_1}, \dots, x_{i_j}]$ also defines a C^1 map from $E_{m+k+1} \oplus \cdots \oplus E_{m+k+1}$ into F_m and this implies that f is of class C^{k+1}. Denoting by $d^{k+1}f(x_1)$ the derivative of f of order $k+1$ we see that the derivative $d(\pi \circ T^kf)(x)$ of $\pi \circ T^kf$ at $x \in E_{m+k+1} \oplus \cdots \oplus E_{m+k+1}$ is a linear combination of terms (1.5) and of the term

$$d^{k+1}f(x_1)[\widehat{x}_1, x_{i_1}, \dots, x_{i_k}] + \sum_{1 \leq l \leq k} d^jf(x_1)[x_{i_1}, \dots, \widehat{x}_{i_l}, \dots x_{i_k}]. \tag{1.6}$$

Hence taking a point $(x, \widehat{x}) = (x_1, \dots, x_{2^k}, \widehat{x}_1, \dots, \widehat{x}_{2^k}) \in E_{m+k+1} \oplus \cdots \oplus E_{m+k+1} \subset (T^{k+1}U)_m$ and evaluating the composition $\pi \circ T^{k+1}f$ at $(x, \widehat{x})$ we see that $\pi \circ T^{k+1}f(x, \widehat{x})$ is a linear combination of terms of the form (1.4). This finishes the induction and hence the proof of the proposition. ■

The following criterion for sc-smoothness will be handy later on.

PROPOSITION 1.10. *Let E be a Banach space with a sc-smooth structure and let $U \subset E$ be an open subset. Assume that $f : U \to \mathbb{R}$ is sc-continuous and that the induced maps*

$$f_m := f|_{U_m} : U_m \to \mathbb{R}, \quad m \geq 0,$$

are of class C^{m+1}. Then f is of class sc^∞.

We observe that the deeper we go down the nested sequence of spaces the higher are the differentiability properties of the sc-smooth functions. The space $\mathbb{R}$ is, as usual, equipped with the constant sc-structure.

PROOF. The proposition follows from the following statement which we shall prove by induction.

(k) The map $f : U \to \mathbb{R}$ is of class sc^k and the iterated tangent map $T^k f : T^k U \to T^k \mathbb{R}$ has the following property for every $m \geq 0$. Let $\pi : T^k \mathbb{R} \to \mathbb{R}$ be the projection on any factor $\mathbb{R}$ of $T^k \mathbb{R}$ and let $x = (x_1, \ldots, x_{2^k}) \in (T^k U)_m$. Then the composition $\pi \circ (T^k f)(x)$ is a linear combination of terms

$$d^j f(x_1)[x_{k_1}, ..., x_{k_j}], \tag{1.7}$$

where $0 \leq j \leq k$ and $x_1 \in U_{m+k}$ and $x_{k_i} \in E_{m_i}$ with $m_i \geq m + (j-1)$.

We start with $k = 1$. Take $x_1 \in U_1$ and define the linear map $Df(x_1) : E_0 \to \mathbb{R}$ by

$$Df(x_1)x_2 = df(x_1)x_2$$

where $df(x_1) : E_0 \to \mathbb{R}$ is the derivative of the map $f : U_0 \to \mathbb{R}$. Clearly, $Df(x_1) \in \mathcal{L}(E_0, \mathbb{R})$. Since $f : U_1 \to \mathbb{R}$ is of class C^2, its derivative $df(x_1) : E_1 \to \mathbb{R}$ is equal to $Df(x_1)|E_1$. Hence part (1) of Definition 1.6 holds. The continuity of the map $U_1 \oplus E_0 \to \mathbb{R}$ given by $(x_1, x_2) \mapsto Df(x_1)x_2$ follows from the continuity of $x_1 \mapsto Df(x_1) = df(x_1) \in \mathcal{L}(E_0, \mathbb{R})$. This also implies the continuity of the map $U_{m+1} \oplus E_m \to \mathbb{R}$ given by $(x_1, x_2) \mapsto Df(x_1)x_2$ since the convergence in $U_{m+1} \oplus E_m$ implies the convergence in $U_1 \oplus E_0$. The tangent map $Tf : TU \to T\mathbb{R}$ has the form

$$Tf(x_1, x_2) = (f(x_1), Df(x_1)x_2) = (f(x_1), df(x_1)x_2)$$

for $(x_1, x_2) \in U_{m+1} \oplus E_m$, $m \geq 0$. We have verified the assertion (k) for $k = 1$.

Next assume that the statement (k) holds for $k \geq 1$. Let $1 \leq j \leq k$. Setting $m = 0$ at first we assume $m_1, \ldots, m_j \geq j - 1$. Abbreviate

$U' = U_k \oplus E_{m_1} \oplus \cdots \oplus E_{m_j}$ and $E' = E_k \oplus E_{m_1} \oplus \cdots \oplus E_{m_j}$. It suffices to show that the map $\Phi : U' \to \mathbb{R}$ defined by

$$\Phi(x) := \Phi(x_1, x_{k_1}, \cdots, x_{k_j}) = d^j f(x_1)[x_{k_1}, \cdots, x_{k_j}] \tag{1.8}$$

is of class sc^1. Take $x = (x_1, x_{k_1}, \ldots, x_{k_j}) \in U_1' = U_{k+1} \oplus E_{m_1+1} \oplus \cdots \oplus E_{m_j+1}$ and define the linear map

$$D\Phi(x) : E_0' = E_k \oplus E_{m_1} \oplus \cdots \oplus E_{m_j} \to \mathbb{R}$$

by setting

$$\begin{aligned} D\Phi(x)(y) &= d^{j+1} f(x_1)[y_1, x_{k_1}, \cdots, x_{k_j}] \\ &\qquad + \sum_{1 \leq i \leq j} d^j f(x_1)[x_{k_1}, \cdots, y_{k_i}, \cdots, x_{k_j}]. \end{aligned} \tag{1.9}$$

Since $f : U_j \to \mathbb{R}$ is of class C^{j+1} and $x_1 \in U_{k+1} \subset U_j$ and $y_1 \subset E_k \subset E_j$ and $x_{k_i} \subset E_{m_i+1} \subset E_j$, it follows that the map

$$E_k \to \mathbb{R}, \quad y_1 \mapsto d^{j+1} f(x_1)[y_1, x_{k_1}, \cdots, x_{k_j}]$$

is a continuous linear operator in y_1. Also the maps

$$E_{m_i} \mapsto \mathbb{R}, \quad y_{k_i} \mapsto d^j f(x_1)[x_{k_1}, \cdots, y_{k_i}, \cdots, x_{k_j}]$$

are continuous linear maps. Consequently, $D\Phi(x) : E_0' \to \mathbb{R}$ is a continuous linear operator. Moreover, the map $\Phi : U_1' \to \mathbb{R}$ is of class C^1 and its derivative $d\Phi(x)$ at the point $x \in U_1'$ coincides with $D\Phi(x)|E_1'$. This shows that condition (1) of Definition 1.6 is satisfied. Note also that if $(x_n, y_n) \in U_1' \oplus E_0'$ converges in $U_1' \oplus E_0'$, then $D\Phi(x_n)y_n$ converges in $\mathbb{R}$ and since the convergence in $U_{m+1}' \oplus E_m$ implies the convergence in $U_1' \oplus E_0'$ we conclude the continuity of the map $(x, y) \mapsto D\Phi(x)y$ from $U_{m+1}' \oplus E_m'$ into $\mathbb{R}$. The tangent map $T\Phi : TU' \to T\mathbb{R}$ has the form

$$T\Phi(x, y) = (\Phi(x), D\Phi(x)(y))$$

where $\Phi(x)$ is given by (1.8) and $D\Phi(x)(y)$ by formula (1.9) for $(x, y) \in TU' = (U')^1 \oplus E' = (U_{k+1} \oplus E_{m_1+1} \oplus \cdots \oplus E_{m_j+1}) \oplus (E_k \oplus E_{m_1} \oplus \cdots \oplus E_{m_j})$. The tangent map Tf is of class sc^0 by the above remark. Inspecting the terms of $\Phi(x)$ and $D\Phi(x)(y)$ we see that the components of $T\Phi(x, y)$ are of the form (1.7) with indices satisfying the conditions in (k) however, with k replaced by $k + 1$. Hence the statement (k+1) holds true and the proof of Proposition 1.10 is complete. ■

The next and more general criterion for sc-smoothness is proved the same way as Proposition 1.10.

PROPOSITION 1.11. *Let E and F be sc-Banach spaces and let $U \subset E$ be open. Assume that the map $f : U \to F$ is sc^0 and that the induced map $f : U_{m+k} \to F_m$ is C^{k+1} for every $m, k \geq 0$. Then $f : U \to E$ is sc-smooth.*

The example coming up has many of the features of situations we are confronted with in the SFT, but is a little bit easier. It illustrates the sharp contrast between the new concept of sc-smoothness and the classical smoothness concept.

EXAMPLE 1.12. The $\mathbb{R}$-action of translation

$$\Phi : \mathbb{R} \oplus E \to E,\ (t, u) \mapsto t * u$$

introduced in (1.1) is an sc-smooth map. The spaces are equipped with the previously defined sc-structures. The tangent map

$$T\Phi : \mathbb{R} \oplus E^1 \oplus \mathbb{R} \oplus E \to E^1 \oplus E$$

is given by

$$(T\Phi)(t, u)[\delta t, \delta u] = \left(t * u, \delta t\left(t * \frac{du}{dt}\right) + t * (\delta u)\right).$$

This more difficult result is proved in Theorem 1.38 below. Here the increasing weights play a decisive role.

We point out that Φ is not differentiable with respect to t in the classical sense due to the loss of derivatives.

Taking the derivative of a function f or g, the target levels drop by 1 according to Proposition 1.7. Therefore, one could expect for the composition $g \circ f$ of the two maps, that the target level should drop by 2 in order to obtain a C^1-map. As it turns out, however, this is not the case.

THEOREM 1.13 (**Chain Rule**). *Assume that E, F and G are sc-smooth Banach spaces and $U \subset E$ and $V \subset F$ are open sets. Assume that $f : U \to F$ and $g : V \to G$ are of class sc^1 and $f(U) \subset V$. Then the composition $g \circ f : U \to G$ is of class sc^1 and the tangent maps satisfy*

$$T(g \circ f) = (Tg) \circ (Tf).$$

PROOF. We shall verifiy properties (1) and (2) in Definition 1.6 for $g \circ f$. From Proposition 1.7 we conclude that the functions $g : V_1 \to G$ and $f : U_1 \to F$ are of class C^1. Moreover, $Dg(f(x)) \circ Df(x) \in \mathcal{L}(E, G)$

if $x \in U_1$. Fix $x \in U_1$ and choose $h \in E_1$ sufficiently small so that $f(x+h) \in V^1$. Then, using the postulated properties of f and g,

$$\begin{aligned}
&g(f(x+h)) - g(f(x)) - Dg(f(x)) \circ Df(x)h \\
&= \int_0^1 Dg(tf(x+h) + (1-t)f(x))\ [f(x+h) - f(x) - Df(x)h]dt \\
&\quad + \int_0^1 \big([Dg(tf(x+h) + (1-t)f(x)) - Dg(f(x))] \circ Df(x)h\big)dt.
\end{aligned}$$

Consider the first term

(1.10)

$$\begin{aligned}
&\frac{1}{\|h\|_1}\int_0^1 Dg(tf(x+h) + (1-t)f(x))[f(x+h) - f(x) - Df(x)h]dt \\
&= \int_0^1 Dg(tf(x+h) + (1-t)f(x)) \cdot \frac{1}{\|h\|_1}[f(x+h) - f(x) - Df(x)h]dt.
\end{aligned}$$

If $h \in E_1$, the maps $[0,1] \to F_m$ defined by $t \to tf(x+h)+(1-t)f(x)$ are continuous and converge in $C^0([0,1], F_1)$ to the constant map $t \to f(x)$ as $\|h\|_1 \to 0$. Moreover, since f is of class sc^1,

$$a(h) := \frac{1}{\|h\|_1}\big[f(x+h) - f(x) - Df(x)h\big]$$

converges to 0 in F_0 as $\|h\|_1 \to 0$. Therefore, by the continuity assumption (2) in Definition 1.6, the map

$$(t, h) \to Dg(tf(x+h) + (1-t)f(x))[a(h)]$$

as a map from $[0,1] \times E_1$ into G_0 converges to 0 as $h \to 0$, uniformly in t. Consequently, the expression in (1.10) converges to 0 in G_0 as $h \to 0$ in E_1. Next consider the second integral

$$(1.11)\quad \int_0^1 \big[Dg(tf(x+h) + (1-t)f(x)) - Dg(f(x))\big] \circ Df(x)\frac{h}{\|h\|_1}\ dt.$$

The set of all $\frac{h}{\|h\|_1} \in E_1$ has a compact closure in E_0, in view of Definition 1.1, so that the closure of the set of all

$$Df(x)\frac{h}{\|h\|_1}$$

is compact in F_0 because $Df(x) \in \mathcal{L}(E_0, F_0)$ is a continuous map by Definition 1.6. Consequently, again by property (2) of Definition 1.1, every sequence h_n converging to 0 in E_1 possesses a subsequence having the property that the integrand of the integral in (1.11) converges to 0 in G_0 uniformly in t. Hence the integral (1.11) also converges to 0 in G_0 as $h \to 0$ in E_1. We have proved that

$$\frac{1}{\|h\|_1}\|g(f(x+h)) - g(f(x)) - Dg(f(x)) \circ Df(x)h\|_0 \to 0$$

as $h \to 0$ in E_1. Consequently, condition (1) of Definition 1.6 is satisfied for the composition $g \circ f$ with the operator

$$D(g \circ f)(x) = Dg(f(x)) \circ Df(x) \in \mathcal{L}(E_0, G_0),$$

where $x \in U_1$. We conclude that the tangent map $T(g \circ f) : TU \to TG$,

$$(x, h) \mapsto (\ g \circ f(x), D(g \circ f)(x)h\ )$$

is sc-continuous and, moreover, $T(g \circ f) = Tg \circ Tf$. The proof of Theorem 1.13 is complete. ■

The reader should realize that in the previous proof all conditions on sc^1 maps were used, i.e. it just works. From Theorem 1.13 one concludes by induction that the composition of two sc^∞-maps is also of class sc^∞ and, for every $k \geq 1$,

$$T^k(g \circ f) = (T^k g) \circ (T^k f).$$

Remark 1.14. There are other possibilities for defining new smoothness concepts. For example, we can drop the requirement of compactness of the embedding operator $E_n \to E_m$ for $n > m$. Then it is necessary to change the definition of smoothness in order to get the chain rule. One needs to replace the second condition in the definition of being sc^1 by the requirement that $Df(x)$ induces a continuous linear operator $Df(x) : E_{m-1} \to F_{m-1}$ for $x \in U_m$ and that the map $U_m \to \mathcal{L}(E_{m-1}, F_{m-1})$ for $m \geq 1$ is continuous. For this theory the sc-smooth structure on E given by $E_m = E$ recovers the usual C^k-theory. However, this modified theory seems not to be applicable to SFT. In particular, the $\mathbb{R}$-action of translation in Example 1.12 would already fail to be smooth using this alternative concept of smoothness.

A **sc-diffeomorphism** $f : U \to V$, where U and V are open subsets of sc-spaces E and F with the induced sc-structure*, is by definition a homeomorphism $U \to V$ so that f and f^{-1} are sc-smooth. Let us note that in view of Theorem 1.13 we can define the pseudogroup of local sc-diffeomorphisms. As a corollary we obtain the category of

*In the following we will sometimes say that a Banach space has a sc-structure rather than a sc-smooth structure.

sc-manifolds consisting of second countable paracompact topological spaces equipped with a maximal atlas of sc-smoothly compatible charts. We will develop this concept later on where we will illustrate it with some examples.

1.3. sc-Operators

We need several definitions.

DEFINITION 1.15. *Consider E equipped with a sc-smooth structure.*

- *An* **sc-smooth subspace** *F of E consists of a closed linear subspace $F \subseteq E$, so that $F_m = F \cap E_m$ defines a sc-smooth structure for F in the sense of Definition 1.1.*
- *An* **sc-smooth subspace** *F* **of** *E* **splits** *if there exists another sc-smooth subspace G so that on every level we have the topological direct sum*

$$E_m = F_m \oplus G_m$$

We shall use the notation

$$E = F \oplus_{sc} G.$$

DEFINITION 1.16. *Let E and F be sc-smooth Banach spaces.*

- *An* **sc-operator** *$T : E \to F$ is a bounded linear operator inducing bounded linear operators on all levels*

$$T : E_m \to F_m.$$

- *An* **sc-isomorphism** *is a surjective sc-operator $T : E \to F$ such that T is invertible and $T^{-1} : E \to F$ is also an sc-operator.*

It is useful to point out that a finite-dimensional subspace K of E which splits the sc-smooth space E necessarily belongs to E_∞.

PROPOSITION 1.17. *Let E be a sc-smooth Banach space and K a finite-dimensional subspace of E_∞. Then K splits the sc-space E.*

PROOF. Take a basis $e_1, ..., e_n$ for K and fix the associated dual basis. By Hahn Banach this dual basis can be extended to continuous linear functionals $\lambda_1, ..., \lambda_n$ on E. Now $P(h) = \sum_{i=1}^n \lambda_i(h) e_i$ defines a continuous projection on E with image in $K \subset E_\infty$. Hence P induces continuous maps $E_m \to E_m$. Therefore P is a sc-projection. Define $Y_m = (Id - P)(E_m)$. Setting $Y = Y_0$ we have $E = K \oplus Y$. We claim that $Y_m = E_m \cap Y_0$. By construction, $Y_m \subset E_m \cap Y_0$. An element

$x \in E_m \cap Y_0$ has the form $x = e - P(e)$ with $x \in E_m$ and $e \in E_0$. Since $P(e) \in E_\infty$ we see that $e \in E_m$, implying our claim. Finally, $Y_\infty = \cap_{m \geq 0} Y_m$ is dense in Y_m for every $m \geq 0$. Indeed, if $x \in Y_m$, we can choose $x_k \in E_\infty$ satisfying $x_k \to x$ in E_m. Then $(Id - P)x_k \in Y_\infty$ and $(Id - P)x_k \to (Id - P)x = x$ in Y_m. ■

We can introduce the notion of a linear Fredholm operator in the sc-setting.

Definition 1.18. *Let E and F be sc-smooth Banach spaces. A sc-operator $T : E \to F$ is called* **Fredholm** *provided there exist sc-splittings $E = K \oplus_{sc} X$ and $F = Y \oplus_{sc} C$ with the following properties.*

- *$K = kernel\ (T)$ is finite-dimensional.*
- *C is finite-dimensional.*
- *$Y = T(X)$ and $T : X \to Y$ defines a linear sc-isomorphism.*

The above definition implies $T(X_m) = Y_m$, that the kernel of $T : E_m \to F_m$ is K and that C spans its cokernel, so that

$$E_m = \ker T \oplus X_m \text{ and } F_m = T(E_m) \oplus C$$

for all $m \in \mathbb{N}$. The following observation, called the regularizing property, should look familiar.

Proposition 1.19. *Assume $T : E \to F$ is sc-Fredholm and*

$$Te = f$$

for some $e \in E_0$ and $f \in F_m$. Then $e \in E_m$.

Proof. Since $F_m = T(E_m) \oplus C$, the element $f \in F_m$ has the representation

$$f = T(x) + c$$

for some $x \in X_m$ and $c \in C$. Similarly, e has the representation

$$e = k + x_0,$$

with $k \in K = \ker T$ and $x_0 \in X_0$ because $E_0 = K \oplus X_0$. From $T(e) = f = T(x) + c$ and $T(x) = T(x_0)$ one concludes $T(x_0 - x) = c$. Hence $c = 0$ because $T(E_0) \cap C = \{0\}$. Consequently, $x_0 - x \in K$. Since $e - x = k + (x_0 - x) \in K$, $x \in E_m$, and $K \subset E_m$ one concludes $e \in E_m$ as claimed. ■

We end this subsection with an important definition and stability result for Fredholm maps.

Definition 1.20. *Let E and F be sc-Banach spaces. An sc-operator $R: E \to F$ is said to be an* **$\mathbf{sc}^+-$operator** *if $R(E_m) \subset F_{m+1}$ and if R induces a sc^0-operator $E \to F_1$.*

Let us note that due to the (level-wise) compact embedding $F_1 \to F$ a sc^+-operator induces on every level a compact operator. This follows immediately from the factorization

$$R: E \to F^1 \to F.$$

The stability result is the following statement.

Proposition 1.21. *Let E and F be sc-Banach spaces. If $T: E \to F$ is a sc-Fredholm operator and $R: E \to F$ a sc^+-operator, then $T+R$ is also a sc-Fredholm operator.*

Proof. Since $R: E_m \to F_m$ is compact for every level we see that $T + R : E_m \to F_m$ is Fredholm for every m. Let K_m be the kernel of $T + R : E_m \to F_m$. We claim that $K_m = K_{m+1}$ for every $m \geq 0$. Indeed, $K_{m+1} \subset K_m$. If $x \in K_m$, then $Tx = -Rx \in F_{m+1}$. Applying Proposition 1.19, $x \in E_{m+1}$ so that $x \in K_{m+1}$. Hence the kernel K_m is independent of m. Set $K = K_0$. By Proposition 1.17, K splits the sc-space E since it is a finite dimensional subset of E_∞. Hence we have the sc-splitting $E = K \oplus X$ for a suitable sc-subspace X. Next define $Y_m = (T + R)(E_m)$. This defines a sc-structure on $Y = Y_0$. Let us show that F induces a sc-structure on Y and that this is the one given by Y_m. For this it suffices to show that

$$Y \cap F_m = Y_m. \tag{1.12}$$

Clearly,

$$Y_m = (T+R)(E_m) \subset F_m \cap (T+R)(E_m) \subset F_m \cap (T+R)(E_0) = Y \cap F_m.$$

Next assume that $y \in Y \cap F_m$. Then there exists $x \in E_0$ with $Tx+Rx = y$. Since R is a sc^+-section it follows that $y - Rx \in F_1$ implying that $x \in E_1$. Inductively we find that $x \in E_m$ implying that $y \in Y_m$ and (1.12) is proved. Observe that we also have

$$F_\infty \cap Y = (\bigcap_{m\in\mathbb{N}} F_m) \cap Y = \bigcap_{m\in\mathbb{N}} (F_m \cap Y) = \bigcap_{m\in\mathbb{N}} Y_m = Y_\infty$$

In view of Lemma 1.22 below, there exists a finite dimensional subspace $C \subset F_\infty$ satisfying $F_0 = C \oplus Y$. From this it follows that

$F_m = C \oplus Y_m$. Indeed, since $C \cap Y_m \subset C \cap Y$, we have $C \cap Y_m = \{0\}$. If $f \in F_m$, then $f = c + y$ for some $c \in C$ and $y \in Y$ since $F_m \subset Y$ and $F_0 = C \oplus Y$. Hence $y = f - c \in F_m$ and using Proposition 1.19 we conclude $y \in F_m$. This implies, in view of (1.12), that $F_m = C \oplus Y_m$. We also have $F_\infty = C \oplus (F_\infty \cap Y) = C \oplus Y_\infty$. It remains to show that Y_∞ is dense in Y_m for every $m \geq 0$. Since $F_m = C \oplus Y_m$ with C finite-dimensional and Y_m closed, the norm $\|c + y\| = \|c\|_m + \|y\|_m$ is equivalent to the norm $\|c+y\|_m$. Take $y \in Y_m$. Then since F_∞ is dense in F_m we find sequences $c_n \in C$ and $y_n \in Y_\infty$ such that $c_n + y_n \to y$ in F_m. From the above remark about equivalent norms we conclude that the sequence c_n is bounded and we may assume $c_n \to c$. Hence the sequence y_n converges to some $y' \in Y_m$ because $Y_\infty \subset Y_m$ and Y_m is closed. Thus, $c + y' = y$ so that $c = 0$ and $y_n \to y$ in F_m, proving our claim. Consequently, we have the sc-splitting

$$F = Y \oplus_{sc} C$$

and the proof of the proposition is complete. ■

Lemma 1.22. *Assume F is a Banach space and $F = D \oplus Y$ with D of finite-dimension and Y a closed subspace of F. Assume, in addition, that F_∞ is a dense subspace of F. Then there exists a finite dimensional subspace $C \subset F_\infty$ such that $F = C \oplus Y$.*

Proof. Denoting by $d_1, \ldots, d_n$ a basis of D we define the finite dimensional subspaces N_i of F by setting $N_i = \operatorname{span}\{d_1, \ldots, \widehat{d_i}, \ldots, d_n\}$ where $\widehat{d_i}$ means that the vector d_i is omitted. The space $N_i \oplus Y$ is closed and since $d_i \notin N_i \oplus Y$ we have $\varepsilon_i := \operatorname{dist}(d_i, N_i \oplus Y) > 0$. Choose $0 < \varepsilon < \min_i \varepsilon_i$. In view of the fact that F_∞ is dense in F we find, for every $1 \leq i \leq n$, an element $c_i \in F_\infty$ satisfying $\|d_i - c_i\| < \varepsilon/(2n)$. The vectors $c_1, \ldots, c_n$ are linearly independent. Indeed, arguing by contradiction we assume $\sum_i \alpha_i c_i = 0$ with $\sum_i |\alpha_i| > 0$. Without loss of generality we may also assume $|\alpha_1| \geq |\alpha_2|, \ldots, |\alpha_n|$. Hence $c_1 = \sum_{i \geq 2} (-\alpha_i/\alpha_1) c_i = \sum_{i \geq 2} \beta_i c_i$ with $|\beta_i| \leq 1$. Then one estimates

$$\begin{aligned}
\frac{\varepsilon}{2n} &> \|d_1 - c_1\| = \|c_1 - \sum_{i \geq 2} \beta_i c_i\| = \|(d_1 - \sum_{i \geq 2} \beta_i d_i) - \sum_{i \geq 2} \beta_i (c_i - d_i)\| \\
&\geq \|d_1 - \sum_{i \geq 2} \beta_i d_i\| - \sum_{i \geq 2} |\beta_i| \|c_i - d_i\| \geq \varepsilon - \frac{n-1}{2n} \varepsilon > \frac{\varepsilon}{2n}.
\end{aligned}$$

This contradiction shows that the vectors c_i are linearly independent. Abbreviating $C := \operatorname{span}\{c_1, \ldots, c_n\}$ we will show that $C \cap Y = \{0\}$. Once again arguing by contradiction we assume that $\sum_i \alpha_i c_i \in Y$ for some constants α_i satisfying $\sum_i |\alpha_i| > 0$. We may assume $|a_1| \geq |a_2|, \ldots, |\alpha_n|$ so that $c_1 + \sum_{i\geq 2} \beta_i c_i \in Y$ where $\beta_i = \alpha_i/\alpha_1$. Note that $(-\sum_{i\geq 2} \beta_i d_i) + (c_1 + \sum_{i\geq 2} \beta_i c_i) \in N_i \oplus Y$. Since $\text{dist}(d_1, N_1 \oplus Y) = \varepsilon_i > \varepsilon$, we reach a contradiction as the following estimates show,

$$\begin{aligned}
\varepsilon < \varepsilon_1 &\leq \|d_1 - \big[-\sum_{i\geq 2} \beta_i d_i + (c_1 + \sum_{i\geq 2} \beta_i c_i)\big]\| \\
&= \|(d_1 - dc_1) + \sum_{i\geq 2} \beta_i (d_i - c_i)\| \leq \sum_{i\geq 1} \|d_i - c_i\| \leq n \cdot \frac{\varepsilon}{2n} = \frac{\varepsilon}{2}.
\end{aligned}$$

Finally, we prove $F = C \oplus Y$. Since C and D have the same dimension there is a linear isomorphism $\psi : C \to D$. Define $\Psi : C \oplus Y \to D \oplus Y$ by $\Psi(c+y) = \psi(c)+y$. Then Ψ is injective and since $\Psi(\psi^{-1}(d)+y) = d+y$ it is also surjective. Hence $C \oplus Y = D \oplus Y = F$ as claimed. The proof of the lemma is complete. ■

1.4. sc-Manifolds

In this section we are going to introduce the spaces needed to formulate the general Fredholm theory.

1.4.1. Topological Considerations. We start by recalling some definitions.

DEFINITION 1.23. *Let X be a topological space.*

- *The space X is said to be second countable if it has a countable basis for its topology.*
- *The space X is said to be completely regular if for every point $x \in X$ and every neighborhood $U(x)$ of x there exists a continuous map $f : X \to [0,1]$ satisfying $f(x) = 0$ and $f(y) = 1$ for all $y \in X \setminus U(x)$.*
- *A space X is said to be normal provided it is Hausdorff and disjoint closed subsets admit disjoint open neighborhoods.*
- *A space X is said to be paracompact if it is Hausdorff and for every open covering of X there exists a finer open covering which is locally finite.*

The crucial result concerning paracompact spaces is the existence of partitions of unity, see for example Dugundji [**6**].

PROPOSITION 1.24. *If X is a paracompact space and $\mathcal{U}$ an open covering of X, then there exists a sub-ordinate partition of unity.*

The following is a useful classical result by Urysohn.

PROPOSITION 1.25. *For a second countable topological space the notions of being normal or completely regular or metrizable are equivalent.*

An obvious consequence is the following corollary.

COROLLARY 1.26. *A second countable Hausdorff space X which is locally homeomorphic to open subsets of Banach spaces is metrizable and hence, in particular, paracompact.*

PROOF. The space X is completely regular since it is locally homeomorphic to open subsets of Banach spaces. Therefore, as a consequence of Proposition 1.25, the space X is metrizable and hence paracompact. ∎

1.4.2. sc-Manifolds. Using the results so far we can define sc-manifolds. This concept will not yet be sufficient to describe the spaces arising in SFT. However, certain components of the ambient spaces have subspaces which will inherit the structure of an sc-manifold.

DEFINITION 1.27. *Let X be a second countable Hausdorff space. An* **sc-chart** *for X consists of a triple (U, φ, E), where U is an open subset of X, E a Banach space with a sc-smooth structure and $\varphi : U \to E$ is a homeomorphism onto an open subset V of E. Two such charts are sc-smoothly compatible provided the transition maps are sc-smooth. An* **sc-smooth atlas** *consists of a family of charts whose domains cover X so that any two charts are sc-smoothly compatible. A maximal sc-smooth atlas is called a* **sc-smooth structure** *on X . The space X equipped with a maximal sc-smooth atlas is called an* **sc-manifold** *.*

Let us observe that a second countable Hausdorff space which admits an sc-smooth atlas has to be metrizable and paracompact since it is locally homeomorphic to open subsets of Banach spaces.

As a side remark we also note that the above construction addresses certain issues arising in the analysis underlying Gromov-Witten theory. For example, if we have a holomorphic curve with an unstable domain component we have to divide out by a non-discrete group. These issues will arise later when we construct the ambient spaces for SFT.

Assume that X has a sc-smooth structure. Then it possesses the filtration given by X_m for all $m \geq 0$, from which every X_m inherits an sc-smooth structure. We shall define the tangent bundle $TX \to X^1$ in a natural way so that the tangent projection is sc-smooth*. In order to do so we use an appropriate modification of the definition found, for example, in Lang's book [**16**]. Namely consider tuples (U, φ, E, x, h) where (U, φ, E) is a sc-smooth chart, $x \in U^1$ and $h \in E$. Call two such tuples equivalent provided $x = x'$ and $D(\varphi' \circ \varphi^{-1})(\varphi(x))h = h'$. An equivalence class $[U, \varphi, E, x, h]$ is called a tangent vector. Denote the whole collection of tangent vectors by TX. We have a canonical projection $p : TX \to X^1$. We define for an open subset $U \subset X$ the set TU by $TU = p^{-1}(U \cap X^1)$. For a chart (U, φ, E) we define the map

$$T\varphi : TU \to E^1 \oplus E$$

by

$$T\varphi([U, \varphi, E, x, h]) = (x, h).$$

One easily checks that the collection of all these maps defines a smooth atlas for TX and that the projection p is sc-smooth.

1.5. A Space of Curves

This section is the first installment of several sections in which we illustrate the abstract concepts in the classical Morse theory. Recall that for a Morse function $f : M \to \mathbb{R}$ on the compact Riemannian manifold M one studies the gradient flow on M defined by the equation

$$\begin{aligned} &\dot{x}(s) = \nabla f(x(s)),\ s \in \mathbb{R} \\ &x(0) = x \in M. \end{aligned}$$

Every solution $x(s)$ converges as $s \to \infty$ and $s \to \infty$ to critical points of the function f. The aim is to investigate the structure of the set of all these orbits connecting critical points and, moreover, the compactifications of these solution spaces consisting of broken trajectories.

For simplicity we first consider a set of curves in $\mathbb{R}^n$ connecting the point $a \in \mathbb{R}^n$ at $-\infty$ with the point $b \in \mathbb{R}^n$ at ∞, where $a \neq b$. We first choose a smooth reference curve $\varphi : \mathbb{R} \to \mathbb{R}^n$ connecting the two points and satisfying

$$\varphi(s) = a \text{ for } \ s \leq -1 \ \text{ and } \ \varphi(s) = b \text{ for } \ s \geq 1.$$

*The definition of sc-smooth maps between sc-manifolds is the obvious modification from standard manifold theory.

Then we equip the Sobolev space $E = H^2(\mathbb{R}, \mathbb{R}^n)$ with the sc-smooth structure $E_m = H^{m+2,\delta_m}$, where $\delta_0 = 0 < \delta_1 < \cdots$ is a strictly increasing sequence of weights. Now we introduce the space

$$\widehat{X} = \{u = \varphi + h |\ h \in E\}$$

of parametrized curves $u : \mathbb{R} \to \mathbb{R}^n$ connecting the point a at $-\infty$ with the point b at $+\infty$ and having Sobolev regularity equal to 2.

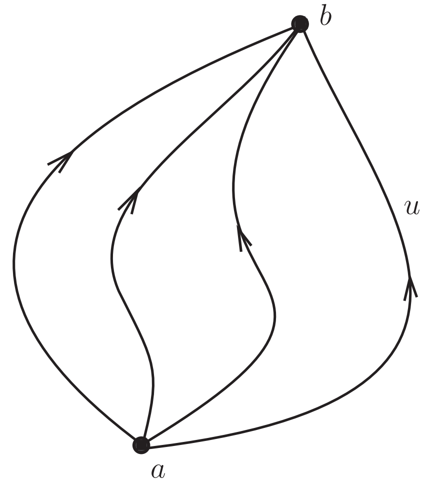


FIGURE 1.1

The topology of $\widehat{X}$ is induced by the complete metric $\widehat{d}$ on $\widehat{X}$,

$$\widehat{d}(u, v) = \|u - v\|_0$$

where the norm on the right hand side is the Sobolev norm on $E = E_0$. The metric is invariant under the $\mathbb{R}$-action of translation,

$$\widehat{d}(t * u, t * v) = \widehat{d}(u, v)$$

for all $t \in \mathbb{R}$ and $u, v \in \widehat{X}$. In order to divide out the $\mathbb{R}$-action we define an equivalence relation calling two elements in $\widehat{X}$ equivalent, $u \sim v$, if there exists a constant $t \in \mathbb{R}$ satisfying

$$u(s) = (t * v)(s) = v(s + t) \text{ for all } \ s \in \mathbb{R}.$$

By $[u]$ we shall denote the equivalence class containing u. The quotient space

$$X = \widehat{X} / \sim$$

consisting of the equivalence classes is equipped with the quotient topology which, as we show next, is determined by a complete metric. We define $d : X \times X \to [0, \infty)$ by

$$d(\alpha, \beta) = \inf\{\widehat{d}(u, v) |\, u \in \alpha, v \in \beta\}.$$

LEMMA 1.28. *The function d is a complete metric on X.*

PROOF. Clearly d is symmetric. Let $[u_1]$, $[u_2]$ and $[u_3]$ be three elements in X. Given $\varepsilon > 0$ we find representatives u_1, u_2 in $\widehat{X}$ satisfying

$$\widehat{d}(u_1, u_2) \leq d([u_1], [u_2]) + \varepsilon$$

and representatives u_2' of $[u_2]$ and u_3 of $[u_3]$ so that

$$\widehat{d}(u_2', u_3) \leq d([u_2], [u_3]) + \varepsilon.$$

Now u_2' and u_2 belong to the same orbit of the $\mathbb{R}$-action. Using the $\mathbb{R}$-invariance of $\widehat{d}$ we may therefore assume, replacing u_3 by some other representative, that $u_2 = u_2'$. Hence,

$$\widehat{d}(u_1, u_3) \leq d([u_1], [u_2]) + d([u_2], [u_3]) + 2\varepsilon.$$

This implies the triangle inequality for d. Finally, we have to show that $d(\alpha, \beta) = 0$ implies $\alpha = \beta$. If $d(\alpha, \beta) = 0$ we find sequences t_n, s_n and representatives u and v of α and β such that

$$\widehat{d}(t_n * u, s_n * v) \to 0.$$

By the $\mathbb{R}$-invariance,

$$\widehat{d}(t_n * u, s_n * v) = \| t_n * u - s_n * v \|_0 = \| (t_n - s_n) * u - v \|_0 .$$

Thus, setting $\tau_n = t_n - s_n$,

$$\|\tau_n * u - v\|_0 \to 0.$$

The sequence τ_n is bounded. Indeed, if $\tau_n \to \infty$ (for some subsequence), then one concludes from the definition of φ, using $a \neq b$, that $\|\tau_n * u - v\|_0 \to \infty$. Similarly, the sequence τ_n cannot have a subsequence converging to $-\infty$. Hence, after taking a subsequence we may assume that $\tau_n \to \tau_0$. This implies $\tau_0 * u = v$ implying $\alpha = \beta$. We have proved that d defines a metric on X implying that X is paracompact.

Next we show that the metric space (X, d) is complete. Given a Cauchy-sequence $\alpha_n \in X$ we can take a fast subsequence α_{n_j} satisfying $d(\alpha_{n_{j+1}}, \alpha_{n_j}) < 2^{-j}$. Pick a representative u_1 for α_{n_1}. Then we find a representative u_2 of α_{n_2} satisfying

$$\widehat{d}(u_2, u_1) < d(\alpha_{n_2}, \alpha_{n_1}) + 2^{-1}.$$

Then we can take a representative u_3 of α_{n_3} (applying the already previously used $\mathbb{R}$-invariance) so that

$$\widehat{d}(u_3, u_2) < d(\alpha_{n_3}, \alpha_{n_2}) + 2^{-2}.$$

Inductively we choose representatives $u_{j+1} \in \widehat{X}$ satisfying

$$\widehat{d}(u_{j+1}, u_j) < d(\alpha_{n_{j+1}}, \alpha_{n_j}) + 2^{-j} \leq 2^{-j+1}.$$

Hence (u_j) is a Cauchy sequence in $\widehat{X}$ and therefore has a limit $w \in \widehat{X}$. By construction,

$$\lim_{j\to\infty} \alpha_{n_j} = [w],$$

implying that the subsequence (α_{n_j}) and hence the whole sequence (α_n) converges, showing that the metric space (X, d) is complete. ■

The projection map

$$p : \widehat{X} \to X : u \to [u]$$

satisfies

$$d([u], [v]) \leq \widehat{d}([u], [v]).$$

Moreover, the map p is open. Indeed,

$$p^{-1}(p(U)) = \bigcup_{t\in\mathbb{R}} t * U.$$

Consequently, $p^{-1}(p(U))$ is open if U is open, implying, by definition of the quotient topology of X, that $p(U)$ is open. We leave it as an exercise to prove that d determines the quotient topology. The space E is separable since it can be viewed as a closed linear subspace of a three-fold product of L^2-spaces which are known to be separable. (For example, rational linear combinations of characteristic functions of closed intervals with rational boundaries constitute a countable dense subset). For a dense sequence u_j in $\widehat{X}$, the sequence $\alpha_j = [u_j]$ is dense in X. Now taking metric balls with rational radii around these points we obtain a countable basis for the topology on X. We have proved the following statement.

Lemma 1.29. *The previously constructed space X is a complete metric space with a countable dense subset. In particular, it is a second countable paracompact space. Moreover, the projection map*

$$p : \widehat{X} \to X$$

is continuous and open.

Next we show that X carries the structure of a sc-manifold. Fix a class $[u]$ which is represented by

$$u = \phi + h_0 \text{ for some } h_0 \in E_\infty. \tag{1.13}$$

Let $\Sigma = \Sigma_u$ be an affine hyperplane in $\mathbb{R}^n$ such that the path u intersects Σ transversally for some parameter value t_0. Using the $\mathbb{R}$-action we may assume that $t_0 = 0$. Recall the continuous embedding $E \hookrightarrow C^1(\mathbb{R}, \mathbb{R}^n)$ by the Sobolev embedding theorem.

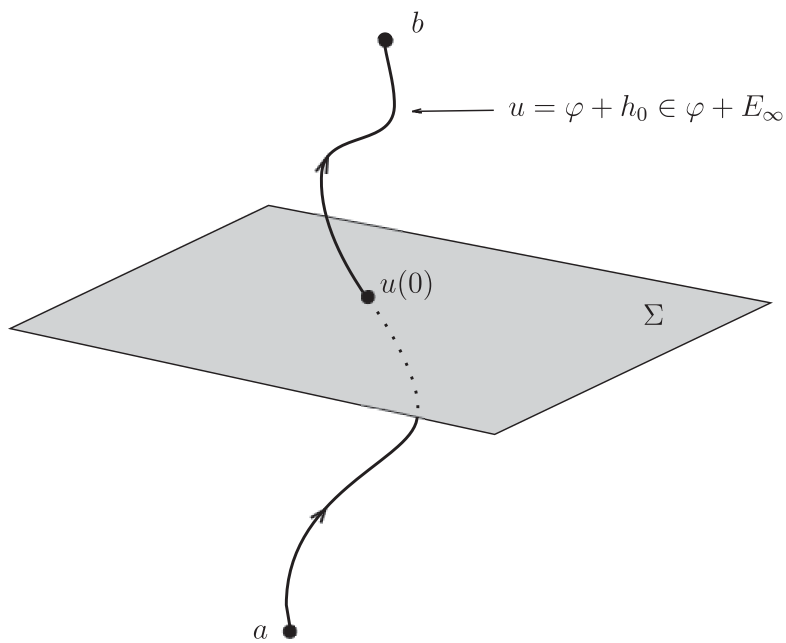


FIGURE 1.2

LEMMA 1.30. *Consider the path u as described in* (1.13). *Then there exists a real number $\varepsilon > 0$ such that for every $h \in E$ satisfying*

$$\|h\|_{C^1([-\varepsilon,\varepsilon],\mathbb{R}^n)} < \varepsilon^2,$$

the path $u + h$ has a unique intersection with Σ at some time $t(h) \in (-\varepsilon, \varepsilon)$. In addition, defining the open subset $U \subset E$ by

$$U = \{h \in E |\ \|h\|_{C^1([-\varepsilon,\varepsilon],\mathbb{R}^n)} < \varepsilon^2\},$$

the map $h \mapsto t(h)$ from U into $\mathbb{R}$ is sc-smooth.

PROOF. We represent the hyperplane Σ as $\Sigma = \{x \in \mathbb{R}^n \,|\, \lambda(x) = c\}$ for a linear map $\lambda : \mathbb{R}^n \to \mathbb{R}$ and a constant $c \in \mathbb{R}$. By assumption, $\lambda(u(0)) = c$ and $\lambda(u'(0)) \neq 0$ since u intersects Σ transversally at time $t = 0$. Now introduce the function $F : [-\varepsilon, \varepsilon] \times C^1([-\varepsilon, \varepsilon], \mathbb{R}^n) \to \mathbb{R}$ by

$$F(t, h) = \lambda(u(t) + h(t)) - c.$$

Then $F(0, 0) = 0$ and the derivative $d_t F(0, 0)$ of F with respect to t at the point $(0, 0)$ is equal to $d_t F(0, 0) = \lambda(u'(0)) \neq 0$. Using the implicit function theorem one finds $\varepsilon > 0$ and, for every h satisfying $\|h\|_{C^1([-\varepsilon,\varepsilon],\mathbb{R}^n)} < \varepsilon^2$, a unique time $t(h) \in (-\varepsilon, \varepsilon)$ solving the equation $F(t(h), h) = 0$ and satisfying $t(0) = 0$. In addition, the map $h \mapsto t(h)$ is of class C^1. This shows that $u + h$ intersects Σ at a unique time in $(-\varepsilon, \varepsilon)$ given by $t(h)$. In order to prove the second statement in the lemma we take the sc-smooth space $F = H^2((-\varepsilon, \varepsilon), \mathbb{R}^n)$ with the nested sequence $F_m = H^{m+2}((-\varepsilon, \varepsilon), \mathbb{R}^n)$ of Sobolev spaces. Define the open subset $V \subset E$ by $V = \{h \in F |\, \|h\|_{C^1([-\varepsilon,\varepsilon],\mathbb{R}^n)} < \varepsilon^2\}$ filtrated by $V^m := V \cap F_m$ for all $m \geq 0$. Since $F \subset C^1([-\varepsilon, \varepsilon], \mathbb{R}^n)$ is continuously embedded, the function $h \mapsto t(h)$ from $V^0 \to \mathbb{R}$ is continuously differentiable. Now consider the map $h \mapsto t(h)$ from V^m into $\mathbb{R}$ and use the fact that $F_m \subset C^{m+1}([-\varepsilon, \varepsilon], \mathbb{R}^n)$ is continuously embedded. Since $u \in C^\infty([-\varepsilon, \varepsilon], \mathbb{R}^n)$, the map $F : [-\varepsilon, \varepsilon] \times C^{m+1}([-\varepsilon, \varepsilon], \mathbb{R}^n) \to \mathbb{R}$ is of class C^{m+1}. By the implicit function theorem and the Sobolev embedding theorem again the map $h \mapsto t(h)$ from V^m into $\mathbb{R}$ is also of class C^{m+1}. Consequently, Proposition 1.10 implies the sc-smoothness of the map $h \mapsto t(h)$ from $V \subset F$ into $\mathbb{R}$. Finally, defining the open set $U \subset E$ by $U = \{h \in E |\, \|h\|_{C^1([-\varepsilon,\varepsilon],\mathbb{R}^n)} < \varepsilon^2\}$, the composition $U \to V \to \mathbb{R}$ given by $h \mapsto t(h|_{[-\varepsilon,\varepsilon]}) =: t(h)$ is also sc-smooth as claimed in Lemma 1.30. ■

LEMMA 1.31. *Given $\delta > 0$ there exists a number $\varepsilon_1 > 0$ so that the following holds. If $h, k \in E$ satisfy $|h(s)|, |k(s)| < \varepsilon_1$ for all $s \in \mathbb{R}$ and*

$$t * (u + h) = u + k$$

for some $t \in \mathbb{R}$, then $|t| < \delta$.

PROOF. Arguing indirectly assume that for a given $\delta > 0$ such a number $\varepsilon_1 > 0$ cannot be found. Then we find sequences (h_j) and (k_j) in E converging uniformly to 0 and a sequence (t_j) of real numbers so that $|t_j| \geq \delta$ and

$$t_j * (u + h_j) = u + k_j.$$

If the sequence (t_j) has a converging subsequence and $\bar{t}$ is its limit, then $\bar{t} * u = u$. This implies that $\bar{t} = 0$ which contradicts $|\bar{t}| \geq \delta$. Hence

$t_j \to \pm\infty$. Consider the case $+\infty$. From

$$(u + h_j)(t_j + s) = (u + k_j)(s)$$

we conclude as $t_j \to \infty$ that

$$b = u(s)$$

for all s. However, if $s \ll 0$ we know that $u(s)$ is close to the point $a \neq b$ giving a contradiction. This completes the proof. ■

Finally, we are able to introduce sc-smooth charts on the metric space X. We fix a class $[u] \in X$ with the smooth representative $u \in \widehat{X}$ as described in (1.13) and recall that u intersects the hyperplane $\Sigma \subset \mathbb{R}^n$ at $u(0) \in \Sigma$ transversally.

We choose $\delta = \varepsilon$ in Lemma 1.31 where $\varepsilon > 0$ is the number guaranteed by Lemma 1.30 and let $\varepsilon_1 > 0$ be the number associated with δ in Lemma 1.31. Let Σ_T be the tangent plane of Σ such that $\Sigma = u(0)+\Sigma_T$. Denote by F the codimension 1 subspace of E consisting of all h satisfying $h(0) \in \Sigma_T$. Let $U \subset F$ be the open neighborhood of $0 \in F$ consisting of all $h \in F$ satisfying

$$\| h \|_{C^1([-\varepsilon,\varepsilon])} < \varepsilon^2$$

and

$$|h(s)| < \varepsilon_1 \text{ for all } s \in \mathbb{R}.$$

Then we define

$$A : U \subset F \to X \quad \text{by} \quad A(h) = [u + h],$$

where u is the distinguished path from (1.13). Geometrically, h represents a small vector field along a smooth curve u such that $h(s) \in T_{u(s)}\mathbb{R}^n$. We have identified the tangent space $T_{u(s)}\mathbb{R}^n$ at the point $u(s)$ with $\mathbb{R}^n$ itself and have used the exponential map of the Euclidean metric on $\mathbb{R}^n$ to present the curves near u by $\exp_{u(s)}(h(s)) = u(s) + h(s)$. All these curves connect the point $a \in \mathbb{R}^n$ at $s = -\infty$ with the point $b \in \mathbb{R}^n$ at $s = \infty$.

Lemma 1.32. *The map A is a homeomorphism of $U \subset F$ onto some open subset of X.*

Proof. In order to prove that A is injective we assume $A(h) = A(k)$. Then

$$t * (u + h) = u + k$$

for some $t \in \mathbb{R}$. Because $|h(s)|, |k(s)| < \varepsilon_1$ for all s we deduce from Lemma 1.31 that $|t| < \delta = \varepsilon$. Since $\|h\|_{C^1([-\varepsilon,\varepsilon])} < \varepsilon^2$ there is, by Lemma 1.30, a unique point $t(h) \in (-\varepsilon, \varepsilon)$ so that $(u+h)(t(h)) \in \Sigma$. Since $h \in F$ and hence $(u+h)(0) \in \Sigma$ we must have $t(h) = 0$ and consequently $h = k$. This shows that the map A is injective. Clearly A is continuous.

Let us finally show that A is open. Assume $[u_0] = A(h_0)$ for some $h_0 \in U$. We find $t_0 \in \mathbb{R}$ such that

$$t_0 * u_0 = u + h_0.$$

Taking the appropriate representative for u_0 we may assume that $t_0 = 0$ so that $u_0(0) \in \Sigma$. Since $h_0 \in U$ we have

$$\begin{aligned} \|h_0\|_{C^1([-\varepsilon,\varepsilon])} &< \varepsilon^2 \\ |h_0(s)| &< \varepsilon_1 \quad \text{for all } s \in \mathbb{R}. \end{aligned}$$

By the definition of the topology on X, an open neighborhood of $[u_0]$ is generated by taking the equivalence classes of the elements of an open neighborhood of the representative u_0. Thus we take a sufficiently small open neighborhood $V_1 \subset \widehat{X}$ of u_0 so that for $v \in V_1$ we still have

$$\|v - u\|_{C^1([-\varepsilon,\varepsilon])} < \varepsilon^2$$

and

$$|v(s) - u(s)| < \varepsilon_1 \quad \text{for all } s \in \mathbb{R}.$$

Since $v = u + (v - u)$ there exists by Lemma 1.30 a unique $t(v) \in (-\varepsilon, \varepsilon)$ such that $v(t(v)) \in \Sigma$. Now define

$$P(v) := t(v) * v - u.$$

If $v = u_0$, so that $v = u + h_0$ we conclude from $h_0(0) \in \Sigma_T$ that $(u + h_0)(0) \in \Sigma$. Hence $t(v) = t(u_0) = 0$ and consequently $P(u_0) = u + h_0 - u_0 = h_0 \in F$. It follows from the definition that $P(v) \in F$. Since by Lemma 1.30 the map $v \mapsto t(v)$ from $V_1 \subset \widehat{X}$ into R is continuous and since the $\mathbb{R}$-action by Theorem 1.38 is sc-smooth, the composition

$$v \mapsto (t(v), v) \mapsto t(v) * v - u = P(v)$$

from V_1 into F is continuous. Therefore, if v is close enough to u_0 then $P(v)$ lies in U. Since V_1 is open in $\widehat{X}$ and since $[v] = [u + (v - u)] = [u + P(v)]$ we have proved that the map A is open, and the proof of Lemma 1.32 is complete. ■

From Lemma 1.32 we conclude that the map

$$\Phi = A^{-1}$$

defines an sc-chart on X whose domain is the open set $A(U) \subset X$. We next show that the transition maps are sc-smooth.

LEMMA 1.33. *Assume that $\Phi : O \to U$ and $\Phi' : O' \to V$ are sc-charts with $O \cap O' \neq \emptyset$ as described above. Then the transition map*

$$\Phi' \circ \Phi^{-1} : \Phi(O \cap O') \to \Phi'(O \cap O')$$

is sc-smooth.

PROOF. In view of the definition of an sc-chart we have $\Phi = A^{-1}$ and $\Phi' = A'^{-1}$ where

$$A : U \to O$$
$$A(h) = [u + h]$$

and where

$$A' : V \to O'$$
$$A'(k) = [v + k]$$

with U and V as described above, and with the paths u and v on the ∞-level. Assume

$$[u + h_0] = [v + k_0]$$

for some h_0 in U and k_0 in V. Then

$$t_0 * (u + h_0) = v + k_0$$

for some real number t_0. Evaluating at $s = 0$ we conclude $(u+h_0)(t_0) = v(0)+k_0(0) \in \Sigma_v$, where Σ_v is the hypersurface used in the construction of the "v-chart". Using the implicit function theorem as in Lemma 1.30 we find an sc-smooth map $h \mapsto t(h)$ defined for h close to h_0 in E so that the path $(u + h)$ intersects Σ_v transversally at the parameter value $t(h)$, and, in addition, satisfies $t(h_0) = t_0$. Now the transition map $\Phi' \circ \Phi^{-1} = A'^{-1} \circ A$ near h_0 is the map

$$h \to t_0(h) * (u + h) - v.$$

Since the $\mathbb{R}$-action is sc-smooth, the composition

$$h \to (t(h), h) \to t(h) * h + (t(h) * u - v)$$

shows the sc-smoothness of the transition map $\Phi' \circ \Phi^{-1}$ as claimed. ∎

We now consider all pairs (u_0, V_{u_0}) having the above properties, define the maps

$$A_{u_0} : V_{u_0} \to X : h \to [u_0 + h]$$

and summarize the discussion in the following two statements.

PROPOSITION 1.34. *The maps A_{u_0} are homeomorphisms onto open subsets of X.*

Denoting the image of A_{u_0} by U_{u_0} and setting $\Phi_{u_0} = A_{u_0}^{-1}$, the main result of this section is as follows.

THEOREM 1.35. *The collection of all (U_{u_0}, Φ_{u_0}) where u_0 varies over all smooth elements in $\widehat{X}$, i.e. elements in $\varphi + E_\infty$, defines an atlas for X whose transition maps are sc-smooth.*

It turns out to be useful for our M-polyfold constructions later on to have the following somewhat sharper version of Lemma 1.30 available.

Consider the distinguished path u in (1.13) which connects, in particular, the point a at $-\infty$ with the point b at ∞. Since $a \neq b$ we can define the positive number σ by

$$\sigma := \tfrac{1}{10} \cdot |a - b|.$$

Then there exists a positive number α such that $|b - u(s)| < \sigma$ if $s \geq \alpha$ and $|a - u(s)| < \sigma$ if $s \leq -\alpha$.

LEMMA 1.36. *Let u and α be as above. Given $\delta > 0$ there exists a number $\varepsilon_2 > 0$ having the following properties. If h and $k \in E$ satisfy $\|h\|_{C^0(\mathbb{R})} < \varepsilon_2$ and $\|k\|_{C^0(\mathbb{R})} < \varepsilon_2$, and if there exists $t \in \mathbb{R}$ satisfying*

$$t * (u + h) = u + k \quad \textit{on } [-\alpha, \alpha],$$

then $|t| < \delta$.

PROOF. Arguing by contradiction we assume that the assertion is wrong. Then there exists $\delta > 0$ and there exist sequences h_n, k_n and t_n so that $|t_n| \geq \delta$, $\|h_n\|_{C^0} \to 0$, $\|k_n\|_{C^0} \to 0$ and

$$t_n * (u + h_n) = u + k_n \quad \text{on} \quad [-\alpha, \alpha]. \tag{1.14}$$

If t_n does not have a bounded subsequence we may assume without loss of generality that $t_n \to \infty$. Consequently, evaluating (1.14) at $s \in [-\alpha, \alpha]$ and taking the limit as $n \to \infty$ we obtain

$$b = u(s) \quad \text{for } s \in [-\alpha, \alpha],$$

which is not possible because the interval $[-\alpha, \alpha]$ necessarily contains points s where $u(s)$ is different from b. We may therefore assume that the sequence t_n is bounded and going over to a subsequence we may assume that $t_n \to t_0$. Hence taking the limit in (1.14) as $n \to \infty$ we obtain

$$t_0 * u = u \quad \text{on } [-\alpha, \alpha].$$

Assume that $t_0 > 0$. If on one hand $t_0 \geq 2\alpha$, then $u(-\alpha) = u(-\alpha + t_0)$. Since $u(-\alpha)$ is σ-close to a while $u(-\alpha + t_0)$ is σ-close to b we have a contradiction. If on the other hand $0 < t_0 < 2\alpha$, then there exists an integer $j \geq 1$ such that $-\alpha + jt_0 \leq \alpha$ and $-\alpha + (j+1)t_0 > \alpha$, so that $u(-\alpha) = u(-\alpha + (j+1)t_0)$, and again we obtain a contradiction. Similarly, $t_0 < 0$ is not possible, so that necessarily $t_0 = 0$. This, however, contradicts $|t_0| \geq \delta$ and completes the proof of Lemma 1.36. ■

We end this section with a recipe for constructing smoothly compatible charts. We formulate it for manifolds and leave it as an exercise to the reader to fill in the details using Lemma 1.36 above.

Recipe 1.37. Let M be a smooth manifold and $a, b \in M$ distinct points. Denote by $\widehat{X}$ the space of maps $u : \mathbb{R} \to M$ which are in H^2_{loc} so that $\lim_{s\to\infty} u(s) = b$ and $\lim_{s\to-\infty} u(s) = a$. We require, moreover, that in local coordinates at a and b the following holds. If φ are local coordinates at the point b satisfying $\varphi(b) = 0$, then the map $\varphi \circ u(s)$ is defined for large s and belongs to $H^2([s_0, \infty), \mathbb{R}^n)$ for some large s_0. Similarly at the point a in which case we choose the Sobolev space $H^2((-\infty, -s_0], \mathbb{R}^n)$. The definition does not depend on the choice of φ. Let X be the quotient of $\widehat{X}$ by the $\mathbb{R}$-action. As before we can distinguish between elements of class H^{m+2,δ_m} for $m \geq 0$ giving a subset X_m of X. Choose $[u_0] \in X_\infty$. Then we find a parameter value s_0 satisfying $u_0'(s_0) \neq 0$. Take a chart φ mapping $u(s_0)$ to 0 and $u_0'(s_0)$ to $e_1 = (1, 0.., 0)$. Then pull back the standard metric on $\mathbb{R}^n$ to a neighborhood of $u(s_0)$ and extend it as a complete Riemannian metric to M. Taking if necessary a different representative we may assume that $s_0 = 0$. We define F_{u_0} to consist of all H^2-sections h of $u_0^*TM \to \mathbb{R}$ so that $h(0) \perp u_0'(0)$. Let Σ be the hyperplane in $T_{u_0(0)}M$ perpendicular to $u_0'(0)$ and let Σ_ε be the image of the ε-ball around 0 in Σ. The definition of H^2-sections depends a priori on the trivialization since $\mathbb{R}$ is non-compact. But it is fixed if we take the ends of the trivialization coming from the chart φ. Now let us denote by exp the exponential map. Then there exist numbers $\alpha > 0, \varepsilon > 0$ and $\varepsilon_1 > 0$ depending on u_0 so that the following holds.

- If h, k are in H^2 with $|h(s)|, |k(s)| < \varepsilon_1$ for all s and $t * \exp_{u_0} h = \exp_{u_0} k$ on $[-\alpha, \alpha]$ for some $t \in \mathbb{R}$, then $|t| < \varepsilon$.
- If h satisfies $|h(s)| + |\nabla h(s)| < \varepsilon^2$ for all $s \in [-\varepsilon, \varepsilon]$ then $\exp_{u_0}(h)$ is transversal to $\exp_{u_0(0)} \Sigma_\varepsilon$ at a unique point parameterized by some $s \in (-\varepsilon, \varepsilon)$.

Denote by O_{u_0} the open neighborhood of $0 \in F_{u_0}$ consisting of those h which satisfy $|h(s)| < \varepsilon_1$ for all $s \in \mathbb{R}$ and $|h(s)| + |\nabla h(s)| < \varepsilon^2$ for all $s \in [-\varepsilon, \varepsilon]$. Then the map

$$A_{u_0} : O_{u_0} \to X : A_{u_0}(h) = [\exp_{u_0}(h)]$$

defines a homeomorphism onto some open neighborhood U_{u_0} of $[u_0]$. The inverse Φ_{u_0} then defines a chart and the union of all these charts will cover X. (Note that by the discussion in the $\mathbb{R}^n$-case this can be guaranteed by taking any element in X and replacing it by a sufficiently close smooth representative.) Moreover, the transition maps are sc-smooth.

1.6. Appendix I

Consider the Hilbert space $E = L^2(\mathbb{R})$ which we equip with the sc-structure $E_m = H^{m,\delta_m}$ consisting of all maps having weak derivatives $D^k u$ up to order m so that $(D^k u)e^{\delta_m |s|}$ belongs to L^2. Here $\delta_0 = 0 < \delta_1 < \ldots$ is a strictly increasing sequence. We are going to study the $\mathbb{R}$-action

$$\mathbb{R} \oplus E \to E : (t, u) \to t * u,$$

defined by the translation $(t * u)(s) = u(s + t)$ and prove the following theorem.

THEOREM 1.38. *The translation map $\mathbb{R} \oplus E \to E : (t, u) \to t * u$ is sc-smooth.*

We begin the proof with a simple observation.

LEMMA 1.39. *The map $(t, u) \to t * u$ is of class sc^0.*

PROOF. Fix a level m. It is easy to see that the smooth maps having compact support are dense in E_m. We can estimate $\| t * u \|_m$ in terms of t and u as follows:

$$\begin{aligned}
\|t * u\|_m^2 &= \sum_{k \leq m} \int_{\mathbb{R}} |u^{(k)}(s + t)|^2 e^{2\delta_m |s|} ds \\
&\leq \sum_{k \leq m} \int_{\mathbb{R}} |u^{(k)}(s + t)|^2 e^{2\delta_m |s+t|} e^{2\delta_m |t|} ds \\
&= e^{2\delta_m |t|} \cdot \|u\|_m^2.
\end{aligned}$$

Hence,

$$\| t * u \|_m \leq e^{\delta_m |t|} \cdot \| u \|_m .$$

If v is smooth and compactly supported, then $t*v \to v$ in C^∞ as $t \to 0$. Let u_0 and u be elements in E^m and v a compactly supported smooth map. Then

$$\begin{aligned}
&\|t*u-u_0\|_m \\
&\quad = \|(t*u-t*u_0)+(t*u_0-t*v)+(t*v-v)+(v-u_0)\|_m \\
&\quad \leq e^{\delta_m|t|}\cdot\big(\|u-u_0\|_m+\|u_0-v\ \|_m\big)+\|\ t*v-v\|_m+\|v-u_0\|_m \\
&\quad \leq (e^{\delta_m|t|}+1)\cdot[\|u-u_0\|_m+\|u_0-v\|_m]+\|t*v-v\|_m.
\end{aligned}$$

Given $\varepsilon>0$ we choose v so that $\|u_0-v\|_m<\varepsilon$. Thus for all $\|u-u_0\|<\varepsilon$ and $|t|$ small enough,

$$\|\ t*u-u_0\ \|_m\leq 10\cdot\varepsilon+\|\ t*v-v\ \|_m$$

and taking $|t|$ even smaller the right-hand side is smaller than $11\cdot\varepsilon$. This proves the continuity of $(t,u)\to t*u$ on level m at the point $(0,u_0)$. Writing

$$t*u-t_0*u_0=(t-t_0)*(t_0*u)-t_0*u_0$$

we obtain continuity on level m at every point as a consequence of the previous discussion. ■

At this point it is useful to recall some rudimentary background on strongly continuous semigroup theory as for example can be found in A. Friedman's book about partial differential equations (page 93-100). As a consequence of the previous lemma we obtain on every level m a strongly continuous group of operators. Let A_m be the associated infinitesimal generator on E^m. By definition it has the domain

$$D(A_m)=\Big\{u\in E^m\ |\ \lim_{h\to 0^+}\frac{h*u-u}{h}\ \text{exists}\Big\}.$$

Moreover, A_m is defined by

$$A_m(u)=\lim_{h\to 0^+}\frac{h*u-u}{h}.$$

From the theory of Sobolev spaces (in the simple case of one variable) we deduce that the operator A_m is equal to the weak derivative $\frac{d}{ds}$. We summarize these facts in the following lemma.

Lemma 1.40. *The infinitesimal generator A_m has as domain all elements in E^m which have weak derivatives up to order m which again belong to E^m. Hence*

$$D(A_m)=H^{m+1,\delta_m}(\mathbb{R}).$$

The standard semigroup theory tells us for $u \in D(A_m)$ that the map

$$\mathbb{R} \to E^m : t \to t * u$$

is C^1 and also gives a continuous map into $D(A_m)$ equipped with its graph norm. Moreover,

$$\frac{d}{dt}(t * u) = A_m(t * u) = t * (A_m u).$$

If $u \in E^{m+1}$, then $u \in D(A_m)$. Indeed, since $\delta_m < \delta_{m+1}$ we have $E^{m+1} = H^{m+1,\delta_{m+1}} \subset H^{m+1,\delta_m} = D(A_m)$. Hence the derivative of $\Phi_{m+1}(t, u) = t * u$, viewed as a map from $\mathbb{R} \oplus E^{m+1}$ into E^m, is given by

$$D\Phi_{m+1}(t, u)(\delta t, \delta u) = \delta t \cdot (t * (A_m u)) + t * (\delta u).$$

We shall show that the map from $\mathbb{R} \oplus E^{m+1}$ into $\mathcal{L}(\mathbb{R} \oplus E^{m+1}, E^m)$ defined by

$$(t, u) \to D\Phi_{m+1}(t, u)$$

is continuous. What will make this true is that $H^{m+1,\delta_{m+1}}$ is a subspace of $D(A_m) = H^{m+1,\delta_m}$. Let $a \in \mathbb{R}$ and $h \in E^{m+1}$. Then

$$\begin{aligned}
&\|a(t * A_m u - t_0 * (A_m u_0)) + t * h - t_0 * h\|_m \\
&\quad \leq |a| \|t * (A_m(u - u_0)) + t * (A_m u_0) - t_0 * (A_m u_0)\|_m \\
&\quad + \| \, t * h - t_0 * h \, \|_m \, .
\end{aligned}$$

We look for an estimate uniform over $|a| + \|h\|_{m+1} \leq 1$. Taking the supremum over all such elements we obtain

$$\begin{aligned}
&\|D\Phi_{m+1}(t, u) - D\Phi_{m+1}(t_0, u_0)\|_{\mathcal{L}(\mathbb{R}\oplus E^{m+1}, E^m)} \\
&\quad \leq \| \, t * (A_m(u - u_0)) + t * (A_m u_0) - t_0 * (A_m u_0) \, \|_m \\
&\quad + \sup_{\|h\|_{m+1}=1} \|t * h - t_0 * h\|_m \\
&\quad =: I + II.
\end{aligned}$$

Clearly, $I \to 0$ as $(t, u) \to (t_0, u_0)$ in $\mathbb{R} \oplus E^{m+1}$. In order to deal with expression II we recall that the embedding $E^{m+1} \to E^m$ is compact which implies that $II \to 0$ as $t \to t_0$. Indeed, arguing indirectly we would find sequences $t_k \to t_0$ and $(h_k) \subset E^{m+1}$ with $\| \, h_k \, \|_{m+1} = 1$ so that for a suitable $\delta > 0$

$$\|t_k * h_k - t_0 * h_k\|_m \ \geq \ \delta.$$

However, using the compact embedding $E^{m+1} \to E^m$, we may assume, perhaps passing to a subsequence, that $h_k \to h_0$ in E^m for some $h_0 \in$

E^m. This implies

$$t_k * h_k - t_0 * h_k \to t_0 * h_0 - t_0 * h_0 = 0$$

in E^m contradicting the estimate above.

REMARK 1.41. We would like to point out that the statement we just proved would fail if we had used the previously discussed different notion of sc-structure (without the compactness assumption).

So far we have verified that $\Phi_{m+1} : \mathbb{R} \oplus E^{m+1} \to E^m$ is C^1. We note that the expression for $D\Phi_{m+1}(t, u)(a, h)$ also defines for $(t, u) \in \mathbb{R} \oplus E^{m+1}$ a bounded linear operator $\mathbb{R} \oplus E^m \to E^m$ because the right-hand side of

$$D\Phi_{m+1}(t, u)(a, h) = a \cdot (t * (A_m u)) + t * h$$

is well defined for such data and $A_m u \in E^m$. Clearly, the map

$$\mathbb{R} \oplus E^{m+1} \oplus \mathbb{R} \oplus E^m \to E^m : (t, u, a, h) \to D\Phi_{m+1}(t, u)(a, h)$$

is continuous. We define the sc-map $\Phi : \mathbb{R} \oplus E \to E$ by setting $\Phi|\mathbb{R} \oplus E_m := \Phi_m : \mathbb{R} \oplus E_m \to E_m$ for all $m \geq 0$. Having found the required extension for the derivative $D\Phi_{m+1}(t, u)$ to the larger space $\mathbb{R} \oplus E^m$ we have proved that Φ is of class sc^1. We introduce the linear sc-operator

$$A : E^1 \to E$$

by setting $A|E_{m+1} = A_m|E_{m+1} : E_{m+1} \to E_m$ for all $m \geq 0$. The tangent map

$$T\Phi : T(\mathbb{R} \oplus E) \to TE : (t, u, a, h) \to (t * u, a \cdot (t * Au) + t * h),$$

explicitly given by

$$(T\Phi)_m : T(\mathbb{R} \oplus E)_m = (\mathbb{R} \oplus E_{m+1}) \oplus (\mathbb{R} \oplus E_m) \to (TE)_m = E_{m+1} \oplus E_m$$
$$(T\Phi)_m(t, u, a, h) = (t * u, a \cdot (t * A_m u) + t * h),$$

is of class sc^0. In order to prove that the map $T\Phi$ is of class sc^1 we observe that the previous discussion proves the assertion with the exception of the term

$$(t, u, a, h) \to a(t * Au).$$

The scalar multiplication by a is smooth and so it suffices to prove that the map

$$\mathbb{R} \oplus E^1 \to E : (t, u) \to t * \frac{du}{dt} = t * (Au)$$

is sc-smooth. Using $t * Au = A(t * u)$ for $u \in E^1$ we can factor the map as

$$\mathbb{R} \oplus E^1 \to \mathbb{R} \oplus E \to E \ : (t,u) \to (t, Au) \to t * (Au).$$

It is the composition of a sc-smooth map and a map we have previously established to be sc^1. Hence we conclude by Theorem 1.13 that $T\Phi$ is sc^1 allowing us to define the sc^0-map $T^2\Phi$. Arguing as above the higher differentiability is reduced by induction to the sc-smoothness of

$$\mathbb{R} \oplus E^k \to E : (t,u) \to t * (A^k u)$$

which can be factored as

$$(t,u) \to (t, A^k u) \to t * (A^k u)$$

the first map being a linear sc-operator

$$\mathbb{R} \oplus E^k \to \mathbb{R} \oplus E.$$

This completes the proof of Theorem 1.38. ∎

CHAPTER 2

M-Polyfolds

In this chapter we introduce a new class of spaces which can have locally varying dimensions but still admit some kind of tangent spaces.

2.1. Cones and Splicings

Let us call a subset C of some finite-dimensional vector space W a **cone** if there is a linear isomorphism $T : W \to \mathbb{R}^n$ mapping C onto $[0, \infty)^n$. If C can be mapped onto $[0, \infty)^k \times \mathbb{R}^{n-k}$, then it is called a **partial cone**.

FIGURE 2.1

DEFINITION 2.1. *Assume V is an open subset of some (partial) cone C and E is a Banach space with a sc-smooth structure. Moreover, let $\pi_v : E \to E$ with $v \in V$, be a family of sc-projections so that the induced map*

$$\Phi : V \oplus E \to E$$
$$\Phi(v, e) = \pi_v(e)$$

is sc-smooth. Then the triple $\mathcal{S} = (\pi, E, V)$ is called an **sc-smooth splicing**.

REMARK 2.2. diffcorner At this point we would like to recall the concept of linearization or differentiation at the boundaries with corners. If $C \subset W$ is a cone or a partial cone and E and F are Banach spaces, and if V is an open set in $C \oplus E$, then the map $f : V \to F$ is called differentiable at the point $x \in V$, if there exists a bounded linear map

$$Df(x) \in \mathcal{L}(W \oplus E, F)$$

which is the linearization at the point x in the sense that

$$\frac{1}{\|h\|} \cdot \|f(x+h) - f(x) - Df(x)h\| \to 0$$

as $h \to 0$ and $x + h \in V$. If the map $x \mapsto Df(x)$ from V into $\mathcal{L}(W \oplus E, F)$ is continuous, then f is of class $C^1(V, F)$, and so on. The same concept of linearization also applies to the definitions and results concerning the sc-smoothness in Section 1.2.

Since π_v is a projection,

$$\Phi(v, \Phi(v, e)) = \Phi(v, e). \tag{2.1}$$

The left-hand side is the composition of Φ with the sc-smooth map $(v, e) \to (v, \Phi(v, e))$. Introduce for fixed $(v, \delta v) \in V \oplus W$ the map

$$\begin{aligned} P_{(v,\delta v)} &: TE \to TE \\ (e, \delta e) &\to (\Phi(v, e), D\Phi(v, e)[\delta v, \delta e]). \end{aligned}$$

It has the property that the induced map

$$TV \oplus TE \to TE : (a, b) \to P_a(b)$$

is sc-smooth because, modulo the identification $TV \oplus TE = T(V \oplus E)$, it is equal to the tangent map of Φ. From (2.1) one deduces by means of the chain rule (Theorem 1.13) at the points $(v, e) \in V \oplus E_1$,

$$D\Phi(v, \Phi(v, e))[\delta v, D\Phi(v, e)(\delta v, \delta e)] = D\Phi(v, e)[\delta v, \delta e]$$

and together with the definition of P one computes

$$\begin{aligned} P_{(v,\delta v)} \circ P_{(v,\delta v)}(e, \delta e) &= P_{(v,\delta v)}(\pi_v(e), D\Phi(v, e)(\delta v, \delta e)) \\ &= (\pi_v^2(e), D\Phi(v, \pi_v(e))(\delta v, D\Phi(v, e)(\delta v, \delta e))) \\ &= (\pi_v(e), D\Phi(v, e)(\delta v, \delta e)) = P_{(v,\delta v)}(e, \delta e). \end{aligned}$$

Consequently, $P_{(v,\delta v)} \circ P_{(v,\delta v)} = P_{(v,\delta v)}$ so that the triple

$$T\mathcal{S} = (P, TV, TE)$$

is a sc-smooth splicing called the **tangent of the splicing $\mathcal{S}$**.

Definition 2.3 (**Splicing Core**). *If $\mathcal{S} = (\pi, E, V)$ is an sc- smooth splicing, then the associated* **splicing core** *is the image bundle of the projection π over V, i.e., it is the subset $K^{\mathcal{S}} \subset V \oplus E$ defined by*

$$K^{\mathcal{S}} := \{(v, e) \in V \oplus E | \, \pi_v(e) = e\}. \tag{2.2}$$

The projection π_v depends smoothly on the parameter $v \in V$. Therefore, if the dimension of E is finite, the images of the projections π_v have all the same rank so that the splicing core is a smooth vector bundle over V. If, however, the dimension of E is infinite, then the ranks of the fibers can change with the parameter v thanks to the definition of sc-smoothness. This truly infinite dimensional phenomenon is crucial for our purposes.

The splicing core of the tangent splicing $T\mathcal{S}$ is the set

$$K^{T\mathcal{S}} = \{(v, \delta v, e, \delta e) \in TV \oplus TE | \; P_{(v,\delta v)}(e, \delta e) = (e, \delta e)\} \tag{2.3}$$

and we have the canonical projection

$$\begin{gathered} K^{T\mathcal{S}} \to K^{\mathcal{S}} \\ (v, \delta v, e, \delta e) \mapsto (v, e). \end{gathered}$$

Clearly the fiber over every point $(v, e) \in V \oplus E^1$ is the sc-Banach space $W \oplus E$.

Definition 2.4. *A local* **m-polyfold model** *consists of a pair $(O, \mathcal{S})$ where O is an open subset of the splicing core $K^{\mathcal{S}} \subset V \oplus E$ associated with the sc-smooth splicing $\mathcal{S}$. The* **tangent** *$T(O, \mathcal{S})$ of the* **local m-polyfold model** *$(O, \mathcal{S})$ is the object defined by*

$$T(O, \mathcal{S}) = (K^{T\mathcal{S}}|O, T\mathcal{S})$$

where $K^{T\mathcal{S}}|O$ denotes the collection of all points in $K^{T\mathcal{S}}$ which project under the canonical projection $K^{T\mathcal{S}} \to K^{\mathcal{S}}$ onto O^1.

The above discussion gives us the natural projection

$$K^{T\mathcal{S}}|O \to O^1 : (v, \delta v, e, \delta e) \to (v, e).$$

In the following we shall write O instead of $(O, \mathcal{S})$, but keep in mind that $\mathcal{S}$ is part of the structure. Hence the tangent $TO = T(O, \mathcal{S})$ of the open subset O of the splicing core $K^{\mathcal{S}}$ is simply the set

$$TO = K^{T\mathcal{S}}|O. \tag{2.4}$$

Note that for an open subset O of a splicing core we have an induced filtration. Hence we may talk about sc^0-maps. We will see in Section 2.3 that there is also a well-defined notion of a sc^1-map in this setting. Before we discuss this, we first study an example which has some of the properties showing up later on. Moreover, it clarifies the close relationship between splicings and gluing. In fact, the concept of splicing captures on a very abstract level the features of gluing constructions. Gluing operations come up in many nonlinear elliptic problems as inverse operations of bubbling-off phenomena.

2.2. Example of a Splicing

Consider the linear space $E = E^+ \oplus E^-$ consisting of pairs (u, v) of maps

$$u \in E^+ = H^2([0, \infty), \mathbb{R}^n) \quad \text{and} \quad v \in E^- = H^2((-\infty, 0], \mathbb{R}^n).$$

We equip the Banach space E with a sc-structure requiring (u, v) on the level m to be of Sobolev class $(m + 2, \delta_m)$ where, as before, δ_m is a strictly increasing sequence of real numbers starting with $\delta_0 = 0$.

We now choose a smooth cut-off function $\beta : \mathbb{R} \to [0, 1]$ having the following properties

$$\begin{aligned} &\bullet\ \beta(-s) + \beta(s) = 1 \quad \text{for all } s \in \mathbb{R} \\ &\bullet\ \beta(s) = 1 \quad \text{for all } s \le -1 \\ &\bullet\ \beta'(s) < 0 \quad \text{for all } s \in (-1, 1) \end{aligned} \tag{2.5}$$

We also need the **gluing profile**

$$\varphi(r) = e^{1/r} - e. \tag{2.6}$$

It establishes a special diffeomorphism $\varphi : (0, 1] \to [0, \infty)$. Next define the real number R and the functions τ and α from $\mathbb{R}$ into $\mathbb{R}$ as follows.

$$\begin{aligned} &\bullet\ R = \varphi(r) \\ &\bullet\ \tau_R(s) \equiv \tau(s) = \beta(s - \tfrac{R}{2}) \\ &\bullet\ \alpha(s) = \tau(s)^2 + [1 - \tau(s)]^2. \end{aligned} \tag{2.7}$$

After all these definitions we can introduce the family of projections

$$\pi_r : E \to E$$

parametrized by $r \in [0, 1)$. If $r = 0$, we set $\pi_0 = \mathrm{Id}$ and if $0 < r < 1$ we define, recalling that $E = E^+ \oplus E^-$,

$$\pi_r(h, k) = (\widehat{h}, \widehat{k}) \in E^+ \oplus E^-$$

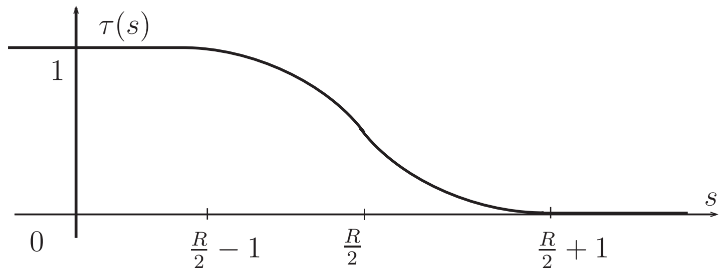


FIGURE 2.2. Graph of the function τ

$$\begin{aligned}\widehat{h}(s) &= \frac{\tau(s)}{\alpha(s)}\big[\tau(s)\cdot h(s) + (1-\tau(s))\cdot k(s-R)\big]\\ \widehat{k}(s') &= \frac{\tau(-s')}{\alpha(-s')}\big[(1-\tau(-s'))\cdot h(s'+R) + \tau(-s')\cdot k(s')\big]\end{aligned} \tag{2.8}$$

for $s \geq 0$ and $s' \leq 0$, respectively. The dependence on r of the right hand side is hidden in $R = \varphi(r)$ and $\tau(s) = \tau_R(s) = \beta(s - \frac{R}{2})$. We point out that near $s = 0$ the functions are not changed. Indeed, in view of the properties of the cut-off function β,

$$\begin{aligned}\widehat{h}(s) &= h(s) \qquad 0 \leq s \leq \frac{R}{2} - 1\\ \widehat{k}(s') &= k(s') \qquad -\frac{R}{2} + 1 \leq s \leq 0\end{aligned}$$

One readily verifies using formulae (2.12) below that

$$\pi_r \circ \pi_r = \pi_r.$$

The maps $\pi_r : E \to E$ have the following useful symmetry. Define the linear operator $S : E \to E$ by

$$S(u, v) = (\widehat{u}, \widehat{v}) \in E^+ \oplus E^-$$

where

$$\widehat{u}(s) = v(-s) \quad \text{and} \quad \widehat{v}(s') = u(-s').$$

Then

$$S \circ \pi_r = \pi_r \circ S. \tag{2.9}$$

The main result in this section is as follows.

THEOREM 2.5. *The triple $(\pi, E, [0,\frac{1}{2}))$ defines a sc-smooth splicing.*

PROOF. In view of the above symmetry we only have to prove that the map

$$[0,\tfrac{1}{2})\oplus E\to E^+$$
$$(r,h,k)\mapsto \widehat{h}$$

is sc-smooth. Here $\widehat{h}$ is defined by the formula in (2.8) for $0<r<\frac{1}{2}$ and by $\widehat{h}=h$ for $r=0$. Recall that $R=\varphi(r)$ if $r>0$ where φ is the gluing profile function (2.6). We note

$$\widehat{h}(s)=h(s)$$

if $s\in[0,\frac{R}{2}-1]\supset[0,1]$. Moreover, $\widehat{h}$ depends only on the values of $k\in E^-$ on the interval $[-\frac{R}{2}-1,-\frac{R}{2}+1]\subset(-\infty,-1]$. Take a smooth function $\sigma:\mathbb{R}\to[0,1]$ satisfying $\sigma(s)=0$ if $s\leq\frac{1}{4}$ and $\sigma(s)=1$ if $s\geq\frac{3}{4}$ while $\sigma'(s)>0$ for $s\in(\frac{1}{4},\frac{3}{4})$. We also note that the support of $\tau\cdot(1-\tau)$ lies in the interval $I_R=[\frac{R}{2}-1,\frac{R}{2}+1]$ and $\sigma(R-s)=1$ if $s\in I_R$. Using this one computes

$$\widehat{h}(s)=(1-\sigma(s))\cdot h(s)+\big[\pi_r(\sigma h,\sigma(-\cdot)k)\big]_1(s) \tag{2.10}$$

for all $s\geq 0$, where $[\,\cdot\,]_1$ stands for the first component of π_r which is an element in E^+. Explicitly,

$$\begin{aligned}&\big[\pi_r(\sigma h,\sigma(-\cdot)k)\big]_1(s)\\ &=\frac{\tau(s)}{\alpha(s)}\big[\tau(s)\cdot\sigma(s)\cdot h(s)+(1-\tau(s))\cdot\sigma(R-s)\cdot k(s-R)\big].\end{aligned}$$

The first part of the map (2.10), namely $h\mapsto(1-\sigma)\cdot h:E^+\to E^+$ is clearly sc-smooth. The second part of (2.10), namely

$$(h,k)\mapsto\big[\pi_r(\sigma h,\sigma(-\cdot)k)\big]_1,$$

from E into E^+ can be factored as follows,

$$[0,\tfrac{1}{2})\oplus E=[0,\tfrac{1}{2})\oplus E^+\oplus E^-\xrightarrow{A}[0,\tfrac{1}{2})\oplus F\oplus F\xrightarrow{B}F\oplus F\xrightarrow{C}F\xrightarrow{D}E^+.$$

Here, $F=H^2(\mathbb{R},\mathbb{R}^n)$ with the sc-structure as before, the level m corresponding to $(m+2,\sigma_m)$. The map A is defined by $(r,h,k)\mapsto(r,\sigma\cdot h,\sigma(-\cdot)\cdot k)$ and B is the map $(r,\widetilde{h},\widetilde{k})\mapsto\pi_r(\widetilde{h},\widetilde{k})$ according to the formula (2.8) but this time for all $s\in\mathbb{R}$ and $s'\in\mathbb{R}$. Finally, C is the projection onto the first component and D the restriction of the domain of definitions of the functions. It remains to prove that B is sc-smooth. For this purpose it is sufficient, by symmetry (2.9) again, to study the first component of B, namely

$$\begin{aligned}\Gamma : [0, \tfrac{1}{2}) \oplus F \oplus F &\to F\\ (r, h, k) &\mapsto \widehat{h}.\end{aligned}$$

We split $\Gamma = \Gamma_1 + \Gamma_2$ into the sum of two maps $\Gamma_j : [0, \frac{1}{2}) \oplus F \to F$ where j=1, 2. Abbreviating

$$\varphi_1(s) = \frac{\tau(s)^2}{\alpha(s)} \quad \text{and} \quad \varphi_2(s) = \frac{\tau(s) \cdot (1 - \tau(s))}{\alpha(s)},$$

we define $\Gamma_j : [0, \frac{1}{2}) \oplus F \to F$ by

$$\begin{aligned}\Gamma_1(r, h)(s) &= \varphi_1(s) \cdot h(s)\\ \Gamma_2(r, k)(s) &= \varphi_2(s) \cdot k(s - R)\end{aligned}$$

if $0 < r < \frac{1}{2}$. For $r = 0$, $\Gamma_1(0, h) = h$ and $\Gamma_2(0, k) = 0$. Recalling $\tau(s) = \beta(s - \frac{R}{2})$ the desired sc-smoothness is now a consequence of the technical Lemma 2.6 below. ■

In order to formulate Lemma 2.6 we define three maps $\Gamma_j : [0, 1) \oplus F \to F$ for $j = 1, 2, 3$.

(a) Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a smooth function which is constant outside of a compact interval so that $f_1(\infty) = 0$. Then we define $\Gamma_1 : [0, 1) \oplus F \to F$ by

$$\begin{aligned}\Gamma_1(0, h)(s) &= f_1(-\infty) \cdot h(s)\\ \Gamma_1(r, h)(s) &= f_1(s - \tfrac{R}{2}) \cdot h(s), \quad 0 < r < 1,\end{aligned}$$

(b) Let $f_2 : \mathbb{R} \to \mathbb{R}$ be a smooth function which is constant outside of a compact interval so that $f_1(-\infty) = 0$, we define $\Gamma_2 : [0, 1) \oplus F \to F$ by

$$\begin{aligned}\Gamma_2(0, h)(s) &= f_1(-\infty) \cdot h(s)\\ \Gamma_2(r, h)(s) &= f_2(s - \tfrac{R}{2}) \cdot h(s), \quad 0 < r < 1.\end{aligned}$$

(c) If $f_3 : \mathbb{R} \to \mathbb{R}$ is a smooth function having compact support, then $\Gamma_3 : [0, 1) \oplus F \to F$ is defined as follows.

$$\begin{aligned}\Gamma_3(0, h)(s) &= 0\\ \Gamma_3(r, h)(s) &= f_3(s - \tfrac{R}{2}) \cdot h(s - R), \quad 0 < r < 1.\end{aligned}$$

LEMMA 2.6. *The three maps* $\Gamma_j : [0, 1) \oplus F \to F$, $j = 1, 2, 3$, *defined above are sc-smooth.*

The proof of the lemma is postponed to Appendix II. We would like to point out that the lemma will be crucial in the proof of the smoothness of the transition maps later on.

Next we explain the relationship between gluing and splicings. Given the gluing profile φ and the cut-off function β we define for $(h,k)\in E=E^+\oplus E^-$ and $r\in[0,\frac{1}{2})$ the **full gluing** $\boxplus_r(h,k)$ as follows. If $r=0$, then

$$\boxplus_0(h,k)=(\oplus_0(h,k),\ominus_0(h,k))=((h,k),0).$$

If $r\in(0,1)$, we set $R=\varphi(r)$ and define

$$\boxplus_r(h,k)=(\oplus_r(h,k),\ominus_r(h,k))$$

by

$$\begin{aligned}\oplus_r(h,k)(s)&=\tau(s)\cdot h(s)+(1-\tau(s))\cdot k(s-R)\\ \ominus_r(h,k)(s)&=-(1-\tau(s))\cdot h(s)+\tau(s)\cdot k(s-R),\end{aligned}\tag{2.11}$$

where

$$\tau(s)=\beta(s-\tfrac{R}{2}).$$

The first equation in (2.11) holds for $s\in[0,R]$ and the second for all $s\in\mathbb{R}$. Recall that $\tau(s)=\tau_R(s)=0$ if $s\geq\frac{R}{2}+1$ and $\tau(s)=1$ if $s\leq\frac{R}{2}-1$ and $R=\varphi(r)\to\infty$ as $r\to 0$. Near the boundaries of the interval $[0,R]$ the functions are not changed,

$$\begin{aligned}\oplus_r(h,k)(s)&=h(s) &&0\leq s\leq\tfrac{R}{2}-1\\ \oplus_r(h,k)(s)&=k(s-R) &&\tfrac{R}{2}+1\leq s\leq R.\end{aligned}$$

Moreover, on $\mathbb{R}$,

$$\begin{aligned}\ominus_r(h,k)(s)&=k(s-R) &&s\leq\tfrac{R}{2}-1.\\ \ominus_r(h,k)(s)&=-h(s) &&\tfrac{R}{2}+1\leq s\end{aligned}$$

From the formulae (2.11) we deduce the following limits as $r\to 0$,

$$\begin{aligned}\lim_{r\to 0}\oplus_r(h,k)(s)&=h(s),\quad s\geq 0\\ \lim_{r\to 0}\oplus_r(h,k)(s'+R)&=k(s'),\quad s'\leq 0.\end{aligned}$$

Moreover, recalling that $(h,k) \in E^+ \oplus E^-$,

$$\begin{aligned} \lim_{r\to 0} \ominus_r(h,k)(s) &= \lim_{R\to\infty} h(s-R) = 0, \quad s \geq 0 \\ \lim_{r\to 0} \ominus_r(h,k)(s'+R) &= -\lim_{R\to\infty} k(s'+R) = 0, \quad s' \leq 0. \end{aligned}$$

The map $\oplus_r(h,k)$ is called the **glued map** for the gluing parameter r. In order to remove ambiguities in the inverse gluing we have added the so called **anti-glued map** $\ominus_r(h,k)$. One should think of $\oplus_r(h,k)$ as defined on the abstract interval I_r obtained by identifying a point $s \in [0,R] \subset [0,\infty)$ (the domain of h) with the point $s' \in [-R,0] \subset (-\infty,0]$ (the domain of k) via $s - R = s'$.

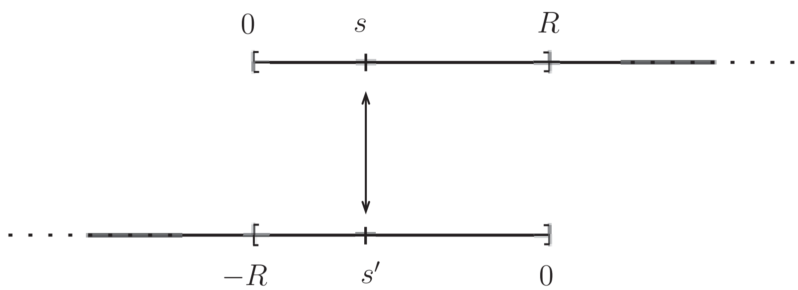


FIGURE 2.3. Identification $-R + s = s'$

From this point of view we have taken the coordinate s (rather than s') on I_r. The formulae, though they do not look symmetric in s and s', are indeed symmetric (the procedure is symmetric around $\frac{R}{2}$). Moreover, the anti-glued map has been defined on the abstract line obtained by gluing the two half-lines together by means of the same identification.

PROPOSITION 2.7. *The relation between splicing and gluing is as follows.*

$$\begin{aligned} \oplus_r \circ (\pi_r - Id) &= 0 \\ \ominus_r \circ \pi_r &= 0 \end{aligned}$$

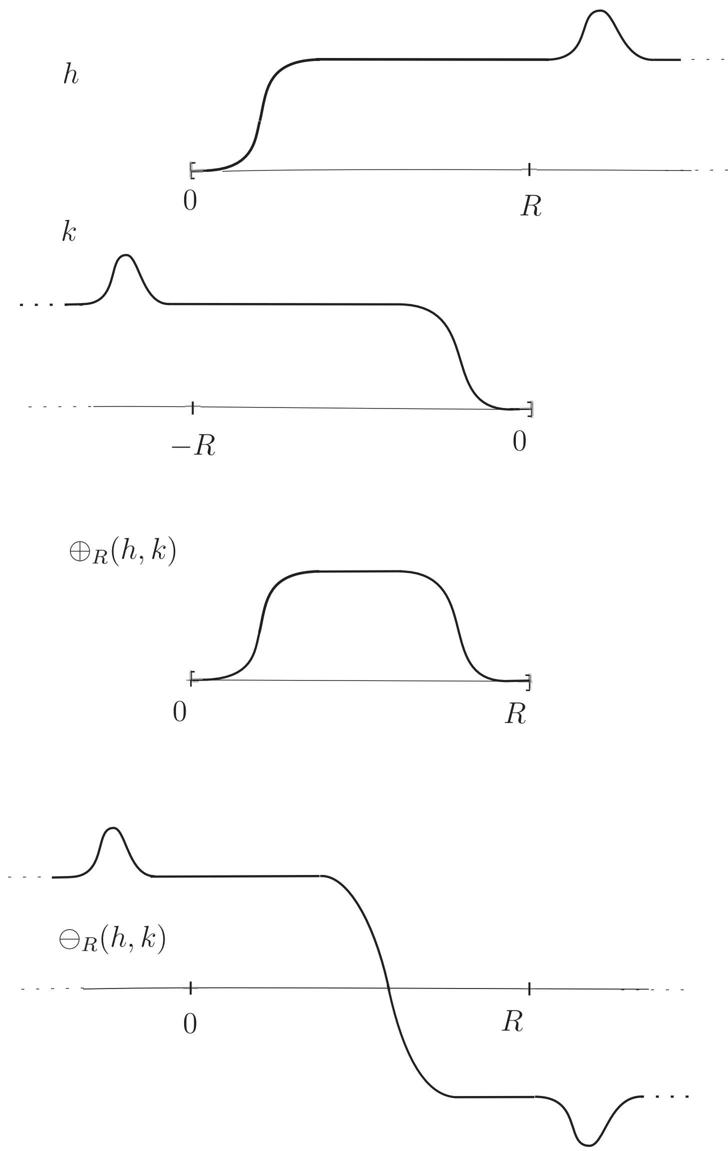


FIGURE 2.4. Graphs of functions h, k and the glued functions $\oplus_r(h,k)$ and the anti-glued map $\ominus_r(h,k)$

on $E = E^+ \oplus E^-$, *for all* $r \in [0,1)$. *In addition, for* $(h^+, h^-) \in H^+ \oplus E^-$,

$$\pi_r(h^+, h^-) = (h^+, h^-) \text{ if and only if } \oplus_r(h^+, h^-) = 0.$$

Proof. Recalling $\beta(s)+\beta(-s)=1$, $\tau(s)=\beta(s-\frac{R}{2})$, and $\alpha(s)=\tau(s)^2+(1-\tau(s))^2$ one finds

$$\begin{aligned}\tau(R-s)&=1-\tau(s)\\ \alpha(R-s)&=\alpha(s).\end{aligned}$$

Hence, setting $s'=s-R$ in (2.8), one obtains

$$\begin{aligned}\widehat{h}(s)&=\frac{\tau(s)}{\alpha(s)}\big[\tau(s)\cdot h(s)+(1-\tau(s))\cdot k(s-R)\big]\\ \widehat{k}(s-R)&=\frac{1-\tau(s)}{\alpha(s)}\big[\tau(s)\cdot h(s)+(1-\tau(s))\cdot k(s-R)\big]\end{aligned} \tag{2.12}$$

and the proposition follows immediately. ■

The geometric meaning of the splicing is now apparent. If $r=0$ we do not do anything. However, if $r\in(0,1)$ there are two distinguished sc-subspaces of E, namely the one consisting of pairs (h,k) which if glued give the zero map and the other subspace consisting of elements which if anti-glued give the zero map. Moreover, E is the sc-direct sum of these subspaces and π_r the projection onto the second along the first. The second subspace has the property that $\oplus_r$-gluing produces a bijective correspondence between this subspace and the maps on $I_r=[0,R]$.

The splicing core $K^{\mathcal{S}}$ of $[0,\frac{1}{2})\oplus E$ associated with the sc-smooth splicing $\mathcal{S}=(\pi,[0,\frac{1}{2}),E)$ consists by definition of all pairs $(r,h^+\oplus h^-)\in[0,\frac{1}{2})\oplus E^+\oplus E^-$ satisfying $\pi_r(h^+\oplus h^-)=h^+\oplus h^-$. It is the following set.

If $0<r<\frac{1}{2}$ the set of points $(r,h^+,h^-)\in K^{\mathcal{S}}$ can be characterized by

$$\ominus_r(h^+,h^-)=0 \tag{2.13}$$

or, explicitly, by

$$[1-\tau(s)]\cdot h^+(s)=\tau(s)\cdot h^-(s-R)$$

for all $s\in\mathbb{R}$, where $R=\varphi(r)$. This follows immediately from Proposition 2.7 and formula (2.11).

2.3. Smooth maps between splicing cores

The aim of this section is to introduce the concept of an sc^1-map between open subsets of splicing cores.

Consider two open subsets $O\subset K^{\mathcal{S}}\subset V\oplus E$ and $O'\subset K^{\mathcal{S}'}\subset V'\oplus E'$ of splicing cores belonging to the splicings $\mathcal{S}=(\pi,V,E)$ and

$\mathcal{S}' = (\pi', V', E')$. The cones V and V' are contained in the vector spaces W resp. W'. Consider an sc^0-map

$$f : O \to O'.$$

If O is an open subset of the splicing core $K^{\mathcal{S}} \subset V \oplus E$ we define the subset $\widehat{O}$ of $V \oplus E$ by

$$\widehat{O} = \{(v,e) \in V \oplus E \mid (v, \pi_v(e)) \in O\}.$$

Clearly, $\widehat{O}$ is open in $V \oplus E$. Indeed, introduce the sc-map $\Phi : V \oplus E \to V \oplus E$ by setting $\Phi(v,e) = (v, \pi_v(e))$. In view of the definition of the splicing, the map Φ is sc-smooth, hence in particular, sc^0-continuous. Since $O = \widetilde{O} \cap K^{\mathcal{S}}$ for some open subset $\widetilde{O}$ of $V \oplus E$ and since $\Phi^{-1}(K^{\mathcal{S}}) = V \oplus E$, we conclude that $\widehat{O} = \Phi^{-1}(O) = \Phi^{-1}(\widetilde{O}) \cap \Phi^{-1}(K^{\mathcal{S}}) = \Phi^{-1}(\widetilde{O}) \cap (V \oplus E)$, proving our claim.

Definition 2.8. *The sc^0-continuous map $f : O \to O'$ is called of* **class sc^1** *if the map*

$$\widehat{f} : \widehat{O} \subset V \oplus E \to W' \oplus E'$$
$$\widehat{f}(v,e) = f(v, \pi_v(e))$$

is of class sc^1.

Splitting the map $\widehat{f}$ according to the splitting of the image space and setting

$$\widehat{f}(v,e) = (\widehat{f}_1(v,e), \widehat{f}_2(v,e)) \in K^{\mathcal{S}} \subset W' \oplus E',$$

the **tangent map** Tf associated with the sc^1-map f is defined as

$$Tf(v, \delta v, e, \delta e) := (T\widehat{f}_1(v, \delta v, e, \delta e), T\widehat{f}_2(v, \delta v, e, \delta e)). \tag{2.14}$$

The map Tf is of class sc^0.

Lemma 2.9.

$$Tf : K^{T\mathcal{S}}|O \to K^{T\mathcal{S}'}|O'$$

hence, in the notation of (2.4), we have $Tf : TO \to TO'$.

Proof. By definition of $K^{\mathcal{S}'}$ we have, with the associated projection $\pi'_w(e')$, the relation

$$\pi'_{\widehat{f}_1(v,e)}(\widehat{f}_2(v,e)) = \widehat{f}_2(v,e).$$

Differentiating this identity in the variable (v,e) we obtain

(2.15)
$$\begin{aligned} D_w\pi'_{\widehat{f}_1(v,e)}(\widehat{f}_2(v,e))\circ D\widehat{f}_1(v,e)[\delta v,\delta e]+\pi'_{\widehat{f}_1(v,e)}\circ D\widehat{f}_2(v,e)[\delta v,\delta e]\\ =D\widehat{f}_2(v,e)[\delta v,\delta e]. \end{aligned}$$

Setting $w=\widehat{f}_1(v,e)$ and $e'=\widehat{f}_2(v,e)$ and $\delta w=D\widehat{f}_1(v,e)[\delta v,\delta e]$ and $\delta e'=D\widehat{f}_2(v,e)[\delta v,\delta e]$ it follows from (2.15) together with the definition of the projection $P'_{(w,\delta w)}$ associated with the splicing $T\mathcal{S}'$ that

$$P'_{(w,\delta w)}(e',\delta e')=(e',\delta e')$$

so that indeed $Tf(v,\delta v,e,\delta e)=(w,\delta w,e',\delta e')\in K^{T\mathcal{S}'}$ as was to be proved. ■

THEOREM 2.10. *Let O,O',O'' be open subsets of splicing cores and $f:O\to O'$ and $g:O'\to O''$ be of class sc^1. Then the composition $g\circ f$ is of class sc^1 and the tangent maps satisfy*

$$T(g\circ f)=Tg\circ Tf.$$

PROOF. This is a consequence of the sc-chain rule (Theorem 1.13), the definition of the tangent map and the fact that our reordering of the terms in our definition (2.14) of the tangent map is consistent. Considering $\widehat{f}$ and $\widehat{g}$ the composition $\widehat{g\circ f}=\widehat{g}\circ\widehat{f}$ is clearly of class sc^1 and $T(\widehat{g}\circ\widehat{f})=T\widehat{g}\circ T\widehat{f}$. From Definition (2.14) we deduce

$$\begin{aligned} T(g\circ f)(v,\delta v,e,\delta e)&=(T(\widehat{g}_1\circ\widehat{f})(v,e,\delta v,\delta e),T(\widehat{g}_2\circ\widehat{f})(v,e,\delta v,\delta e))\\ &=((T\widehat{g}_1)\circ(T\hat{f})(v,e,\delta v,\delta e),(T\widehat{g}_2)\circ(T\widehat{f})(v,e,\delta v,\delta e))\\ &=(Tg)(T\widehat{f}_1(v,e,\delta v,\delta e),T\widehat{f}_2(v,e,\delta v,\delta e))\\ &=(Tg)\circ(Tf)(v,\delta v,e,\delta e). \end{aligned}$$

■

Given a sc^1-map $f:O\to O'$ between open sets of splicing cores we obtain, in view of Lemma 2.9, an induced tangent map $Tf:TO\to TO'$. Since TO and TO' are again open sets in the splicing cores $K^{T\mathcal{S}}$ and $K^{T\mathcal{S}'}$ we can iteratively define the notion of f being of class sc^k or even sc-smooth.

Assume now that $f:O\to O'$ is a homeomorphism and f and f^{-1} are sc-smooth maps between the open sets of splicing cores. Then, setting $g=f^{-1}$,

$$Tf \circ Tg = T(Id) \text{ and } Tg \circ Tf = T(Id).$$

Hence Tf is sc-smooth as is its inverse.

DEFINITION 2.11. *Let O be an open subset of a splicing core $K^{\mathcal{S}}$ and $(v,e) \in O_1$. The* **tangent space** *to O at the point (v,e) is the sc-Banach space*

$$T_{(v,e)}O = \{(\delta v, \delta e) \in W \oplus E \mid (v, \delta v, e, \delta e) \in TO\}. \tag{2.16}$$

We then have

$$TO = \bigcup_{(v,e)\in O_1} T_{(v,e)}O.$$

If $f : O \to O'$ is a homeomorphism our tangent map Tf defined in (2.14) induces the linear sc-isomorphism

$$Tf(v,e) : T_{(v,e)}O \to T_{f(v,e)}O'.$$

2.4. M-Polyfolds

Now we are able to introduce the notion of an M-polyfold*.

DEFINITION 2.12. *Let X be a second countable Hausdorff space. An* **M-polyfold chart** *for X is a triple $(U, \varphi, \mathcal{S})$, where U is an open subset of X and $\varphi : U \to K^{\mathcal{S}}$ a homeomorphism onto an open subset of a splicing core. Two charts are called compatible if the transition maps between open subsets of splicing cores are sc-smooth in the sense defined in Section 2.3. A maximal atlas of sc-smoothly compatible M-polyfold charts is called a* **M-polyfold structure** *on X.*

An M-polyfold is necessarily metrizable by using an argument similar to the one used already for sc-Manifolds.

Each splicing core $K^{\mathcal{S}}$ carries the structure of a M-polyfold with the global chart being the identity.

EXAMPLE 2.13. The example illustrates that the set $((\mathbb{R} \setminus \{0\}) \times \mathbb{R}) \cup \{(0,0)\}$ can be equipped with the structure of a M-polyfold. Let $E = L^2(\mathbb{R})$ and define E_m to be $H^{m,\delta_m}(\mathbb{R})$, where δ_m is a strictly

*The "M" indicates the "manifold flavor" of the polyfold. A general polyfold will be a generalization of an orbifold.

increasing sequence starting with $\delta_0 = 0$. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a smooth function having compact support and L^2-norm equal to 1. With $\langle \cdot, \cdot \rangle$ being the L^2-inner product we define the family of projections $\pi_t : E \to E$ by setting $\pi_0 = 0$ and

$$\pi_t f = \langle f, \gamma_t \rangle \cdot \gamma_t \tag{2.17}$$

if $t \neq 0$ where have abbreviated $\gamma_t(s) = \gamma(s + e^{\frac{1}{|t|}})$. Then $(\pi, \mathbb{R}, V)$ is an sc-smooth splicing and the splicing core is homeomorphic to $((\mathbb{R} \setminus \{0\}) \times \mathbb{R}) \cup \{(0, 0)\}$.

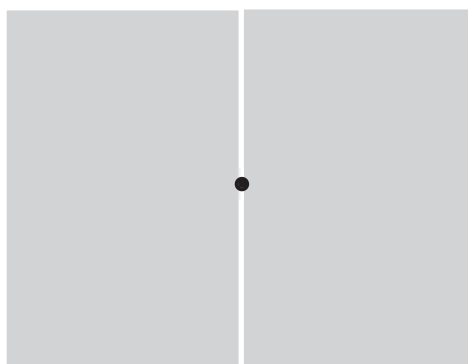


FIGURE 2.5. The splicing core in the example is homeomorphic to $((\mathbb{R} \setminus \{0\}) \times \mathbb{R}) \cup \{(0, 0)\}$

Indeed, assume for the moment that $\pi : \mathbb{R} \oplus E \to E$ is sc-smooth. Clearly, $\pi_0(f) = f$ only when $f = 0$. If $\pi_t(f) = f$ with $t \neq 0$, then $|\langle f, \gamma_t \rangle|^2 \cdot |\gamma_t|^2 = |f|^2$ a.e. so that after integrating, using $\|\gamma_t\|_0 = 1$, we obtain $|\langle f, \gamma_t \rangle| = \|f\|_0 = \|f\|_0 \cdot \|\gamma_t\|_0$ and this implies that f is a multiple of γ_t.

It remains to show that π is of class sc^∞. We only show that the map is of class sc^1 leaving the sc-smoothness to the reader. The proof follows from the following two estimates. Consider two smooth functions $\phi, \psi : R \to \mathbb{R}$ having their supports in the interval $I = [-a, a]$. Introducing the intervals $I_t = [-a - e^{\frac{1}{|t|}}, a - e^{\frac{1}{|t|}}]$ and the function

$\phi_t(s) = \phi(s + e^{\frac{1}{|t|}})$, we estimate for $f \in E_{m+1}$,

$$\begin{aligned}
\langle f, \phi_t \rangle^2 &= \int_{\mathbb{R}} |f(s)|^2 \cdot |\phi(s + e^{\frac{1}{|t|}})|^2 ds \\
&= \int_{I_t} |f(s)|^2 \cdot |\phi(s + e^{\frac{1}{|t|}})|^2 \cdot e^{-2\delta_{m+1} s} \cdot e^{2\delta_{m+1} s} ds \qquad (2.18)\\
&\leq C e^{-2\delta_{m+1} e^{\frac{1}{|t|}}} \cdot \|f\|^2_{m+1}.
\end{aligned}$$

Next we consider the map $(t, f) \mapsto p(t)\langle f, \phi\rangle \psi_t$ from $\mathbb{R} \oplus E_{m+1}$ into E_{m+1} where $p(t)$ is a polynomial of degree k in variable $\frac{1}{|t|} e^{\frac{1}{|t|}}$. Using (2.18) we estimate

$$\begin{aligned}
\|p(t)\langle f, \phi_t\rangle \psi_t\|^2_m &= |p(t)\langle f, \phi_t\rangle|^2 \sum_{j \leq m} \int_{I_t} |D^j \psi(s + e^{\frac{1}{|t|}})|^2 e^{2\delta_m |s|} ds \\
&\leq |p(t)\langle f, \phi_t\rangle|^2 \sum_{j \leq m} \int_I |D^j \psi(s)|^2 e^{-2\delta_m (s - e^{\frac{1}{|t|}})} ds \qquad (2.19)\\
&\leq C|p(t)|^2 e^{-2(\delta_{m+1} - \delta_m) e^{\frac{1}{|t|}}} \|f\|^2_m.
\end{aligned}$$

Since $\delta_{m+1} - \delta_m > 0$, the last estimate tends to 0 as $t \to 0$. With these two observations we prove our claim. The function γ and hence its derivatives have compact support. If $t \neq 0$ and $f \in E_{m+1}$, then by (2.18),

$$\|\pi(t, f)\|_m \to 0$$

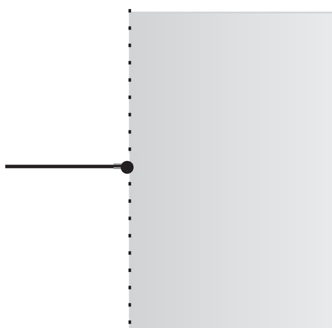
as $t \to 0$, and since $\pi(0, f) = 0$, we conclude that the map π is differentiable at the point $(0, f)$ and its derivative is equal to 0. For $t \neq 0$, the derivative $d\pi(t, f) : \mathbb{R} \oplus E_{m+1} \to E_m$ is given by

$$\begin{aligned}
d\pi(t, f)[\delta t, \delta f] = \langle \delta f, \gamma_t\rangle \gamma_t &- \operatorname{sign}(t) \tfrac{1}{|t|^2} e^{\frac{1}{|t|}} \langle f, \dot{\gamma}_t\rangle \gamma_t \cdot [\delta t] \\
&- \operatorname{sign}(t) \tfrac{1}{|t|^2} e^{\frac{1}{|t|}} \langle f, \gamma_t\rangle \dot{\gamma}_t \cdot [\delta t]
\end{aligned}$$

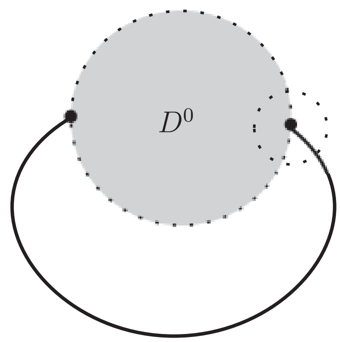
where $\dot{\gamma}_t$ denotes the derivative of γ_t with respect to the variable s. Since the function γ has a compact support we may apply (2.19) to estimate each term. It follows each term tends to 0 in E_m as $t \to 0$ showing that the map $(t, f) \mapsto d\pi(t, f)$ from $\mathbb{R} \oplus E_{m+1}$ into E_m is continuous and the map $\pi : \mathbb{R} \oplus E_{m+1} \to E_m$ is of class C^1. Note that, in view of (2.19), the map $\delta f \mapsto \langle \delta f, \gamma_t\rangle \gamma_t$ from E_{m+1} into E_m has an extension to a bounded linear map from $E_m \to E_m$ so that $d\pi(t, f) : \mathbb{R} \oplus E_{m+1} \to E_m$ has an extension to a linear bounded operator $D\pi(t, f) : \mathbb{R} \oplus E_m \to E_m$ having the property that

$(t, f, \delta t, \delta f) \mapsto D\pi(t, f)(\delta t, \delta f)$ from $\mathbb{R} \oplus E_{m+1} \oplus \mathbb{R} \oplus E_m \to E_m$ is continuous. In view of Proposition 1.7 we have verified that $\pi : \mathbb{R} \oplus E \to E$ is of class sc^1.

What is $K^{T\mathcal{S}}$? One can, of course generalize these ideas. For example, take the open unit disk D° in $\mathbb{R}^2$ and add a closed interval $[-1, 1]$ and identify the ends ± 1 with the points $(\pm 1, 0)$ in the boundary of the closed disk. A small open disc centered at $(\pm 1, 0)$ intersected with Figure 2.6 (b) is homeomorphic to the splicing core in Figure 2.6 (a). The splicing core represented by Figure 2.6 (a) can be obtained by taking E as above and setting $\pi_t = 0$ for $t \leq 0$ and for $t > 0$ by defining the projection π_t by (2.17). This can be generalized to higher dimensions.



(a)



(b)

FIGURE 2.6

Let us note the following useful result about sc-smooth partitions of unity.

THEOREM 2.14. *Let X be an M-polyfold with local models being splicing cores build in sc-Hilbert spaces. Assume that $(U_\lambda)_{\lambda \in \Lambda}$ is an open covering. Then there exists a subordinate sc-smooth partition of unity $(\beta_\xi)_{\xi \in \Xi}$.*

Let us note that the product $X \times Y$ of two M-polyfolds is in a natural way a M-polyfold. For charts $(U, \varphi, \mathcal{S})$ and $(W, \psi, \mathcal{T})$ of X and Y, respectively, we define the product chart $(U \times W, \phi \times \psi, \mathcal{S} \oplus \mathcal{T})$. Here

$$\mathcal{S} \times \mathcal{T} = (\pi, V, E) \times (\rho, V', F) = (\sigma, V \times V', E \oplus F),$$

with

$$\sigma(v, v') = \pi_v \oplus \rho_{v'}.$$

We call $\mathcal{S} \times \mathcal{T}$ the product of the splicings. There are several possible notions of subpolyfolds (we suppress the M in the notation) and we describe them in a later chapter.

2.5. Corners and Boundary points

In this section we will prove the extremely important fact that sc-smooth maps are able to recognize corners. This will be crucial for SFT since most of the algebra is a consequence of the corner structure.

Let X be a M-polyfold. For a point $x \in X$ we take a M-polyfold chart $\varphi : U \to K^{\mathcal{S}}$ where $K^{\mathcal{S}}$ is the splicing core associated with the splicing $\mathcal{S} = (\pi, V, E)$. Here V is an open subset of a cone C contained in an n-dimensional vector space W. By definition there exists a linear isomorphism from W to $\mathbb{R}^n$ mapping C onto $[0, \infty)^n$. Identifying the cone C with $[0, \infty)^n \subset \mathbb{R}^n$ we shall use the notation $\varphi = (\varphi_1, \varphi_2) \in [0, \infty)^n \oplus E$ according to the splitting of the target space of φ. We associate with $x \in U$ the integer $d(x)$ defined by

$$d(x) = \sharp\{\text{coordinates of } \varphi_1(x) \text{ which are equal to } 0\}. \tag{2.20}$$

Theorem 2.15. *The map $d : X \to \mathbb{N}$ is well-defined and does not depend on the choice of the compatible polyfold chart. Moreover, every point $x \in X$ has an open neighborhood O satisfying*

$$d(y) \leq d(x) \text{ for all } \ y \in O.$$

The map d is called the **degeneracy index**. A point with $d(x) = 0$ is an interior point. A point with $d(x) = 1$ is called a **good boundary point**. A point with $d(x) \geq 2$ is called a **corner**. In general, $d(x)$ is the order of the corner .

Proof of Theorem 2.15. Consider two M-polyfold charts $\varphi : \widehat{U} \subset X \to K^{S}$ and $\varphi' : \widehat{U}' \subset X \to K^{\mathcal{S}'}$ such that $x \in \widehat{U} \cap \widehat{U}'$. Introducing the open subsets $U = \varphi(\widehat{U} \cap \widehat{U}')$ and $U' = \varphi'(\widehat{U} \cap \widehat{U}')$ of $K^{\mathcal{S}}$and $K^{\mathcal{S}'}$ resp., and setting $\varphi(x) = (r, e)$ and $\varphi'(x) = (r', e')$ we define the sc-diffeomorphism $\Phi : U \to U'$ by $\Phi = \varphi' \circ \varphi^{-1}$. Obviously, $\Phi(r, e) = (r', e')$. Now the proof of Theorem 2.15 reduces to the following proposition.

Proposition 2.16. *Let $\mathcal{S} = (\pi, V, E)$ and $\mathcal{S}' = (\pi', V', E')$ be two splicings having the parameter sets $V = [0, \infty)^k$ and $V' = [0, \infty)^{k'}$.*

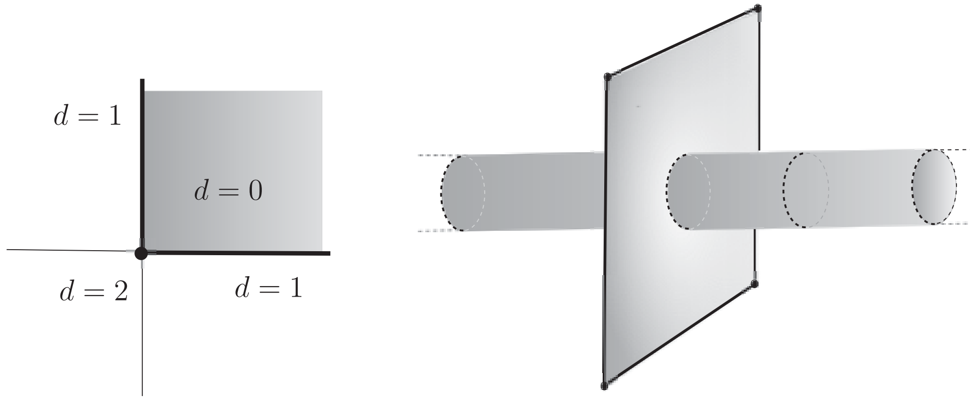


FIGURE 2.7

Assume that U and U' are open subsets of the splicing cores $K^{\mathcal{S}}$ and $K^{\mathcal{S}'}$ containing the points (r,e) and (r',e') and assume that the map

$$\Phi : U \to U'$$

is an sc-diffeomorphism mapping (r,e) to (r',e'). Then r and r' have the same number of vanishing coordinates.

PROOF. We first prove the assertion of the proposition under the additional assumption that the point $p_0 = (r,e)$ belongs to U_∞. Then the image point $q_0 = (r',e') = \Phi(p_0)$ belongs to U'_∞. Denote by J the subset of $\{1,\cdots,k\}$ consisting of those indices j for which $x_j = 0$. Similarly, $j \in J' \subset \{1,\cdots,k'\}$ if $r'_j = 0$. Denoting by $\sharp r$ and $\sharp r'$ the cardinalities of J and J' we claim that $\sharp r = \sharp r'$. Since Φ is a sc-diffeomorphism it suffices to prove the inequality $\sharp r \geq \sharp r'$ since the inequality has to also hold true for the sc-diffeomorphism $\Psi = \Phi^{-1}$. If $\pi_r(e) = e$, then differentiating $\pi_r \circ \pi_r(e) = \pi_r(e)$ in r one finds $\pi_r \circ D_r(\pi_r(e)) = 0$ so that $D_r(\pi_r(e)) \cdot [\delta r]$ is contained in the range of $\mathrm{id} - \pi_r$. Therefore, given $(r,e) \in U_\infty$ satisfying $\pi_r(e) = e$ and given $\delta r \in \mathbb{R}^k$, there exists $\delta e \in E_\infty$ solving

$$\delta e = \pi_r(\delta e) + D_r(\pi_r(e))[\delta r]. \tag{2.21}$$

In particular, taking $\delta r \in \mathbb{R}^l$ with $(\delta r)_j = 0$ for $j \in J$, there exists $\delta e \in E_\infty$ solving the equation (2.21). This is equivalent to $(\delta r, \delta e) \in (T_{(r,e)}U)_\infty$. Introduce the path

$$\tau \mapsto p_\tau = (r + \tau\delta r, \pi_{r+\tau\delta r}(e + \tau\delta e))$$

for $|\tau|<\rho$ and ρ small. From $(r,e)\in U_\infty$ and $\delta e\in E_\infty$ one concludes $p_\tau\in U_\infty$. Moreover, considering $\tau\to p_\tau$ as a map into U_m for $m\geq 0$, its derivative at $\tau=0$ is equal to $(\delta r,\delta e)$. Fix a level $m\geq 1$ and consider the map

$$(-\tau_0,\tau_0)\to\mathbb{R}^k\times F_m:\tau\to\Phi(p_\tau).$$

The map $\Phi:U\to U'$ is C^1 as a map from $U_{m+1}\subset\mathbb{R}^k\oplus E_{m+1}$ into $\mathbb{R}^{k'}\oplus F_m$. Its derivative $d\Phi(r,e):\mathbb{R}^k\oplus E_{m+1}\to\mathbb{R}^{k'}\oplus F_m$ has an extension to the continuous linear operator $D\Phi(r,e):\mathbb{R}^k\oplus E_m\to R^{k'}\oplus E_m$. Since Φ is a sc-diffeomorphism the extension $D\Phi(r,e):\mathbb{R}^k\oplus E_m\to R^{k'}\oplus E_m$ is a bijection. Thus, since $\delta e\in E_\infty$,

$$\begin{aligned}\Phi(p_\tau)&=\Phi(p_0)+\tau\cdot d\Phi(p_0)[\delta r,\delta e]+o_m(\tau)\\&=q_0+\tau\cdot D\Phi(p_0)[\delta r,\delta e]+o_m(\tau)\end{aligned}\tag{2.22}$$

where $o_m(\tau)$ is a function taking values in $\mathbb{R}^{k'}\oplus F_m$ and satisfying $\frac{1}{\tau}o_m(\tau)\to 0\quad$ as $\tau\to 0$. Introduce the sc-continuous linear functionals $\lambda_j:\mathbb{R}^k\times F\to\mathbb{R}$ by

$$\lambda_j(s,h)=s_j$$

where $j\in J'$. Then

$$\lambda_j\circ\Phi(p_\tau)\geq 0$$

for $|\tau|<\delta_0$. Applying λ_j to both sides of (2.22) we conclude for $\tau>0$

$$\begin{aligned}0\leq\frac{1}{\tau}\cdot\lambda_j\big[\Phi(p_\tau)\big]&=\frac{1}{\tau}\cdot\lambda_j\big[\Phi(p_0)+\tau\cdot d\Phi(p_0)[\delta r,\delta e]+o_m(\tau)\big]\\&=\lambda_j\big[D\Phi(p_0)[\delta r,\delta e]\big]+\lambda_j\big(\frac{o_m(\tau)}{\tau}\big).\end{aligned}$$

Passing to the limit $\tau\to 0^+$ we find

$$0\leq\lambda_j(D\Phi(r,e)[\delta r,\delta e])$$

and replacing $(\delta r,\delta e)$ by $(-\delta r,-\delta e)$ we obtain the equality sign. Consequently,

$$\lambda_j(D\Psi(r,e)[\delta r,\delta e])=0,\quad j\in J'\tag{2.23}$$

for all $[\delta r,\delta e]\in\mathbb{R}^k\oplus E_\infty$ satisfying $\pi_r(e)+D_r(\pi_r(e))[\delta r]=\delta e$ and $(\delta r)_i=0$ for all $i\in J$. Introduce the codimension $\sharp r$ subspace L of the tangent splicing core $K^{T\mathcal{S}}\subset\mathbb{R}^k\oplus E_\infty$, defined as

$$\begin{aligned}L=\{(\delta r,\delta r)\in\mathbb{R}^k\oplus E_\infty|\ &\pi_r(\delta e)+D_r(\pi_r(e))[\delta r]=\delta e\\&\text{and }\ (\delta r)_i=0\ \text{ for all } i\in J\ \}.\end{aligned}$$

Then, in view of (2.23),

$$D\Phi(r,e)L\subset\{[\delta r',\delta e']\in K^{T\mathcal{S}'}|\,(\delta e')_j=0\text{ for all }\ j\in J'\}.$$

Because the subspace on the right hand side has codimension $\sharp r'$ in $K^{T\mathcal{S}'}$ and since $D\Phi(r,e)$, being a bijection, maps L onto a codimension $\sharp r$ subspace of $K^{T\mathcal{S}'}$, it follows that $\sharp r' \leq \sharp r$, as claimed.

Next we shall prove the general case. For this we take $p_0 = (r,e)$ in U_0, so that the image point $(r',e') = \Phi(r,e)$ belongs to U_0'. Arguing by contradiction we may assume without loss of generality that $\sharp r > \sharp r'$, otherwise we replace Φ by Φ^{-1}. Since U_∞ is dense in U_0 we find a sequence $(r,e_n) \in U_\infty$ satisfying $\pi_r(e_n) = e_n$ and $(r,e_n) \to (r,e)$ in U_0. By the previous discussion $\sharp r = \sharp r_n'$ where $(r_n', e_n') = \Phi(r,e_n)$. Since Φ is sc-smooth, we have $(r_n', e_n') \to (r',e')$ in U_0' and $\pi'_{r'}(e') = e'$. From this convergence we deduce $\sharp r' \geq \sharp r_n'$ so that $\sharp r' \geq \sharp r$ contradicting our assumption. The proof of Proposition 2.16 is complete. ■

To finish the proof of Theorem 2.15 it remains to show that the function d is lower semicontinuous. Arguing indirectly assume that there exists a sequence of points x_k converging to x so that $d(x_k) > d(x)$. Since φ is continuous, we have the convergence $\varphi_1(x_k) = (r_1^k, \cdots, r_n^k) \to \varphi_1(x) = (r_1, \cdots, r_n)$. If for a given coordinate index j the coordinate $r_j^k = 0$ for all but finitely many k, then $r_j = 0$, and if $r_j^k > 0$ for all but finitely many k, then $r_j \geq 0$. Hence $d(x_k) \leq d(x)$ contradicting our assumption. The proof of Theorem 2.15 is complete. ■

The results about corner recognition remain true if the projections in a splicing are parameterized by open subsets of a partial cone. Recall that a partial cone P in a finite-dimensional vector space is a closed subset which by a linear isomorphism to some $\mathbb{R}^n$ is mapped onto $[0,\infty)^k \times \mathbb{R}^{n-k}$. In that case the degeneracy index counts the vanishing of the first k coordinates. These types of parameter sets will arise in SFT. In fact there we will have $[0,\infty)^k \times \mathbb{C}^l$, where the first k coordinates will be related to the corner structure arising from breaking of trajectories, whereas the other arise from bubbling-off of spheres.

Let us observe that the result about corner recognition allows us to define a degeneracy index $d : X \to \mathbb{N}$ as follows. Take for a point $x \in X$ a local chart

$$\varphi : U \to K^{\mathcal{S}} \subset ([0,\infty)^k \times \mathbb{R}^{n-k}) \oplus E.$$

Then define $d(x)$ to be the number of the first k coordinates of $\varphi(x)$ which are vanishing. This definition is independent of the choices involved. We call $d = d_X$ the degeneracy index.

Assume that $X \times Y$ is a product of two M-polyfolds. It follows immediately from the definition of the product structure that

$$d_{X\times Y}(x,y) = d_X(x) + d_Y(y).$$

If A a subpolyfold of the M-polyfold X then A has an induced degeneracy index d_A since it carries an induced M-polyfold structure. Clearly $d_A \leq d_X|A$.

2.6. Appendix II

In this appendix we will prove the important technical Lemma 2.6, which we have used in showing the smoothness of the splicing. For the reader's convenience we recall the statement of the lemma, which we formulate here as a technical theorem.

THEOREM 2.17. *Let $E = H^2(\mathbb{R}, \mathbb{R}^n)$ be equipped with the sc-structure where level m consists of regularity $(m+2, \delta_m)$-maps. Using the exponential gluing profile φ let $R = \varphi(r)$. Then the following three maps*

$$\Gamma_i : [0,1) \oplus E \to E, \ i = 1,2,3$$

are sc-smooth.

(a) *Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a smooth map which is constant outside of a compact interval so that $f_1(+\infty) = 0$. Define $\Gamma_1(0,h)(s) = f_1(-\infty)h(s)$ and $\Gamma_1(r,h)(s) = f_1(s - \frac{R}{2})h(s)$.*

(b) *Let $f_2 : \mathbb{R} \to \mathbb{R}$ be a smooth map which is constant outside of a compact interval so that $f_2(-\infty) = 0$. Define $\Gamma_2(0,h) = 0$ and $\Gamma_2(r,h)(s) = f_2(s - \frac{R}{2})h(s)$.*

(c) *Let $f_3 : \mathbb{R} \to \mathbb{R}$ be a compactly supported smooth map and define $\Gamma_3(0,h) = 0$ and $\Gamma_3(r,h)(s) = f_3(s - \frac{R}{2})h(s - R)$.*

We first observe that the statements (a) and (b) of the theorem are equivalent. Indeed, assuming (a) holds true let f_2 be a function as in (b). Define the linear sc-operator $L : E \to E$ by $h \to f_2(+\infty)h$. Clearly, L is sc-smooth. The function $f_1(s) = f_2(s) - f_2(+\infty)$ satisfies the hypothesis of (a) giving the sc-smooth map Γ_1. Since

$$\Gamma_2(r,h) = \Gamma_1(r,h) + L(h),$$

(a) implies (b). Similar argument shows that (b) implies (a) proving our claim.

In the following we will prove statements (b) and (c). In the first step we verify sc^0-continuity.

LEMMA 2.18. *The maps Γ_2 and Γ_3 are of class sc^0.*

PROOF. We begin with Γ_2 and consider the induced map $[0,1) \oplus E_m \to E_m$. The only difficulty arising could be at $r = 0$. The compactly supported smooth functions are dense in E_m. If h_0 is a compactly supported smooth function and $r > 0$ is close to 0, then $\Gamma_3(r, h_0) = 0$. Now our assertion follows from the observation that the norm of $\Gamma_3(r, \cdot) : E_m \to E_m$ is uniformly bounded in r,

$$\|\Gamma_3(r,h)\|_m \leq C \cdot \sup_{0 \leq k \leq m} \|f^k\|_{C^0} \cdot \|h\|_m,$$

where C is a constant independent of r and u.

Next consider the map Γ_3. Again the difficulty is at $r = 0$. For a compactly supported smooth function h_0 and r close to 0 we have $\Gamma_3(r, h_0) = 0$. Using the density of smooth compactly supported functions in E_m it suffices to show as before a uniform bound for the operator norm of $\Gamma_3(r, .) : E_m \to E_m$. Assuming supp $f_2 \subset [-a, a]$ we take R so large that the interval $I_R = [-a - \frac{R}{2}, a - \frac{R}{2}]$ is contained in $(-\infty, 0]$. Denoting by C a generic constant independent of r and u we estimate

$$\begin{aligned}
\|\Gamma_3(r,h)\|_m &\leq C \cdot \|h(\cdot - R)e^{\delta_m \cdot}\|_{H^m(I_{-R})} \\
&\leq C \cdot \|h \cdot e^{\delta_m(\cdot + R)}\|_{H^m(I_R)} \\
&\leq C \cdot e^{\delta_m \frac{R}{2}} \cdot \|h\|_{H^m(I_{-R})} \\
&\leq C \cdot \|h\|_m.
\end{aligned}$$

The proof of Lemma 2.18 is complete. ■

Here are some additional observations which are used in the proof of the theorem.

LEMMA 2.19. *Let f_2 and f_3 be as described in the theorem. If $m, k \geq 0$, we set $d_{m+k,m} = \frac{1}{2}[\delta_{m+k} - \delta_m] > 0$. Then there exists a constant $C = C(m, k) > 0$ independent of u and r so that the following estimates hold true.*

$$\begin{aligned}
\|\Gamma_2(r,u)\|_m &\leq C \cdot e^{-d_{m+k,m} \cdot R} \cdot \|u\|_{m+k} \\
\|\Gamma_3(r,u)\|_m &\leq C \cdot e^{-d_{m+k,m} R} \cdot \|u\|_{m+k}
\end{aligned}$$

for all $u \in E_{m+k}$, where $R = \varphi(r)$.

PROOF. We start with the map Γ_3. Since supp $f_3 \subset [-a, a]$, the support of the function $s \mapsto f_3(s - \frac{R}{2})u(s - R)$ is contained in the interval $I_{-R} = [-a + \frac{R}{2}, a + \frac{R}{2}]$. Denoting by C a generic constant we estimate

$$\begin{aligned}
&\|f_3(\cdot - \tfrac{R}{2})u(\cdot - R)\|_m \leq C \cdot \|f_2(\cdot - \tfrac{R}{2})u(\cdot - R)e^{\delta_m s}\|_{H^m(I_{-R})}\\
&= C \cdot \|u \cdot e^{\delta_m (s+R)}\|_{H^m(I_R)} = C \cdot \|u \cdot e^{-\delta_{m+k} s} e^{(\delta_m + \delta_{m+k})s + \delta_m R}\|_{H^{m+k}(I_R)}\\
&\qquad \leq C \cdot e^{-d_{m+k,m} R} \cdot \|u\|_{m+k}.
\end{aligned}$$

To prove the estimate for Γ_1 let $v \in E_{m+k}$ and assume that the support of the function f_2 is contained in $[-a, \infty)$. Then the support of the function $s \to f_2(s - \frac{R}{2})v(s)$ is contained in $[\frac{R}{2} - a, \infty)$. Then one estimates

$$\|f_2(\cdot - \tfrac{R}{2})v\|_m \leq C \cdot e^{-d_{m+k,m} \cdot R} \cdot \|u\|_{m+k}.$$

The proof of the lemma is complete. ∎

The following result follows from the sc-smoothness of the $\mathbb{R}$-action.

Lemma 2.20. *The maps*

$$\mathbb{R} \oplus E \to E$$

defined by $(R, u) \to f_2(\cdot - \frac{R}{2})u$ *and* $(R, v) \to f_3(\cdot - \frac{R}{2})v(\cdot - R)$ *are sc-smooth.*

Proof. For the first map this is obvious. To see that the second map is sc-smooth observe that it can be viewed as the composition of the following maps

$$(R, v) \to (R, \tfrac{-R}{2} * v) \to (R, f_2 \cdot (\tfrac{-R}{2} * v)) \to (-\tfrac{R}{2}) * (f_2 \cdot (\tfrac{-R}{2} * v)).$$

Since each of the above maps is sc-smooth and the composition of sc-smooth maps is sc-smooth the lemma follows. ∎

We need some estimates for the derivatives of the function

$$R(r) = e^{\frac{1}{r}} - e$$

which agrees with the gluing profile introduced in (2.6). The next lemma can be proved by induction.

Lemma 2.21. *For every* $k \geq 0$ *there exists a constant* $C_k > 0$ *such that*

$$\left| \frac{d^k R}{dx^k}(r) \right| \leq C_k \cdot R(r) \cdot [\ln R(r)]^{2k}$$

for all $r \in (0, \frac{1}{2})$.

Now given two positive integers m and n satisfying $m \geq n$ consider the sc-Banach spaces $E^m = (E_{m+i})_{i \geq 0}$ and $E^n = (E_{n+i})_{i \geq 0}$. Let f :

$\mathbb{R} \to \mathbb{R}$ be a compactly supported smooth function and let $k \geq 0$ be an integer. Then we introduce the map

$$A : [0, \tfrac{1}{2}) \times \mathbb{R}^n \times E^m \to E^n \tag{2.24}$$

by setting $A(r, y, u) = 0$ if $r = 0$. If $r > 0$ we set

$$A(r, y, u)(s) = y_1 \cdot y_2 \cdots y_n \cdot \widehat{R}(r) \cdot D^j f(s - \tfrac{R(r)}{2}) \cdot D^i u(s - R(r))$$

where $(y_1, \ldots, y_n) = y \in \mathbb{R}^n$, and

$$\widehat{R}(r) = D^{k_1} R(r) \cdot D^{k_2} R(r) \cdots D^{k_n} R(r).$$

The indices satisfy the conditions $m \geq i + n$ and $k_1 + \cdots + k_m = k$. If $k = 0$, we set $\widehat{R}(r) = 1$. Abbreviating

$$\alpha_k(r) = [R(r)]^k \cdot [\ln R(r)]^{2k}$$

one derives from Lemma 2.21 the estimate

$$|\widehat{R}(r)| \leq C \cdot \alpha_k(r). \tag{2.25}$$

If $k \geq 0$ and $d > 0$, then clearly,

$$\alpha_k(r) \cdot e^{-d \cdot R(r)} \to 0 \quad \text{as } r \to 0.$$

The proof of Theorem 2.17 will follow from the next lemma.

LEMMA 2.22. *The map A viewed as a map from $[0, \frac{1}{2}) \oplus \mathbb{R}^n \oplus E_{m+1}$ into E_n is of class C^1. At every point $(r, y, u) \in [0, \frac{1}{2}) \oplus \mathbb{R}^n \oplus E_{m+1}$, the derivative $dA(r, y, u) : \mathbb{R} \oplus \mathbb{R}^n \oplus E_{m+1} \to E_n$ has an extension to the bounded linear map $DA(r, y, u) : \mathbb{R} \oplus \mathbb{R}^n \oplus E_m \to E_n$. Moreover, the sc-map $A : [0, \frac{1}{2}) \oplus \mathbb{R}^n \oplus E^m \to E^n$ is of class sc^1. In particular, the tangent map $TA : TE^m \to TE^n$,*

$$TA(r, y, u, \widehat{r}, \widehat{y}, \widehat{u}) = (A(r, y, u), DA(r, y, u)[\widehat{r}, \widehat{y}, \widehat{u}])$$

is of class sc^0.

PROOF. Since $R = \varphi(r)$ is smooth for $r > 0$ it follows that A, viewed as a map from $[0, \frac{1}{2}) \oplus \mathbb{R}^n \oplus E_{m+1}$ into E_n, is continuously differentiable on the set $(0, \frac{1}{2}) \oplus \mathbb{R} \oplus E_{m+1}$. We will show that A is differentiable at every point $(0, y, u)$ and its derivative $dA(0, y, u)$ is equal to 0. In the following we write $d_{m,n} = \frac{1}{2}[\delta_m - \delta_n]$ which is positive when $m > n$.

If $(\widehat{r},\widehat{y},\widehat{u})\in(0,\infty)\oplus\mathbb{R}^n\oplus E_{m+1}$, then we derive from Lemma 2.19 and (2.25) the following estimate

$$
\begin{aligned}
&\|A(\widehat{r},y+\widehat{y},u+\widehat{u})\|_n\\
&=|y_1+\widehat{y}_1|\cdots|y_n+\widehat{y}_n|\cdot|\widehat{R}(\widehat{r})|\cdot\|D^jf(\cdot-\tfrac{R}{2})\cdot D^iu(\cdot-R)\|_n\\
&\leq C\alpha_k(\widehat{r})\cdot e^{-d_{m,k+1}\cdot R(r)}\cdot|y_1+\widehat{y}_1|\cdots|y_n+\widehat{y}_n|\cdot\|u+\widehat{u}\|_{m+1}.
\end{aligned}
$$

Dividing both sides by $|\widehat{r}|+|\widehat{y}|+\|\widehat{u}\|_{m+1}$ and noticing that $\frac{1}{\widehat{r}}\cdot\alpha_k(\widehat{r})\cdot e^{-d_{m,k+1}\cdot R(\widehat{r})}\to 0$ as $\widehat{r}\to 0$ we conclude that

$$
\frac{1}{|\widehat{r}|+|\widehat{y}|+\|\widehat{u}\|_{m+1}}\cdot\|A(\widehat{r},y+\widehat{y},u+\widehat{u})\|_n\to 0
$$

as $(\widehat{r},\widehat{y},\widehat{u})\to 0$ in $(0,\infty)\oplus\mathbb{R}^n\oplus E_{m+1}$. Recalling that $A(0,y,u)=0$ our claim follows.

At any other point (r,y,u) with $r>0$ the derivative $dA(r,y,u):\mathbb{R}\oplus\mathbb{R}^n\oplus E_{m+1}\to E_n$ evaluated at $[\widehat{r},\widehat{y},\widehat{u}]\in\mathbb{R}\oplus\mathbb{R}^n\oplus E_{m+1}$ is a linear combination of the following terms

(1) $y_1\cdots\widehat{y}_l\cdots y_n\cdot\widehat{R}(r)\cdot D^jf(s-\frac{R}{2})\cdot D^iu(s-R)$
(2) $\widehat{r}\cdot y_1\cdots y_n\cdot D\widehat{R}(r)\cdot D^jf(s-\frac{R}{2})\cdot Du^i(s-R)$
(3) $\widehat{r}\cdot y_1\cdots y_n\cdot\widehat{R}(r)\cdot DR(r)\cdot D^{j+1}(s-\frac{R}{2})\cdot D^iu(s-R)$
(4) $\widehat{r}\cdot y_1\cdots y_n\cdot\widehat{R}(r)\cdot DR(r)\cdot D^jf(s-\frac{R}{2})\cdot D^{i+1}u(s-R)$
(5) $y_1\cdots y_n\cdot\widehat{R}(r)\cdot D^jf(s-\frac{R}{2})\cdot D^i\widehat{u}(s-R)$.

Assuming $|\widehat{r}|+|\widehat{y}|+\|\widehat{u}\|_{m+1}\leq 1$ and recalling our notation $\widehat{R}(r)=D^{k_1}R(r)\cdot D^{k_2}R(r)\cdots D^{k_n}R(r)$ with $k_1+\cdots+k_n=k$ and the assumption $m\geq i+n$ we use Lemma 2.19 and Lemma 2.21 to estimate each of the above terms and arrive at the following bound,

$$
\begin{aligned}
\|dA(r,y,u)[\widehat{r},\widehat{y},\widehat{u}]\|_n\leq C\cdot|\widehat{R}(r)|^{k+1}\cdot e^{-d_{m+1,n}\cdot R(r)}\\
\cdot(1+\|u\|_{m+1})\cdot\sum_{1\leq l\leq n}|y_1\cdot y_2\cdots y_{l-1}\cdot(1+y_l)\cdot y_{l+1}\cdots y_n|.
\end{aligned}
$$

In particular, the operator norm of $dA(r,y,u)$ satisfies

$$
\|dA(r,y,u)\|\to 0
$$

as $(r,y,u)\to(0,z,v)$ in $[0,\frac{1}{2})\oplus\mathbb{R}^n\oplus E_{m+1}$. Consequently, the derivative $dA(r,y,u)$ depends continuously on $(r,y,u)\in[0,\frac{1}{2})\oplus\mathbb{R}^n\oplus E_{m+1}$ so that $A:[0,\frac{1}{2})\oplus\mathbb{R}^n\oplus E_{m+1}\to E_n$ is of class C^1 as claimed.

Next we claim the derivative $dA(r,y,u):\mathbb{R}\oplus\mathbb{R}^n\oplus E_{m+1}\to E_n$ at the point $(r,y,u)\in[0,\frac{1}{2})\oplus\mathbb{R}^n\oplus E_{m+1}$ has an extension to a continuous linear map $DA(r,y,u):\mathbb{R}\oplus\mathbb{R}^n\oplus E_m\to E_n$. If $r=0$, then we set $DA(0,y,u)=0$. If $r>0$ inspecting the terms (1)-(5) we see that we only have to consider term (5). By Lemma 2.19 and the assumption $m \geq i+n$, term (5) with $\widehat{u} \in E_m$ can be estimated as follows

$$
\begin{aligned}
&\|y_1 \cdot y_2 \cdots y_n \cdot \widehat{R}(r) \cdot D^j f(\cdot - \tfrac{R}{2}) \cdot D^i \widehat{u}(\cdot - R)\|_n \\
&\leq |y_1| \cdots |y_n| \cdot |\widehat{R}(r)| \cdot \|D^j f(\cdot - \tfrac{R}{2}) \cdot D^i \widehat{u}(\cdot - R)\|_n \\
&\leq C \cdot \alpha_m(r) \cdot e^{-[\delta_m - \delta_n]\cdot \frac{R(r)}{2}} \cdot |y_1| \cdots |y_n| \cdot \|\widehat{u}\|_m.
\end{aligned}
\tag{2.26}
$$

Thus term (5) defines a bounded linear operator in $\widehat{u}$ from E_m into E_n and so it is also a bounded linear operator from $\mathbb{R} \oplus \mathbb{R}^n \oplus E_m$ into E_n. Therefore, the derivative $dA(r,y,u) : \mathbb{R} \oplus \mathbb{R}^n \oplus E_{m+1} \to E_n$ has an extension to a continuous linear map $DA(r,y,u) : \mathbb{R} \oplus \mathbb{R}^n \oplus E_m \to E_n$.

In addition, if $m > n$ so that $\delta_m - \delta_n > 0$, the left hand side of (2.26) converges to 0 as $(r,y,\widehat{u}) \to (0,z,\widehat{u}_0)$ in $[0,\frac{1}{2}) \oplus \mathbb{R}^n \oplus E_m$. In the case $m = n$ so that $i = 0$, term (5) has, in view of our convention that in this case $\widehat{R}(r) = 1$, the form $y_1 \cdots y_n \cdot D^j f(s - \frac{R(r)}{2}) \cdot \widehat{u}(s - R)$. Thus the continuity of the map

$$
(r,y,\widehat{u}) \mapsto y_1 \cdots y_n \cdot D^j f(s - \tfrac{R(r)}{2}) \cdot \widehat{u}(s-R)
$$

from $\mathbb{R}^n \oplus E_m \to E_n$ follows from Lemma 2.18. Since

$$
\begin{aligned}
&[0,\tfrac{1}{2}) \oplus \mathbb{R}^n \oplus E_{m+1} \oplus \mathbb{R} \oplus \mathbb{R}^n \oplus E_m \to E_n \\
&(r,y,u,\widehat{r},\widehat{y},\widehat{u}) \mapsto DA(r,y,u)[\widehat{r},\widehat{y},\widehat{u}]
\end{aligned}
$$

is a linear combination of maps of the form (1)-(5), it is continuous. Taking $l \geq 1$ and $(r,y,u,\widehat{r},\widehat{y},\widehat{u}) \in [0,\frac{1}{2}) \oplus \mathbb{R}^n \oplus E_{m+l+1} \oplus \mathbb{R} \oplus \mathbb{R}^n \oplus E_{m+l}$ we observe that $DA(r,y,u)[\widehat{r},\widehat{y},\widehat{u}] \in E_{n+l}$ and it follows from the estimates in Lemma 2.18 and Lemma 2.19 that the evaluation map

$$
\begin{aligned}
&[0,\tfrac{1}{2}) \oplus \mathbb{R}^n \oplus E_{m+1} \oplus \mathbb{R} \oplus \mathbb{R}^n \oplus E_m \to E_n \\
&(r,y,u,\widehat{r},\widehat{y},\widehat{u}) \mapsto DA(r,y,u)[\widehat{r},\widehat{y},\widehat{u}]
\end{aligned}
$$

is continuous. We have verified that the sc-continuous map $A : [0,\frac{1}{2}) \oplus \mathbb{R}^n \oplus E^m \to E^n$ is of class sc^1. The finishes the proof of the lemma. ■

We are ready to prove Theorem 2.17. We only prove part (c) since the proofs of the two other cases are similar. We drop the subscripts in f_3 and Γ_3 and simply write f and Γ so that

$$
\Gamma(r,u)(s) = f(s - \tfrac{R(r)}{2}) \cdot u(s - R(r)).
$$

We prove the theorem by proving by induction that the following assertion holds for all positive integers k.

(k) The map $\Gamma:[0,\frac{1}{2})\oplus E\to E$ is of class sc^k and the iterated tangent map $T^k\Gamma:T^k([0,\frac{1}{2})\oplus E)\to T^kE$ has the following property. Choose any factor E_n of the tangent space T^kE and let π be the projection of T^kE onto E_n. Let $(r,y,u)\in (T^k([0,\frac{1}{2})\oplus E))_0$ where $r\in[0,\frac{1}{2})$, $y=(y_1,y_2,\ldots,y_{2^k})\in\mathbb{R}^{2^k}$ and $u=(u_1,u_2,\ldots u_{2^k})\in E_{m_1}\oplus E_{m_2}\oplus\cdots\oplus E_{m_{2^k}}$. If $r=0$, then $T^k\Gamma(0,y,u)=0$ and if $r>0$, then the composition $\pi\circ T^k\Gamma(r,y,u)$ is a linear combination of maps having the form

$$y_{l_1}\cdot y_{l_2}\cdots y_{l_n}\cdot\widehat{R}(r)\cdot D^jf(s-\tfrac{R(r)}{2})\cdot D^iu_l(s-R(r)) \tag{2.27}$$

where $u_l\in E_{m_l}$ and $\widehat{R}(r)=D^{k_1}R(r)\cdot D^{k_2}R(r)\cdots D^{k_n}R(r)$ and where the indices satisfy $k_1+\cdots+k_n\leq k$ and $m_l\geq i+n$.

We begin with $k=1$. We view Γ as a map from $[0,\frac{1}{2})\oplus E_1$ into E_0. The map is of class C^1 on $(0,\frac{1}{2})\oplus E_1$. At the point $(r,u)\in(0,\frac{1}{2})\oplus E_1$ the derivative $d\Gamma(r,u):\mathbb{R}\oplus E_1\to E_0$ is equal to

$$\begin{aligned} d\Gamma(r,u)[y_1,\widehat{u}](s) &= -\tfrac{1}{2}\cdot y_1\cdot Df(s-\tfrac{R(r)}{2})\cdot u(s-R(r))\\ &\quad - y_1\cdot f(s-\tfrac{R(r)}{2})\cdot Du(s-R(r))\\ &\quad + f(s-\tfrac{R(r)}{2})\cdot\widehat{u}(s-R(r)). \end{aligned}\tag{2.28}$$

Notice that each of the three summands has the form of the operator A in Lemma 2.22. By the proof of Lemma 2.22 we have $d\Gamma(0,u)=0$ and the derivative $d\Gamma(r,u):\mathbb{R}\oplus E_1\to E_0$ extends to a bounded linear operator $D\Gamma(r,u):\mathbb{R}\oplus E_0\to E_0$. In addition, the evaluation map

$$\begin{gathered}[0,\tfrac{1}{2})\oplus E_1\oplus\mathbb{R}\oplus E_0\to E_0\\ (r,u,y_1,\widehat{u})\mapsto D\Gamma(r,u)[y_1,\widehat{u}]\end{gathered}$$

is continuous. The same holds true if we consider $D\Gamma(r,u)[y_1,\widehat{u}]\in E_l$ with $(r,u,y_1,\widehat{u})\in[0,\frac{1}{2})\oplus E_{l+1}\oplus\mathbb{R}\oplus E_l$ for every $l\geq 0$. Thus, Γ is of class sc^1. The tangent map $T\Gamma:[0,\frac{1}{2})\oplus E^1\oplus\mathbb{R}\oplus E\to E^1\oplus E$ is the map

$$T\Gamma(r,u,y_1,\widehat{u})=(\Gamma(r,u),D\Gamma(r,u)[y_1,\widehat{u}]).$$

Taking a projection π of the tangent $TE=E^1\oplus E$ onto either factor, the composition $\pi\circ T\Gamma(r,u,y_1,\widehat{u})$ is a linear combination of terms (2.27) in the assertion (k) with k=1. Thus the statement is proved for $k=1$.

Next, assuming that the statement (k) with $k\geq 1$ holds true, we will show that the statement $(k+1)$ also holds true. We start by showing that the tangent map $T^k\Gamma:T^k([0,\frac{1}{2})\oplus E)\to T^kE$ is of class sc^1. Choose a factor E_n of the tangent space T^kE and let π be the projection of T^kE onto E_n. At a point $(r, y, u) \in (T^k([0, \frac{1}{2}) \oplus E))_0$ we have $T^k\Gamma(r, y, u) = 0$ if $r = 0$ and otherwise $T^k\Gamma(r, y, u)$ is a linear combination of maps of the form,

$$y_{i_1} \cdot y_{l_2} \cdots y_{l_n} \cdot \widehat{R}(r) \cdot D^j f(s - \tfrac{R(r)}{2}) \cdot D^i u_l(s - R(r)) \tag{2.29}$$

where $u_l \in E_{m_l}$ and $\widehat{R}(r) = D^{k_1}R(r) \cdot D^{k_2}R(r) \cdots D^{k_n}R(r)$ and where the indices satisfy $k_1 + \cdots + k_n \leq k$ and $m_l \geq i + n$. Abbreviate $u = u_l$, $y_1 = y_{l_1}, \ldots, y_n = y_{l_n}$ and $m_l = m$. Then, denoting by $A(r, y, u)$ the map defined by (2.29), we conclude from Lemma (2.22) that A is sc^1 from the sc-space $[0, \frac{1}{2}) \oplus \mathbb{R}^n \oplus E^{m_l}$ with the sc-structure $[0, \frac{1}{2}) \oplus \mathbb{R}^n \oplus E_{m_l+\alpha}$ for all integers $\alpha \geq 0$ into the sc-Banach space E^n equipped with the sc-structure $E_{n+\alpha}$, $\alpha \geq 0$. Moreover, $DA(0, y, u) = 0$. Taking $(r, y, u) \in (0, \frac{1}{2}) \oplus \mathbb{R}^n \oplus E_{m+1}$ and evaluating the linear operator $DA(r, y, u) : [0, \frac{1}{2}) \oplus \mathbb{R}^n \oplus E_m \to E_n$ at $(\widehat{r}, \widehat{y}, \widehat{u}) \in \mathbb{R} \oplus \mathbb{R}^n \oplus E_m$ we see that it is a linear combination of the terms (1)-(5) in the proof of Lemma 2.22. Since $u \in E_{m+1}$ and $\widehat{u} \in E_m$, the indices in the terms (1)-(3) and (4) satisfy the condition $m+1 > i+n$. In the term (5) we have $m+1 \geq (i+1)+n$ because, by assumption, $m \geq i + n$. Since $T^k\Gamma$ is a linear combination of terms of the form (2.29) we conclude that $T^{k+1}\Gamma$ satisfies the conditions of the statement (k+1). Thus, the statement (k+1) holds true and the proof of Theorem 2.17 is complete. ■

CHAPTER 3

The space of curves as an M-Polyfold

For two different points a and b in $\mathbb{R}^n$ we have introduced in Section 1.5 the metric space $\widehat{X}(a,b)$ of parametrized curves $u : \mathbb{R} \to \mathbb{R}^n$ connecting the point a at $-\infty$ with the point b at $+\infty$. The metric is denoted by $\widehat{d}(u,v) = \|u-v\|$. Quotienting out the $\mathbb{R}$-action of translation we have obtained the space

$$X(a,b) = \widehat{X}(a,b)/\sim$$

of equivalence classes $[u]$. Its quotient topology is determined by the complete metric d. The space of curves $X(a,b)$ is equipped with an sc-manifold structure.

We now consider three different points a, b and c in $\mathbb{R}^n$ and study the set of curves

$$X = X(a,c) \cup [X(a,b) \times X(b,c)]. \tag{3.1}$$

We are going to define a second countable paracompact topology on X which induces the previously constructed topology on $X(a,c)$ and on $X(a,b) \times X(b,c)$ so that $X(a,c)$ is open and dense in X. After this we shall equip the space X with an M-polyfold structure for which the degeneracy function d introduced in Section 2.5 takes on the values $d=0$ on $X(a,c)$ and $d=1$ on $X(a,b) \times X(b,c)$. The polyfold structure on the open subset $X(a,c)$ of X will be induced by the previously constructed sc-manifold structure. Since this particular case explains all major technical issues we will formulate at the end a situation frequently arising in Morse theory.

3.1. The Topology on X

We first introduce some notation. We fix a smooth cut-off function $\beta : \mathbb{R} \to [0,1]$ having the properties listed in (2.5), and define for $R \geq 0$ the cut-off function

$$\tau_R(s) = \beta(s - \tfrac{R}{2}).$$

We shall associate with a pair of parametrized curves $(u,v) \in \widehat{X}(a,b) \times \widehat{X}(b,c)$ the **glued parametrized curve** $\oplus_R(u,v) \in \widehat{X}(a,c)$ by defining

$$\oplus_R(u,v)(s) = \tau_R(s) \cdot u(s) + (1 - \tau_R(s)) \cdot v(s - R) \tag{3.2}$$

where $0 \leq R < \infty$. If $R = \infty$ we define

$$\oplus_\infty(u,v) = (u,v) \in \widehat{X}(a,b) \times \widehat{X}(b,c).$$

We have chosen the half line $(-\infty, 0]$ as the negative region of $\mathbb{R}$ and $[0, \infty)$ as the positive region of $\mathbb{R}$ for both of our two curves $u, v : \mathbb{R} \to \mathbb{R}^n$. The gluing is the following procedure. We identify the intervals $[0, R]$ of the positive part $[0, \infty)$ of u with interval $[-R, 0]$ of the negative part $(-\infty, 0]$ of v by $s \in [0, R] \sim s' \in [-R, 0]$ if and only if $-R + s = s'$.

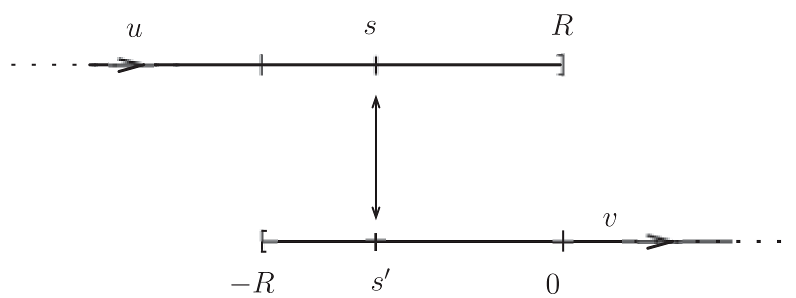


Figure 3.1

From the properties of the cut-off function $\tau_R(s) = \beta(s - \frac{R}{2})$ one reads off that

$$\oplus_R(u,v)(s) = \begin{cases} u(s) & s \leq \frac{R}{2}{}_1 \\ v(s-R) & s \geq \frac{R}{2} + 1. \end{cases}$$

We point out the limits as $R \to \infty$,

$$\lim_{R \to \infty} \oplus_R(u,v)(s) = u(s)$$
$$\lim_{R \to \infty} \oplus_R(u,v)(s + R) = v(s)$$

for all $s \in \mathbb{R}$.

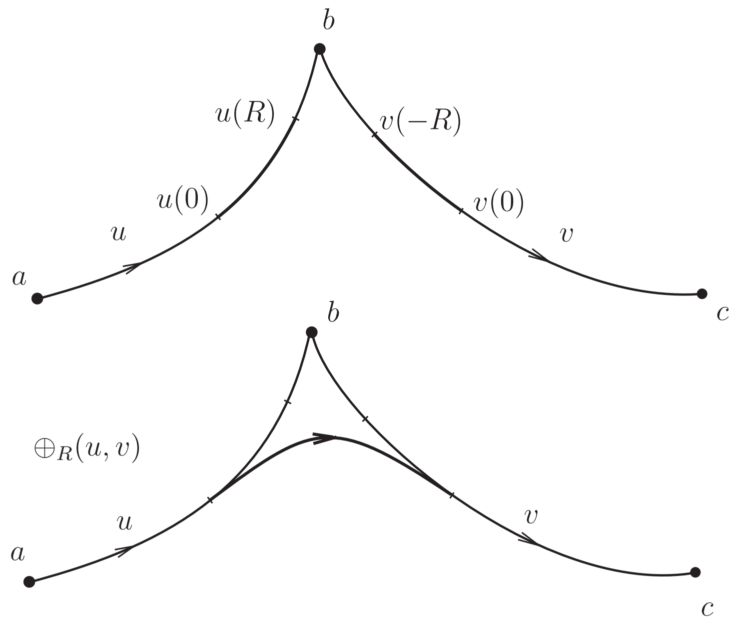


FIGURE 3.2

We shall use the following notation

$$[\oplus_\infty(u,v)] := ([u],[v]) \in X(a,b) \times X(b,c)$$

where the bracket $[u]$ stands for the equivalence class containing the representative u. We use the same formula (3.1) in order to glue the two vector fields

$$s \mapsto h(s) \in T_{u(s)}\mathbb{R}^n = \mathbb{R}^n$$
$$s' \mapsto k(s') \in T_{v(s')}\mathbb{R}^n = \mathbb{R}^n$$

along the curves u and v, where $s \in \mathbb{R}$ and $s' \in \mathbb{R}$. We define the vector field $\oplus_R(h,k)$ along the glued curve $\oplus_R(u,v)$ by

$$\oplus_R(h,k)(s) = \tau_R(s) \cdot h(s) + (1 - \tau_R(s)) \cdot k(s - R)$$

for all $s \in \mathbb{R}$, where $\tau(s) = \beta(s - \frac{R}{2})$. The identification for the gluing is as above.

In order to keep the information about the vector fields $h(s)$ and $k(s')$ also beyond the gluing intervals for $s \geq R$ and $s' \leq -R$ we introduce the **anti-gluing** of the vector fields h and k by the formula

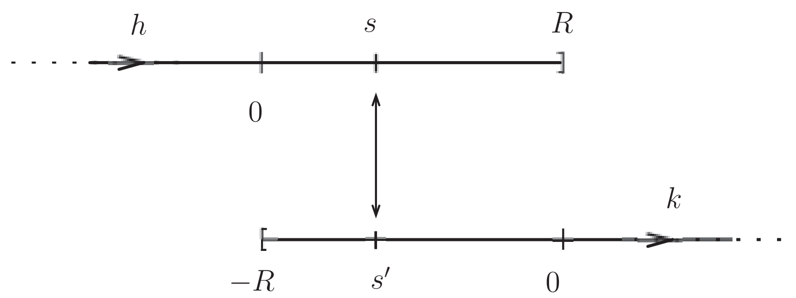


FIGURE 3.3

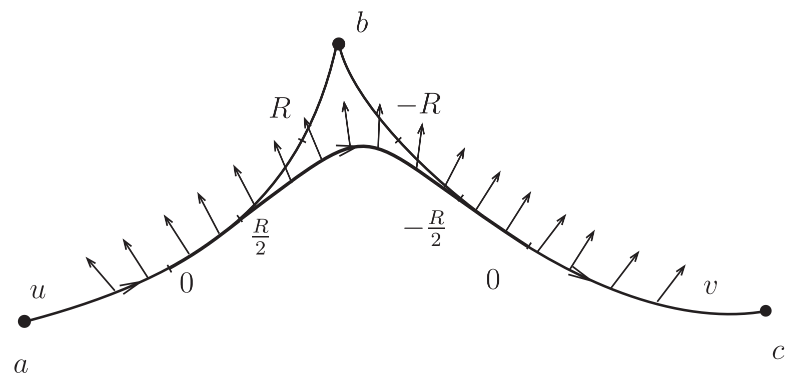


FIGURE 3.4

$$\ominus_R(u,v)(s) = -(1-\tau_R(s))\cdot u(s) + \tau_R(s)\cdot v(s-R) \tag{3.3}$$

for all $s \in \mathbb{R}$. Here we use the following identification.

In view of properties of the cut-off function β,

$$\oplus_R(h,k)(s) = \begin{cases} h(s) & s \leq \frac{R}{2}-1 \\ k(s-R) & s \geq \frac{R}{2}+1 \end{cases}$$

$$\ominus_R(h,k)(s) = \begin{cases} k(s-R) & s \leq \frac{R}{2}-1 \\ -h(s) & s \geq \frac{R}{2}+1 \end{cases}$$

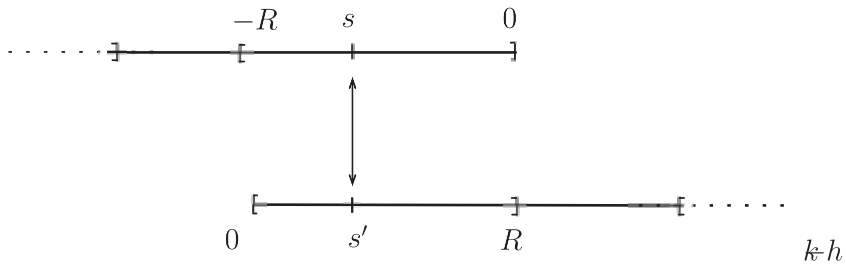


FIGURE 3.5

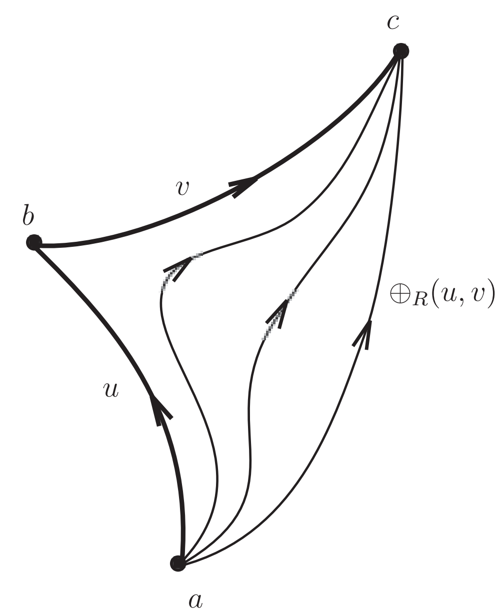


FIGURE 3.6

Next we define the topology of X by defining its basis $\mathcal{B}$. We shall distinguish between sets of type 1 and sets of type 2 in $\mathcal{B}$. In order to define a type 1 set we take a path $w_0 \in \widehat{X}(a, c)$ and a number $\delta > 0$ and define the set of curves connecting the point a with the point c by

$$O(w_0, \delta) = \{[w] \; | w \in \widehat{X}(a, c) \text{ and } d(w_0, w) < \delta\}. \tag{3.4}$$

Our notation indicates that we take the representative w_0 of $[w_0]$. In order to define a type 2 set of $\mathcal{B}$ we take a pair $(u_0, v_0) \in \widehat{X}(a,b) \times \widehat{X}(b,c)$ of curves as well as real numbers $\delta > 0$ and $R_0 > 1$ and define the neighborhood of broken curves (they are broken at $R = \infty$) as the set

(3.5)

$$O(u_0, v_0, R_0, \delta_0) = \{[\oplus_R(u,v)] \,|(u,v) \in \widehat{X}(a,b) \times \widehat{X}(b,c)$$
$$\text{with } \widehat{d}(u_0,u) < \delta,\ \widehat{d}(v_0,v) < \delta \text{ and } R_0 < R \leq \infty\}.$$

The aim of this section is the proof of the following theorem.

Theorem 3.1. *The collection $\mathcal{B}$ consisting of sets of type 1 and type 2 is the basis for a second countable Hausdorff topology for X. It induces on the subsets $X(a,c)$ and $X(a,b) \times X(b,c)$ the given topology. The subset $X(a,c)$ is open and dense in X.*

We start the proof with the following observation.

Lemma 3.2. *Fix real numbers t_0, t_1 and a pair of curves $(u,v) \in \widehat{X}(a,b) \times \widehat{X}(b,c)$. Then*

$$\widehat{d}(t_0 * \oplus_R(u,v), \oplus_{R+t_1-t_0}(t_0 * u, t_1 * v)) \to 0$$

as $R \to \infty$. Moreover,

$$d([\oplus_R(u,v)], [\oplus_{R+t_1-t_0}(t_0 * u, t_1 * v)]) \to 0$$

as $R \to \infty$.

Proof. Abbreviating $a = \frac{1}{2}[t_0 + t_1]$ and $R = R' + (t_0 - t_1)$ we compute

$$\begin{aligned}
&(t_0 * \oplus_R(u,v))(s) = \oplus_R(u,v)(t_0+s)\\
&= \tau_R(t_0+s) \cdot u(t_0+s) + (1-\tau_R(t_0+s)) \cdot v(t_0+s-R)\\
&= \tau_{R'}(a+s) \cdot (t_0 * u)(s) + (1-\tau_{R'}(a+s)) \cdot (t_1 * v)(s-R')\\
&= \oplus_{R'}(t_0 * u, t_1 * v) + [\tau_{R'}(a+s) - \tau_{R'}(s)] \cdot (t_0 * u)(s)\\
&\quad - [\tau_{R'}(a+s) - \tau_{R'}(s)] \cdot (t_1 * v)(s-R')\\
&=: \oplus_{R'}(t_0 * u, t_1 * v)(s) + \delta(R)(s).
\end{aligned}$$

The compact supports of the functions $s \mapsto [\tau_{R'}(a+s) - \tau_{R'}(s)]$ move to ∞ as $R \to \infty$ so that if u and v are fixed, then $\delta(R) \to 0$ in $H^2(\mathbb{R}, \mathbb{R}^n)$

as $R \to \infty$. Therefore,

$$\widehat{d}(t_0 * \oplus_R(u, v), \oplus_{R'}(t_0 * u, t_1 * v)) \to 0$$

as $R \to \infty$. The second assertion now follows from the inequality $d(\alpha, \beta) \le \widehat{d}(u, v)$ for any two equivalence classes α, $\beta \in X$ and representatives $u \in \alpha$, $v \in \beta$. The proof of the lemma is complete. ■

We need the following technical lemmata in which $\|\cdot\|$ denotes the norm on $E = H^2(\mathbb{R}, \mathbb{R}^n)$, and $B_\delta(0)$ denotes the open δ-ball around 0 in E.

Lemma 3.3. *Given $\varepsilon > 0$ there exists $\delta > 0$ so that for every $R \ge 2$,*

$$B_\delta(0) \subset \{\oplus_R(h, k) | \; \|h\|, \; \|k\| < \varepsilon\}.$$

Proof. Fix $u \in E$ and consider the system of equations

$$\begin{aligned} \oplus_R(h, k) &= u \\ \ominus_R(h, k) &= 0. \end{aligned}$$

The system can be written in the following matrix form

$$\begin{bmatrix} \tau(s) & 1 - \tau(s) \\ \tau(s) - 1 & \tau(s) \end{bmatrix} \cdot \begin{bmatrix} h(s) \\ k(s - R) \end{bmatrix} = \begin{bmatrix} u(s) \\ 0 \end{bmatrix}.$$

Multiplying by the inverse matrix we obtain

$$\begin{aligned} h(s) &= \frac{\tau(s)}{\alpha(s)} \cdot u(s) \\ k(s - R) &= \frac{1 - \tau(s)}{\alpha(s)} \cdot u(s). \end{aligned}$$

Setting $s' = s - R$ the last formula takes the form

$$k(s') = \frac{\tau(-s')}{\alpha(-s')} \cdot u(s' + R).$$

Using the translation invariance of the norm, we find a constant $C > 0$ independent of $R \ge 2$, depending only on bounds up to the second derivatives of the cut off function β, so that $\|h\| + \|k\| \le C \cdot \|u\|$. Consequently, for given ε we may take $\delta = \frac{\varepsilon}{2C}$ and our assertion is proved. ■

Lemma 3.4. *Given $\varepsilon > 0$ there exists $\delta > 0$ so that for all $R \ge 2$,*

$$\{\oplus_R(h, k) | \; \|h\|, \|k\| < \delta\} \subset B_\varepsilon(0).$$

Proof. This follows from a straightforward estimate. ■

Next we prove three lemmata which will imply Theorem 3.1

LEMMA 3.5. *The collection of sets $\mathcal{B}$ is the basis for a topology. The induced topologies on $X(a,c)$ and $X(a,b)\times X(b,c)$ are the previously defined ones.*

PROOF. Assume $[w_0]\in O_1\cap O_1'$ where O_1, O_1' are sets of type 1. Since the type 1 sets are a basis for the topology on $X(a,c)$ we find a set O_1'' of type 1 so that $[w_0]\in O_1''\subset O_1\cap O_1'$. Next assume that $[w_0]\in O_1\cap O_2$, where O_i is of type i. By Lemma 3.3, there exists O_1' with $[w_0]\in O_1'\subset O_1\cap O_2$. This also implies that the topology induced on $X(a,c)$ is the previously defined one. If $[w_0]$ belongs to $O_2\cap O_2'$ we can pick O_1 of type 1 containing $[w_0]$ and apply the previous consideration to $O_1\cap O_2$ and $O_1\cap O_2'$ in order to find a type 1 set with $[w_0]\in O_1'\subset O_2\cap O_2'$. Finally assume that $([u_0],[v_0])\in O_2\cap O_2'$. We may assume without loss of generality that

$$O_2 = O(u_0,v_0,R_0,\delta) \text{ and } O_2' = O(t_0 * u_0, t_1 * v_0, R_0,\delta).$$

It suffices to show for a suitable $S_0 >> 0$ and a suitable $\sigma > 0$ that

$$O(u_0,v_0,S_0,\sigma)\subset O(t_0 * u_0, t_1 * v_0, R_0,\delta).$$

Arguing indirectly we find sequences $S_k\to\infty$, $u_k\to u_0$ and $v_k\to v_0$ so that

$$[\oplus_{S_k}(u_k,v_k)]\notin O(t_0 * u_0, t_1 * v_0, R_0,\delta). \tag{3.6}$$

We estimate

$$\begin{aligned}
&\widehat{d}(t_0 * \oplus_{S_k}(u_k,v_k), \oplus_{S_k+t_1-t_0}(t_0 * u_k, t_1 * v_k))\\
&\leq\ \widehat{d}(t_0 * \oplus_{S_k}(u_k,v_k), t_0 * \oplus_{S_k}(u_0,v_0))\\
&\quad + \widehat{d}(t_0 * \oplus_{S_k}(u_0,v_0), \oplus_{S_k+t_1-t_0}(t_0 * u_0, t_1 * v_0))\\
&\quad + \widehat{d}(\oplus_{S_k+t_1-t_0}(t_0 * u_0, t_1 * v_0), \oplus_{S_k+t_1-t_0}(t_0 * u_k, t_1 * v_k))\\
&=\ \|t_0 * \oplus_{S_k}(u_k-u_0, v_k-v_0)\|\\
&\quad + \|\oplus_{S_k+t_1-t_0}(t_0 * (u_k-u_0), t_1 * (v_k-v_0))\|\\
&\quad + \widehat{d}(t_0 * \oplus_{S_k}(u_0,v_0), \oplus_{S_k+t_1-t_0}(t_0 * u_0, t_1 * v_0))\\
&=:\ I_k + II_k + III_k.
\end{aligned}$$

Using the invariance of the norm under the $\mathbb{R}$-action and Lemma 3.4 one sees that $I_k\to 0$. Lemma 3.4 also implies $II_k\to 0$ and from Lemma 3.2 one deduces $III_k\to 0$. Therefore,

$$\widehat{d}(t_0 * \oplus_{S_k}(u_k,v_k), \oplus_{S_k+t_1-t_0}(t_0 * u_k, t_1 * v_k))\to 0$$

as $k \to \infty$. Using Lemma 3.3 we find $\sigma > 0$ so that

$$B_\sigma(0) \subset \oplus_R(B_{\delta/2}(0), B_{\delta/2}(0))$$

for all $R \geq 2$. For large k the element $t_0 * \oplus_{S_k}(u_k, v_k)$ belongs to $\oplus_{S_k+t_1-t_0}(t_0 * u_k, t_1 * v_k) + B_\sigma(0)$ which in turn belongs to $\oplus_{S_k+t_1-t_0}(B_\delta(t_0 * u_0), B_\delta(t_1 * v_0))$. Now passing to equivalence classes we conclude

$$[\oplus_{S_k}(u_k, v_k)] \in O(t_0 * u_0, t_1 * v_0, R_0, \delta),$$

contradicting (3.6). This shows that $\mathcal{B}$ is a basis for a topology. From the form of the sets of type 2 in (3.5) one reads off immediately that the induced topology on $X(a,b) \times X(b,c)$ is the previously defined one. The proof of Lemma 3.5 is complete. ∎

LEMMA 3.6. *The topology generated by $\mathcal{B}$ is Hausdorff.*

PROOF. Arguing by contradiction we assume the points $([u_0],[v_0])$ and $([a_0],[b_0])$ in $X(a,b) \times X(b,c)$ do not have disjoint neighborhoods. Then there are sequences S_k and $R_k \to \infty$, $t_k \in \mathbb{R}$ and $(u_k, v_k) \to (u_0, v_0)$ and $(a_k, b_k) \to (a_0, b_0)$ such that

$$t_k * \oplus_{R_k}(u_k, v_k) = \oplus_{S_k}(a_k, b_k).$$

of Lemma 3.4 this implies that

$$t_k * \oplus_{R_k}(u_0, v_0) - \oplus_{S_k}(a_0, b_0) \to 0 \tag{3.7}$$

in $E = H^2(\mathbb{R}, \mathbb{R}^n)$. From $S_k \to \infty$ we deduce $\oplus_{S_k}(a_0, b_0)(s) \to a_0(s)$ as $k \to \infty$ for every $s \in \mathbb{R}$. We claim that the sequence t_k is bounded. If not we assume first that $t_k \to -\infty$. Then for every fixed s,

$$t_k * \oplus_{R_k}(u_0, v_0)(s) = u_0(s + t_k) \to a$$

as $k \to \infty$. Hence $a = a_0(s)$ for all s, contradicting $a_0(s) \to b \neq a$ as $s \to \infty$. If $t_k \to \infty$, then at $s - t_k$ with k large,

$$t_k * \oplus_{R_k}(u_0, v_0)(s - t_k) = u_0(s)$$

while

$$\oplus_{S_k}(a_0, b_0)(s - t_k) = a_0(s - t_k) \to a.$$

This implies $u_0(s) = a$ for all $s \in \mathbb{R}$ contradicting $u_0(s) \to b \neq a$ as $s \to \infty$. Thus the sequence t_k is bounded and we may assume $t_k \to t_0$. We next claim that the sequence $R_k - S_k$ is bounded. Indeed, if $R_k - S_k \to \infty$, then for fixed s as $k \to \infty$

$$t_k \oplus_{R_k} (u_0, v_0)(s + R_k) = u_0(s + t_k + R_k) \to b$$

while

$$\oplus_{S_k}(a_0, b_0)(s + R_k) = b_0(s + (R_k - S_k)) \to c \neq b,$$

contradicting (3.7). If $R_k - S_k \to -\infty$, then

$$t_k \oplus_{R_k} (u_0, v_0)(s + S_k) = v_0(s + t_k + (S_k - R_k)) \to c$$

while

$$\oplus_{S_k}(a_0, b_0)(s + S_k) = b_0(s).$$

Hence $b_0(s) = c$. But this contradicts $b_0(s) \to b \neq c$ as $s \to \infty$. Thus the sequence $R_k - S_k$ is bounded and we may assume that it converges to τ. Then evaluating $t_k \oplus_{R_k} (u_0, v_0)$ and $\oplus_{S_k}(a_0, b_0)$ at s and passing to the limit as $k \to \infty$ we find $t_0 * u_0 = a_0$. Evaluating both sequences at $s + R_k$ we conclude

$$t_k \oplus_{R_k} (u_0, v_0)(s + R_k) = v_0(s + t_k) \to t_0 * v_0(s)$$

and

$$\oplus_{S_k}(a_0, b_0)(s + R_k) = b_0(s + (R_k - S_k)) \to \tau * b_0(s).$$

Hence $([u_0], [v_0]) = ([a_0], [b_0])$ contradicting our assumption that these points are different.

If we have two different points $[w_0] \neq [w_0']$ in $X(a, c)$, the existence of separating open neighborhoods is immediate. The remaining case where the points $[w_0] \in X(a, c)$ and $([u_0], [v_0]) \in X(a, b) \times X(b, c)$ can be separated by open sets is left to the reader. ■

LEMMA 3.7. *The topology generated by $\mathcal{B}$ on X is second countable.*

PROOF. Recall from Section 1.5 that $X(a, c)$ is second countable. Take a countable basis for its topology, say W_i for $i \geq 1$. We also know that $\widehat{X}(a, b)$ and $\widehat{X}(b, c)$ are second countable. Take a countable basis U_i for the first and a countable basis V_j for the second space. Define the subsets $O_{i,j,k}$ of X by

$$O_{i,j,k} = \{[\oplus_R(u, v)] \mid u \in U_i \ v \in V_j, \ R \in (k, \infty]\}.$$

This collection of sets is a countable basis on X. ■

In view of Lemmata 3.5, 3.6 and 3.7 the proof of Theorem 3.1 is complete. ■

We would like to point out that the definition of $\mathcal{B}$ does not depend on the choice of the cut-off function β used in all our constructions.

3.2. M-Polyfold Charts on X

In this section we will construct a M-polyfold structure on X. For points in $X(a, c)$ we can take the sc-charts with domains disjoint from $X(a, b) \times X(b, c)$ which we constructed in Section 1.5. Clearly sc-charts are also M-polyfold charts the splicing being defined by $V = \{0\}$ and

$\pi_0 = Id$ so that the splicing core is $\{0\} \oplus E = E$. Before constructing M-polyfold charts for points in $X(a,b) \times X(b,c)$ we have to recall some facts from Section 1.5.

If a pair $([u'],[v']) \in X(a,b) \times X(b,c)$ is given, then there are positive numbers ε and ε_1 and a pair $([u],[v]) \in X(a,b) \times X(b,c)$ having smooth representatives $(u,v) \in \widehat{X}(a,b) \times \widehat{X}(b,c)$ and satisfying

$$\begin{aligned} \|u-u'\|_{C^0(\mathbb{R})} < \varepsilon_1 \quad &\text{and} \quad \|u-u'\|_{C^1([-\varepsilon,\varepsilon])} < \varepsilon^2 \\ \|v-v'\|_{C^0(\mathbb{R})} < \varepsilon_1 \quad &\text{and} \quad \|v-v'\|_{C^1([-\varepsilon,\varepsilon])} < \varepsilon^2. \end{aligned} \tag{3.8}$$

In addition, there are affine hyperplanes Σ_u and Σ_v in $\mathbb{R}^n$ so that u intersects Σ_u and v intersects Σ_v transversally at the parameter value $s=0$. Moreover, considering parametrized paths nearby we take $h,k \in E$ satisfying the estimates

$$\|h\|_{C^1([-\varepsilon,\varepsilon])} < \varepsilon^2 \quad \text{and} \quad \|k\|_{C^1([-\varepsilon,\varepsilon])} < \varepsilon^2. \tag{3.9}$$

Then the paths $u+h \in \widehat{X}(a,b)$ resp. $v+k \in \widehat{X}(b,c)$ intersect Σ_u resp. Σ_v transversally at unique points parametrized by s satisfying $|s| < \varepsilon$. In addition, there exist positive numbers α and ε_2 such that if

$$\|h\|_{C^0(\mathbb{R})},\ \ \|h'\|_{C^0(\mathbb{R})},\ \ \|k\|_{C^0(\mathbb{R})},\ \ \|k\|_{C^0(\mathbb{R})} < \varepsilon_2$$

and

$$u+h = t*(u+h') \quad \text{and} \quad v+k = t'*(v+k') \tag{3.10}$$

on $[-\alpha,\alpha]$ for some $t,t' \in \mathbb{R}$, then necessarily $|t|,|t'| < \varepsilon$. Recall furthermore the following construction of an sc-chart at $[u] \in X(a,b)$. Introduce the closed subspace $F_u \subset E$ consisting of functions h satisfying $h(0) \in \Sigma_u - u(0)$. Define the open subset U of F_u by

$$U = \{h \in F_u \mid \|h\|_{C^0(\mathbb{R})} < \varepsilon_1 \ \text{and} \ \|h\|_{C^1([-\varepsilon,\varepsilon])} < \varepsilon^2\}. \tag{3.11}$$

Then the map

$$\begin{aligned} U &\to X(a,b) \\ h &\mapsto [u+h] \end{aligned}$$

defines an sc-chart. It should be recalled that E is the Banach space $E = H^2(\mathbb{R},\mathbb{R}^n)$ equipped with the sc-smooth structure introduced in Section 1.5.

Similarly, an sc chart at $[v] \in X(b,c)$ is defined by the map

$$\begin{aligned} V &\to X(b,c) \\ k &\mapsto [v+k] \end{aligned}$$

from the open subset $V \subseteq F_v$ defined by

$$V = \{h \in F_v \mid \|k\|_{C^0(\mathbb{R})} < \varepsilon_1 \text{ and } \|k\|_{C^1([-\varepsilon,\varepsilon])} < \varepsilon^2\} \tag{3.12}$$

where the subspace F_v consists of functions k in E satisfying $k(0) \in \Sigma_v - v(0)$.

LEMMA 3.8. *Let the data be as just described. Then there exists a real number $R_0 > 2$ so that for $R, R' \geq R_0$, for $h, h' \in U$ and $k, k' \in V$ and $t \in \mathbb{R}$, the equality*

$$\oplus_R(u+h, v+k) = t * \oplus_{R'}(u+h', v+k') \tag{3.13}$$

implies $t = 0$ and $R = R'$.

PROOF. We choose $R_0 = 2\alpha + 2$ with α as in (3.10) . Pick $R, R' \geq R_0$ and assume

$$\oplus_R(u+h, v+k) = t * \oplus_{R'}(u+h', v+k')$$

for some $r \in \mathbb{R}$. We may assume $t \leq 0$, interchanging if necessary the roles of (h, k) and (h', k'). In view of the choice of R_0 and since the function τ_R is equal to 1 for $s \leq \frac{R}{2} - 1$, it follows from (3.13) that

$$(u+h)(s) = (u+h')(t+s) = (t * (u+h'))(s)$$

for all $s \in [-\alpha, \alpha]$. Thus (3.10) implies $|t| < \varepsilon$. Setting $s = 0$ and recalling that $h \in F_u - u(0)$ we obtain $u(0) + h(0) = u(t) + h'(t) \in \Sigma_u$. Since $h' \in F_u$ we also know $u(0) + h'(0) \in \Sigma_u$ and conclude $t = 0$ from the uniqueness of the intersection under assumption (3.9). Hence

$$\oplus_R(u+h, v+k) = \oplus_{R'}(u+h', v+k')$$

and we will show that $R = R'$. We may assume $R \geq R'$. Evaluating the above paths at $s + R$ with $s \in [-\alpha, \alpha]$ we find

$$(v+h)(s) = (v+h')(s - R' + R) = (R - R') * (v+h')(s)$$

if $s \in [-\alpha, \alpha]$. Again applying 3.10 we conclude $R = R'$. The proof is complete. ∎

Choose $R_0 = 2\alpha + 2$ as in Lemma 3.8. If $r_0 \in (0, 1)$ satisfies $\varphi(r_0) > R_0$, where φ is the gluing profile $\varphi(r) = e^{\frac{1}{r}} - e$, we define the gluing map

$$A : [0, r_0) \oplus U \oplus V \to X \tag{3.14}$$

for $r = 0$ by

$$A(0, h, k) = ([u+h], [v+k])$$

and if $r \in (0, r_0)$, we put

$$A(r, h, k) = [\oplus_R(u + h, v + k)]$$

where $R = \varphi(r)$. By construction, the map A is continuous.

LEMMA 3.9. *If*

$$A(r, h, k) = A(r', h', k')$$

and $\ominus_R(h, k) = 0$ *and* $\ominus_{R'}(h', k') = 0$ *where* $R = \varphi(r)$ *and* $R' = \varphi(r')$, *then* $(r, h, k) = (r', h', k')$.

PROOF. If $A(r, h, k) = A(r', h', k')$ and one of the two numbers r, r' vanishes, then the other vanishes as well. If $r = r' = 0$, then

$$([u + h], [v + k]) = ([u + h'], [v + k'])$$

for $h, h' \in U$ and $k, k' \in V$, implying $h = h'$ and $k = k'$ because the data define charts for $X(a, b)$ and $X(b, c)$. Now assume $r, r' \in (0, r_0)$. From $A(r, h, k) = A(r', h', k')$ we conclude the existence of $t \in \mathbb{R}$ satisfying

$$\oplus_R(u + h, v + k) = t * \oplus_{R'}(u + h', v + k'). \tag{3.15}$$

Here $R = \varphi(r)$, $R' = \varphi(r')$ and, by construction, $R > R_0$ and $R' > R_0$. Hence $t = 0$ and $R = R'$ in view of Lemma 3.8, so that

$$\oplus_R(u + h, v + k) = \oplus_R(u + h', v + k')$$

implying

$$\oplus_R(h, k) = \oplus_R(h', k').$$

This together with $\ominus_R(h, k) = \ominus_R(h', k') = 0$ imply $(h, k) = (h', k')$. The proof of the lemma is complete. ■

Define the splicing

$$\mathcal{S} = (\pi, F_u \oplus F_v, [0, r_0))$$

as in Section 2.2 by the formulae in (2.8) replacing E^+ by F_u and E^- by F_v. The formulae (2.8) holds true this time for all $s \in \mathbb{R}$ and all $s' \in \mathbb{R}$. Hence, if $(h, k) \in F_u \oplus F_v$, then $\pi_r(h, k) = (\widehat{h}, \widehat{k}) \in F_u \oplus F_v$ is defined by the formulae

$$\begin{aligned}
\widehat{h}(s) &= \frac{\tau(s)}{\alpha(s)}\big[\tau(s) \cdot h(s) + (1 - \tau(s)) \cdot k(s - R)\big] \\
\widehat{k}(s') &= \frac{\tau(-s')}{\alpha(-s')}\big[(1 - \tau(-s')) \cdot h(s' + R) + \tau(-s') \cdot k(s')\big]
\end{aligned}$$

for all $s \in \mathbb{R}$ and $s' \in \mathbb{R}$. Near 0 the functions are not changed,

$$
\begin{aligned}
\widehat{h}(s) &= \begin{cases} h(s) & 0 \leq s \leq \frac{R}{2} - 1 \\ 0 & s \geq \frac{R}{2} + 1 \end{cases} \\
\widehat{k}(s') &= \begin{cases} 0 & s' \leq -\frac{R}{2} - 1 \\ k(s') & s' \geq -\frac{R}{2} + 1. \end{cases}
\end{aligned}
\tag{3.16}
$$

The associated splicing core $K^{\mathcal{S}}$ is characterized as the collection of points $(r, h, k) \in [0, r_0) \oplus F_u \oplus F_v$ satisfying $\ominus_R(h, k) = 0$ where $R = \varphi(r)$. Take the open set O of the splicing core $K^{\mathcal{S}}$ defined by

$$
O = ([0, r_0) \oplus U \oplus V) \cap K^{\mathcal{S}}
$$

with U and V according to (3.11) and (3.12). In view of Lemma 3.9, the map

$$
A : O \to X \tag{3.17}
$$

is injective.

Lemma 3.10. *The map $A : O \to X$ is a homeomorphism onto some open subset of X.*

Proof. We already know that A is continuous and injective. It remains to show that A is open. Take an open subset O' of O. We will show that every point of $A(O')$ is an interior point of $A(O)$. Assume first that for $(r, h, k) \in O'$, the image point $A(r, h, k)$ belongs to $X(a, c)$. Taking the representative w of $A(r, h, k)$ we have the path

$$
\oplus_R(u + h, v + k) = w \in \widehat{X}(a, c). \tag{3.18}
$$

Here $R = \varphi(r)$ with $r \in (0, r_0)$ and the pair $(h, k) \in U \oplus V$ satisfies $\pi_r(h, k) = (h, k)$ or equivalently $\ominus_R(h, k) = 0$. Evaluating both sides of (3.18) at 0 and R we conclude from the definition of the sets U and V that

$$
w(0) = (u + h)(0) \in \Sigma_u \quad \text{and} \quad w(R) = (v + k)(0) \in \Sigma_v.
$$

It follows from the implicit function theorem, arguing as in Lemma (1.30), that if $g \in E$ is close to 0 in E, then there exists a unique time $t_0(g)$ so that $(w + g)(t_0(g)) \in \Sigma_u$. The map $g \mapsto t_0(g)$ is sc-smooth and $t_0(0) = 0$. Similarly, there exists a sc-smooth map $g \mapsto t_1(g)$ satisfying $t_1(0) = R$ and $(w + g)(t_1(g)) \in \Sigma_v$. Set $R' = t_1(g) - t_0(g)$ and define r' via $R' = \varphi(r')$.

We look for the pair $(h',k') \in U \times V$ solving the following two equations for given $g \in E$ close to 0

$$\oplus_{R'}(h',k') = t_0(g) * (w+g) - \oplus_{R'}(u,v) \quad \text{and} \quad \ominus_{R'}(h',k') = 0.$$

Abbreviating the functions $\tau' = \beta(\cdot - \frac{R'}{2})$ and $\alpha' = (\tau')^2 + (1-\tau')^2$, the solution (h',k') can be written explicitly as

$$\begin{bmatrix} h'(s) \\ k'(s-R') \end{bmatrix} = \frac{1}{\alpha'} \begin{bmatrix} \tau' & -(1-\tau') \\ 1-\tau' & \tau' \end{bmatrix} \cdot \begin{bmatrix} t_0(g) * (w+g) - \oplus_{R'}(u,v) \\ 0 \end{bmatrix}.$$

It follows immediately that if $g \in E$ is close to 0 in E then (r',h',k') is close to (r,h,k) in $\mathbb{R} \oplus E \oplus E$. The condition $\ominus_{R'}(h',k') = 0$ implies $(h',k') \in K^{\mathcal{S}}$. Substituting $s=0$ and then $s=R$ in the above equation we find

$$\begin{aligned} h'(0) &= (w+g)(t_0(g)) - u(0) \in \Sigma_u - u(0) = F_u \\ k'(0) &= (w+g)(t_1(g)) - v(0) \in \Sigma_v - v(0) = F_v. \end{aligned}$$

Summing up, we have proved for every $g \in E$ near 0 in E that there exists $(r',h',k') \in O'$ satisfying $A(r',h',k') = [w+g]$. Thus $A(r,h,k)$ is an interior point of $A(O')$.

Next we consider the case $([u_0],[v_0]) \in A(O')$. In view of the definition of A we have $A(0,h_0,k_0) = ([u_0],[v_0])$ for some $(h_0,k_0) \in U \oplus V$. Taking representatives we may assume $u_0 = u+h_0$ and $v_0 = v+k_0$. Recall that by construction of the topology on X, an open neighborhood of $([u_0],[v_0])$ consists of the set of equivalence classes $[\oplus_R(u_0+h, v_0+k)]$ where R is large including $+\infty$ and h and k are close to 0 in E. Using the implicit function theorem as above we find sc-smooth functions $h \mapsto t_0(h)$ and $k \mapsto t_1(k)$ defined for h and k close to 0 in E such that $t_0(0) = 0$ and $t_1(0) = 0$ and

$$(u_0+h)(t_0(h)) \in \Sigma_u \quad \text{and} \quad (v_0+k)(t_1(k)) \in \Sigma_v.$$

Introduce the path

$$w := t_0(h) * \oplus_R(u_0+h, v_0+k) \in \widehat{X}(a,c).$$

Obviously, $[w] = [\oplus_R(u_0+h, v_0+k)]$. Taking R large and setting $R' = R + [t_1(k) - t_0(h)]$ we deduce from the definition of w that

$$w(0) = (u_0+h)(t_0(h)) \in \Sigma_u \quad \text{and} \quad w(R') = (v_0+k)(t_1(k)) \in \Sigma_v.$$

Now we look for solutions (h',k') of the equations

$$\oplus_{R'}(h',k') = w - \oplus_{R'}(u,v) \quad \text{and} \quad \ominus_{R'}(h',k') = 0.$$

Explicitly, the solution (h', k') is given by

$$\begin{bmatrix} h'(s) \\ k'(s-R') \end{bmatrix} = \frac{1}{\alpha'} \begin{bmatrix} \tau' & -(1-\tau') \\ 1-\tau' & \tau' \end{bmatrix} \cdot \begin{bmatrix} w - \oplus_{R'}(u,v) \\ 0 \end{bmatrix}$$

where as before $\tau' = \beta(\cdot - \frac{R'}{2})$ and $\alpha' = (\tau')^2 + (1-\tau')^2$. It follows for R large and hence for R' large that the resulting solution (h', k') is close to (h_0, k_0) in $E \oplus E$. Moreover, $(h', k') \in K^{\mathcal{S}}$ since $\ominus_{R'}(h', k') = 0$. Furthermore, substituting $s = 0$ and $s = R'$ into the the above equation we find

$$h'(0) = w(0) - u(0) \in \Sigma_u - u(0) = F_u$$

and $k'(0) \in F_v$. Hence taking R large and defining r' via $R' = \varphi(r')$ we conclude $(r', h', k') \in O'$ and $A(r', h', k') = [\oplus_{R'}(u + h', v + k')] = [w]$ showing that $([u_0], [v_0])$ is an interior point of $A(O')$. Thus the map A is open and the proof of the lemma is complete. ■

This implies the following result.

Lemma 3.11. *The topological space X is a second countable metrizable space.*

Proof. We already know from Theorem 3.1 that the space X is second countable and Hausdorff. Since by Lemma 3.10 X is locally homeomorphic to open sets in splicing cores, we know that it is completely regular which implies also assertion by Proposition 1.25. ■

In view of Lemma 3.10, the map $A : O \to X$ defined by (3.17) is a homeomorphism onto an open subset $A(O)$ of the topological space X. Abbreviating this open set by

$$U = A(O) \subset X$$

we denote by

$$\Phi := A^{-1} : U \to O$$

the inverse of the map A. Then the collection of all charts (U, Φ) or more precisely $(U, \Phi, (O, \mathcal{S}))$, as well as the previously constructed sc-charts for $X(a, c)$, have the property that the union of their domains covers X. Our main result in this section is the following theorem.

Theorem 3.12. *The atlas consisting of all sc-charts for $X(a, c)$ and all charts $(U, \Phi, (O, \mathcal{S}))$ around smooth points in $X(a, b) \times X(b, c)$ covers X and the transition maps are sc-smooth. In other words the atlas defines a sc-smooth M-polyfold structure.*

PROOF. Recall that we distinguish between the charts of type 1 which are inherited from $X(a,c)$ and charts of type 2 which involve the gluing construction. We already have proved in Lemma 1.33 that the transition maps between two charts of type 1 are sc-smooth.

We now consider the transition maps from charts of type 1 to charts of type 2. Take a chart $\Phi_1 : U_1 \to O_1$ of type 1 and a chart $\Phi_2 : U_2 \to O_2$ of type 2. Then $A_1 = \Phi_1^{-1} : O_1 \to U_1$ and $A_2 = \Phi_2^{-1} : O_2 \to U_2$. Assuming that the domain of $A_2^{-2} \circ A_1$ is nonempty we will show that it is sc-smooth. Assume $A_1(f_0) = A(r_0, h_0, k_0)$ for some $f_0 \in O_1$ and $(r_0, h_0, k_0) \in O_2 \subset K^{\mathcal{S}}$. Then there are curves $u \in \widehat{X}(a,b)$ and $v \in \widehat{X}(b,c)$ and $w \in \widehat{X}(a,c)$ and real numbers $R_0 \geq 2$ and $t_0 \in \mathbb{R}$ satisfying

$$\oplus_{R_0}(u + h_0, v + k_0) = t_0 * (w + f_0)$$

where $R_0 = \varphi(r_0)$. In addition, $\ominus_{R_0}(h_0, k_0) = 0$ since $(h_0, k_0) \in K^{\mathcal{S}}$.

In view of the definition of O_2 the functions h_0 and k_0 belong to the spaces F_u and F_v, respectively, from which we conclude after evaluating the above equation at 0 and R_0 that $(u + h_0)(0) = (w + f_0)(t_0) \in \Sigma_u$ and $(v + k_0)(0) = (w + f_0)(R_0 + t_0) \in \Sigma_v$. Using the implicit function theorem we find two sc-smooth functions $f \mapsto t_0(f)$ and $f \mapsto t_1(f)$ defined for f close to f_0 in E satisfying

$$(w + f)(t_0(f)) \in \Sigma_u \quad \text{and} \quad (w + f)(R_0 + t_1(f)) \in \Sigma_v,$$

moreover, $t_0(f_0) = t_0$ and $t_1(f_0) = t_0$. Define the positive number $R(f) = R_0 + [t_1(f) - t_0(f)]$. Then the curve $t_0(f) * (w + f)$ intersects Σ_u and Σ_v at the times 0 and $R(f)$. We define $r(f)$ via the formula

$$R(f) = \varphi(r(f)). \tag{3.19}$$

We look for a pair (h, k) solving the following two equations

$$\oplus_{R(f)}(h, k) = t_0(f) * (w + f) - \oplus_{R(f)}(u, v) \quad \text{and} \quad \ominus_{R(f)}(h, k) = 0.$$

Introducing $\widehat{w}_{R(f)} := t_0(f) * (w + f) - \oplus_{R(f)}(u, v)$, the solution pair (h, k) can be represented explicitly as

$$\begin{bmatrix} h(s) \\ k(s - R(f)) \end{bmatrix} = \frac{1}{\alpha(f)} \begin{bmatrix} \tau(f) & -(1 - \tau(f)) \\ (1 - \tau(f)) & \tau(f) \end{bmatrix} \cdot \begin{bmatrix} \widehat{w}_{R(f)} \\ 0 \end{bmatrix}$$

where we have abbreviated $\tau(f) = \beta(\cdot - \frac{R(f)}{2})$ and $\alpha(f) = \tau(f)^2 + (1 - \tau(f))^2$. We denote the solution as $(h(f), k(f))$. We only consider $h(f)$ since the results for $k(f)$ are proved in a similar way. We set

$$\gamma(f) = \frac{\tau(f)}{\alpha(f)}.$$

and define the map

$$f \mapsto h(f) = \gamma(f) \cdot \big[t_0(f) * (w+f) - \oplus_{R(f)}(u,v)\big] \tag{3.20}$$

from E into E.

LEMMA 3.13. *The map* (3.20) *is sc-smooth.*

PROOF. The curves $u \in \widehat{X}(a,b)$, $v \in \widehat{X}(b,c)$ and $w \in \widehat{X}(a,c)$ have the form $u = \phi + h_0'$, $v = \psi + k_0'$ and $w = \eta + g_0'$ where h_0', k_0' and $g_0' \in E$ and ϕ, ψ and η are smooth functions which are constant outside the interval $[-1,1]$. We represent the map under consideration as a sum,

$$\begin{aligned}\gamma(f) \cdot \big[t_0(f) * (w+f) - \oplus_R(u,v)\big] &= \gamma(f) \cdot t_0(f) * f \\ &\quad + \gamma(f) \cdot t_0(f) * g_0' - \gamma(f) \cdot \oplus_{R(f)}(h_0', k_0') \\ &\quad + \gamma(f) \cdot [t_0(f) * \eta - \oplus_{R(f)}(\phi, \psi)].\end{aligned} \tag{3.21}$$

The sc-smoothness of the first two maps follows from the sc-smoothness of the $\mathbb{R}$-action together with Theorem 2.17. The third map is equal to

$$\gamma(f) \cdot \tau_{R(f)} h_0' + \gamma(f) \cdot (1 - \tau_{R(f)}) \cdot k_0'(\cdot - R(f)).$$

Sc-smoothness follows by applying Theorem 2.17 to each of the above summands. Finally, the last term can be represented as

$$\begin{aligned}&\gamma(f) \cdot [t_0(f) * \eta - \oplus_{R(f)}(\phi, \psi)] \\ &\qquad = \gamma(f) \cdot \tau_{R(f)} \cdot [t_0(f) * \eta(s) - \phi(s)] \\ &\qquad\quad + \gamma(f) \cdot (1 - \tau_{R(f)}) \cdot [t_0(f) * \eta(s) - \psi(s - R(f))].\end{aligned} \tag{3.22}$$

The maps $f \mapsto t_0(f)$ and $f \mapsto t_1(f)$ are bounded, hence, $R(f) = R_0 + [t_1(f) - t_0(f)]$ is also bounded. Since $\phi(s) = \eta(s) = a$ for $s \leq -1$ and $\phi(s) = b$ and $\eta(s) = c$ for $s \geq 1$, the first map on the right hand side can be written as

$$\gamma(f) \cdot \tau_{R(f)} \cdot [t_0(f) * \eta - \phi] = \gamma(f) \cdot \tau_{R(f)} \cdot \beta(\cdot - C) \cdot [t_0(f) * \eta - \phi]$$

where C is a large positive constant.

Let $\delta : \mathbb{R} \to E$ be the map defined by $\delta(x) = \beta(\cdot - C) \cdot [\eta(\cdot - x) - \phi(\cdot)]$. Since δ is sc-smooth, the composition $f \mapsto \delta(t_0(f))$ from E into E is also sc-smooth. Define the function γ by

$$\gamma(s) = \frac{\beta(s)}{\beta(s)^2 + (1 - \beta(s))^2}, \tag{3.23}$$

where β is the cut-off function from (2.5). With the function $\widehat{\gamma} = \gamma \cdot \beta$ having its support in $(-\infty, 1]$, the map $\Gamma(r,g) = \widehat{\gamma}(\cdot - \frac{R}{2}) \cdot g$ for $g \in E$

is sc-smooth by Theorem 2.17. Thus, the composition

$$f \longrightarrow (R(f), t_0(f)) \xrightarrow{(\varphi^{-1},\delta)} (r(f), \delta(t_0(f)) \xrightarrow{\Gamma} \Gamma(r(f), \delta(t_0(f)))$$

is sc-smooth. The first term on the right hand side of (3.22) is precisely this composition and so it defines an sc-smooth map.

In view of the boundedness of the maps $f \mapsto t_0(f)$ and $f \mapsto t_1(f)$, and hence also of $R(f) = R_0 + [t_1(f) - t_0(f)]$, the second term from on the right hand side of (3.22) can be written as

$$\begin{aligned}\gamma(f)\cdot&(1-\tau_{R(f)}) \cdot [t_0(f) * \eta(s) - \psi(s - R(f))]\\ &= \gamma(f) \cdot (1-\tau_{R(f)}) \cdot \beta(s-C) \cdot [t_0(f) * \eta(s) - \psi(s - R(f))]\end{aligned}$$

where C is a large positive constant C.

The map ξ from $\mathbb{R}^2$ to E defined by $\xi(x, y) = \beta(\cdot - C) \cdot [\eta(\cdot + x) - \psi(\cdot - y)$ is sc-smooth implying that $f \mapsto \xi(t_0(f), R(f))$ is also of class sc^∞.

The function $\widehat{\gamma} = \gamma \cdot (1-\beta)$ has a compact support so that, in view of Theorem 2.17, the map $\Gamma_1(r, g) := \widehat{\gamma}(\cdot - \frac{R}{2}) \cdot g$ is sc-smooth. Thus, the composition,

$$\begin{aligned}f \longrightarrow (R(f), t_0(f)) &\xrightarrow{(\varphi^{-1},\xi)} (r(f), \xi(R(f), t_0(f))\\ &\xrightarrow{\Gamma_1} \Gamma_1[r(f), \xi(R(f), t_0(f))]\end{aligned}$$

is sc-smooth. The second term of (3.22) is precisely the above composition and so the proof of the lemma is complete. ■

The lemma holds also true for the map $f \mapsto k(f)$. Since the transition map $A_2^{-2} \circ A_1$ is given by $f \mapsto (h(f), k(f))$ we conclude that $A_2^{-2} \circ A_1$ is sc-smooth.

We next consider the transition map $A_1^{-1} \circ A_2$. Assume that $A_1(f_0) = A_2(r_0, h_0, k_0)$ for some $f_0 \in O_1$ and for $(r_0, h_0, k_0) \in O_2 \subset K^{\mathcal{S}}$ with $r_0 > 0$. Then there are curves $u \in \widehat{X}(a, b)$ and $v \in \widehat{X}(b, c)$ and $w \in \widehat{X}(a, c)$ and a real number t_0 satisfying

$$t_0 * \oplus_{R_0}(u + h_0, v + k_0) = w + f_0 \tag{3.24}$$

where $R_0 = \varphi(r_0)$. Evaluating both sides at time $s = 0$ we find

$$\oplus_{R_0}(u + h_0, v + k_0)(t_0) = (w + f_0)(0) \in \Sigma_w.$$

Lemma 3.14. *There exists an sc-smooth map* $(R,h,k)\mapsto t(R,h,k)$ *defined for* R, h *and* k *close to* R_0, h_0 *and* k_0 *respectively so that at the parameter value* $t(R,h,k)$ *the curve* $\oplus_R(u+h,v+k)$ *intersects* Σ_w *transversally and satisfies, in addition,* $t(R_0,h_0,k_0)=t_0$.

Proof. To see this we proceed as in Lemma 1.30 and represent $\Sigma:=\Sigma_w$ as $\Sigma=\{x\in\mathbb{R}^n|\ \lambda(x)=c\}$ for some linear map $\lambda:\mathbb{R}^n\to\mathbb{R}$ and some constant $c\in\mathbb{R}$, and define the map $F:\mathbb{R}\oplus(0,\infty)\oplus E\oplus E\to\mathbb{R}$ by setting

$$F(t,R,h,k)=\lambda(\oplus_R(u+h,v+k)(t+t_0))-c. \tag{3.25}$$

At the point $(0,R_0,h_0,k_0)$ we have $F(0,R_0,h_0,k_0)=0$. Because of the Sobolev embedding $E\subset C^1(\mathbb{R})$ the map F is of class C^1. From (3.24) we conclude that the derivative of F with respect to t at the point $(0,R_0,h_0,k_0)$ is equal to

$$d_tF(0,R_0,h_0,k_0)[\delta t]=\lambda(w'(0)+f_0'(0))\cdot[\delta t]$$

for all $\delta t\in\mathbb{R}$. Since the curve $w+f_0$ intersects Σ transversally at the parameter value 0, the derivative $\lambda(w'(0)+f_0'(0))$ does not vanish. So, $d_tF(0,R_0,h_0,k_0):\mathbb{R}\to\mathbb{R}$ is invertible. Applying the implicit function theorem, there exist $\varepsilon>0$ and, for every $(R,h,k)\in(0,\infty)\oplus E\oplus E$ satisfying $|R-R_0|<\varepsilon$ and $\|h-h_0\|,\|k-k_0\|<\varepsilon$, a unique time $t(R,h,k)\in\mathbb{R}$ solving the equation $F(t(R,h,k),R,h,k)=0$. In addition, the map $(R,h,k)\mapsto t(R,h,k)$ is of class C^1 and satisfies $t(R_0,h_0,k_0)=t_0$. Thus, $\oplus_R(u+h,v+k)$ intersects Σ transversally at the unique parameter values $t(R,h,k)$ if $|R-R_0|<\varepsilon$ and $\|h-h_0\|,\|k-k_0\|<\varepsilon$. To see that the mapping $(R,h,k)\mapsto t(R,h,k)$ is sc-smooth we define an open subset V of $(0,\infty)\oplus E\oplus E$ by $V=\{(R,h,k)|\,|R-R_0|,\,\|h-h_0\|,\,\|k-k_0\|<\varepsilon\}$, which is filtrated by $V_m:=V\cap(\mathbb{R}\oplus E_m\oplus E_m)$ for all $m\geq 0$. The function $(R,h,k)\mapsto t(R,h,k)$ viewed as a function from V_m to $\mathbb{R}$ is of class C^1. In view of the Sobolev embedding theorem, $F_m\subset C^{m+1}$. and hence the map F in (3.25) viewed as a map from $\mathbb{R}\oplus(0,\infty)\oplus E_m\oplus E_m$ into $\mathbb{R}$ is of class C^{m+1}. Then the implicit function theorem shows that $(R,h,k)\mapsto t(R,h,k)$ from V_m into $\mathbb{R}$ is of class C^{m+1}. Consequently, Proposition 1.10 implies the sc-smoothness of $(R,h,k)\mapsto t(R,h,k)$ from V into $\mathbb{R}$.

■

If $(R,h,k)\in V$, we define the map f by

$$f(R,h,k)=t(R,h,k)*\oplus_R(u+h,v+k)-w. \tag{3.26}$$

Then $f(R_0,h_0,k_0)=f_0$, in view of Lemma 3.14. Abbreviating $f(s)=f(R,h,k)(s)$,

$$\begin{aligned}(w+f)(0)&=t(R,h,k)*\oplus_R(u+h,v+k)(0)\\&=\oplus_R(u+h,v+k)(t(R,h,k))\in\Sigma_w.\end{aligned}$$

It follows $f(0)\in\Sigma_w-w(0)$ and so f belongs to F_w.

Lemma 3.15. *The map* (3.26) *is sc-smooth.*

Proof. The curves $u\in\widehat{X}(a,b)$, $v\in\widehat{X}(b,c)$ and $w\in\widehat{X}(a,c)$ have the form $u=\phi+h_0$, $v=\psi+k_0$ and $w=\eta+g_0$ where h_0,k_0 and $g_0\in E$ and ϕ, ψ and η are smooth functions which are constant outside the interval $[-1,1]$. Using this we represent the map as

$$\begin{aligned}t(R,h,k)*\oplus_R(u+h,v+k)-w&=t(R,h,k)*\oplus_R(h,k)\\&+[t(R,h,k)*\oplus_R(h_0,k_0)-g_0]+[t(R,h,k)*\oplus_R(\phi,\psi)-\eta].\end{aligned}$$

Since the $\mathbb{R}$-action by translation is sc-smooth and the composition of two sc-smooth maps is again sc-smooth, we verify, by applying Proposition 2.17, that the first two maps on the right-hand side are sc-smooth. In order to see that the third term also defines an sc-map we write

$$t*\oplus_R(\phi,\psi)-\eta=t*[\oplus_R(\phi,\psi)-(-t)*\eta].$$

Thus it suffices to show that $(r,h,k)\mapsto\oplus_R(\phi,\psi)-(-t(R,h,k))*\eta$ is sc-smooth. We write

$$\begin{aligned}\oplus_R(\phi,\psi)(s)-(-t)*\eta(s)&=\tau_R(s)\cdot[\phi(s)-\eta(s-t)]\\&+(1-\tau_R(s))\cdot[\psi(s-R)-\eta(s-t)].\end{aligned}\tag{3.27}$$

Since the map $(r,h,k)\mapsto t(R,h,k)$ is bounded and R is close to R_0 and since $\phi(s)=\eta(s)=a$ for $s\leq-1$ and $\phi(s)=b$ and $\eta(s)=c$ for $s\geq1$, the first summand in (3.27) can be written as

$$\tau_R(s)\cdot[\phi(s)-\eta(s-t)]=\tau_R(s)\cdot\beta(s-C)\cdot[\phi(s)-\eta(s-t)].$$

The map δ defined by $x\mapsto\beta(\cdot-C)\cdot[\phi(\cdot)-\eta(\cdot-x)]$ from $\mathbb{R}$ into E is sc-smooth so that $(r,h,k)\mapsto\delta(t(R,h,k))$ is also sc-smooth as a composition of sc-smooth maps. With the sc-smooth map Γ defined by $\Gamma(r,g)=\tau_R\cdot g$, the composition

$$\begin{aligned}(r,h,k)\xrightarrow{(\mathrm{Id},t\circ(\varphi,\mathrm{Id}))}(r,t(R,h,k))&\xrightarrow{(\mathrm{Id},\delta)}(r,\delta(t(R,h,k))0\\&\xrightarrow{\Gamma}\Gamma(r,\delta(t(R,h,k)))\end{aligned}$$

is of class sc^∞. Thus the first summand on the right hand side in (3.27) is sc-smooth.

Due to the boundedness of $(r,h,k)\mapsto t(R,h,k)$ and in view of the properties of the functions ψ and η, and since we consider only R close to R_0, the second map in (3.27) can be represented as

$$\begin{aligned}&(1-\tau_R(s))\cdot[\psi(s-R)-\eta(s-t)]\\&=(1-\tau_R(s))\cdot\tau_R(s-C)\cdot(1-\beta(s+C))\cdot[\psi(s-R)-\eta(s-t)]\end{aligned}$$

with some positive constant C. The map ξ from $\mathbb{R}^2$ to E defined by $\xi(x,y)=(1-\beta(\cdot+C))\cdot[\psi(\cdot-x)-\eta(\cdot-y)$ is sc-smooth implying that $(r,h,k)\mapsto\xi(R,t(R,h,k))$ is of class sc^∞. The function $\gamma_1=(1-\beta)\cdot\beta(\cdot-C)$ has a compact support so that, in view of Proposition 2.17, the map $\Gamma_1(r,g):=\gamma_1(\cdot-\frac{R}{2})\cdot g$ is sc-smooth. Consquently, the composition

$$\begin{aligned}(r,h,k)\xrightarrow{(\mathrm{Id},t\circ(\varphi,\mathrm{Id}))}(r,t(R,h,k))\xrightarrow{(\mathrm{Id},\xi)}&(r,\xi(R,t(R,h,k)))\\&\xrightarrow{\Gamma_1}\Gamma_1(r,\xi(R,t(R,h,k)))\end{aligned}$$

is sc-smooth. Since this composition is precisely the second summand in (3.27), the proof of the lemma is complete. ■

Note that with $R=\varphi(r)$ the transition map $A_1^{-1}\circ A_2$ near (R_0,h_0,k_0) is precisely the map (3.26). Consequently, it is sc-smooth.

Continuing with the proof of Theorem 3.12 we study finally the transition map between two charts of type 2. We only need to study the transition at an element of the form $([u_0],[v_0])\in X(a,b)\times X(b,c)$. Indeed, if the element has the form $[w]$ with $w\in\widehat{X}(a,c)$, we can introduce an auxiliary chart of type 1 and represent the transition as the composition of a 2 to 1 with a 1 to 2 transition which we already know to be sc-smooth.

We consider two charts $\Phi:U\to O$ and $\Psi:U'\to O'$ of type 2 such that $U\cap U'\neq\emptyset$. The maps $A=\Phi^{-1}:O\to U$ and $B=\Psi^{-1}:O'\to U'$ are given by

$$A(r,h,k)=[\oplus_R(u+h,v+k)]\quad\text{and}\quad B(r',h',k')=[\oplus_{R'}(u'+h',v'+k')]$$

where $R=\varphi(r)$ and $R'=\varphi(r')$. We have to show that the composition $B^{-1}\circ A$ is sc-smooth. As already pointed out we only have to consider the special case in which $(0,h_0,k_0)$ is in the domain of the transition map, so that

$$A(0,h_0,k_0)=([u_0],[v_0])=B(0,h_0',k_0')$$

with $(u_0,v_0)\in\widehat{X}(a,b)\times\widehat{X}(b,c)$. Therefore,

$$u'+h_0'=t_0*(u+h_0) \quad \text{and} \quad v'+k_0'=t_1*(v+k_0) \tag{3.28}$$

for uniquely determined real numbers t_0 and t_1. Evaluating (3.28) at the parameter value $s=0$ we find

$$(u'+h_0')(0)=(u+h_0)(t_0)\in\Sigma_{u'} \quad \text{and} \quad (v'+k_0')(0)=(v+k_0)(t_1)\in\Sigma_{v'}.$$

Hence applying the implicit function theorem together with the Sobolev embedding theorem as in Lemma 1.30 we find two sc-smooth maps $h\mapsto t_0(h)$ and $k\mapsto t_1(k)$ defined for h,k close to 0 in E having the following properties. At $t_0(h)$ and $t_1(k)$ the curves $u+h$ and $v+k$ intersect $\Sigma_{u'}$ and $\Sigma_{v'}$,

$$(u+h)(t_0(h))\in\Sigma_{u'} \quad \text{and} \quad (v+k)(t_1(k))\in\Sigma_{v'}.$$

In addition, $t_0(h_0)=t_0$ and $t_1(k_0)=t_1$. For large R we define

$$R'=R'(r,h,k)=R+[t_1(k)-t_0(h)] \tag{3.29}$$

where $R=\varphi(r)$. Notice that $t_0(h)*\oplus_R(u+h,v+k)(0)=(u+h)(t_0(h))\in\Sigma_{u'}$ and $t_0(h)*\oplus_R(u+h,v+k)(R'(r,h,k))=(v+k)(t_1(k))\in\Sigma_{v'}$. We also define $r'(r,h,k)$ via $R'(r,h,k)=\varphi(r'(r,h,k))$ so that

$$r'(r,h,k)=\varphi^{-1}(R'(r,h,k))=\varphi^{-1}(\varphi(r)+[t_1(k)-t_0(h)]).$$

Since the map $(h,k)\mapsto t_1(k)-t_0(h)$ is sc-smooth, it follows from Lemma 3.17 below, that

$$(r,h,k)\mapsto r'(r,h,k)$$

is sc-smooth. Taking (h,k) close to (h_0,k_0) in $E\oplus E$ we now consider the system of equations for (h',k'),

$$\oplus_{R'}(h',k')=t_0(h)*\oplus_R(u+h,v+k)-\oplus_{R'}(u',v') \quad \text{and} \quad \ominus_{R'}(h',k')=0$$

which has the solution

$$\begin{bmatrix} h'(s)\\ k'(s-R')\end{bmatrix}=\frac{1}{\alpha'}\begin{bmatrix}\tau' & -(1-\tau')\\ (1-\tau') & \tau')\end{bmatrix}\cdot\begin{bmatrix} w\\ 0\end{bmatrix}$$

where we abbreviated $w=t_0(h)*\oplus_R(u+h,v+k)-\oplus_{R'}(u',v')$ and $\tau'=\beta(\cdot-\frac{R'}{2})$ and $\alpha'=(\tau')^2+(1-\tau')^2$.

The solutions $h'=h'(r,h,k)$ and $k'=k'(r,h,k)$ belong to $F_{u'}$ and $F_{v'}$, respectively. Indeed, evaluating h' at $t=0$ we get $h'(0)=(u+h)(t_0(h))\in\Sigma_{u'}$ and evaluating k' at $t=0$ we get $k'(0)=k'(R'-R')=t_0(h)*(v+k)(R'-R)=(v+k)(t_1(k))\in\Sigma_{v'}$. From $\ominus_{R'}(h',k')=0$ we conclude $(h',k')\in K^{\mathcal{S}}$. Moreover, if (h,k) is close to (h_0,k_0) in $E\oplus E$, then (h', k') is close to $(h'_0, k'_0) \in E \oplus E$. Summing up, if (h, k) belongs to O and is close to (h_0, k_0), then (h', g') belongs to O' and is close to (h'_0, k'_0). It remains to show that the transition map

$$(r, h, k) \mapsto (r'(r, h, k), h', k')$$

is sc-smooth. Since we already know that $(r, h, k) \mapsto r'(r, h, k)$ is sc-smooth, it suffices to show that $(r, h, k) \mapsto h'(r, h, k)$ and $(r, h, k) \mapsto k'(r, h, k)$ are sc-smooth. By symmetry we only have to consider the first map. The solution $h' = h'(r, h, k)$ is given by the formula

$$h' = \frac{\tau'}{\alpha'}\big[t_0(h) * \oplus_R(h, k) + t_0(h) * \oplus_R(u, v) - \oplus_{R'}(u', v')\big]. \tag{3.30}$$

The map h' is the sum of three summands and we start with the first summand, namely

$$(r, h, k) \mapsto \frac{\tau'}{\alpha'} \cdot \big[t_0(h) * \oplus_R(h, k)].$$

The $\mathbb{R}$-action is sc-smooth (Theorem 1.38) and $h \mapsto t_0(h)$ is sc-smooth (Lemma 1.30), so it suffices to show that the map

$$(r, h, k) \mapsto (-t_0(h)) * \left[\frac{\tau'}{\alpha'} \cdot \Big(t_0(h) * \oplus_R(h, k)\Big)\right]$$

is sc-smooth. Recalling the function γ from (3.23), the right hand side above can be written as

$$\gamma(s - t_0(h) - \tfrac{R'}{2}) \cdot \big[\beta(s - \tfrac{R}{2}) \cdot h(s) + (1 - \beta(s - \tfrac{R}{2})) \cdot k(s - R)\big]. \tag{3.31}$$

Define the map Γ_1 into E by

$$\Gamma_1(r, h) = \beta(\cdot - \tfrac{R}{2}) \cdot h. \tag{3.32}$$

Because β has a support contained in $(-\infty, 1]$ we may apply Theorem 2.17 to conclude that Γ_1 is an sc-smooth map. Applying Theorem 2.17 again, the map

$$\widehat{\Gamma}_1(r, g) = \gamma(\cdot - \tfrac{R}{2}) \cdot g \quad \text{for } g \in E \tag{3.33}$$

is also sc-smooth since γ has its support in $(-\infty, 1]$. The function σ defined by

$$\begin{aligned} \sigma(r, h, k) &= \varphi^{-1}(\varphi(r) + t_1(k) + t_0(h)) \\ &= \varphi^{-1}(R + t_1(k) + t_0(h)) = \varphi^{-1}(R' + 2t_0(h)) \end{aligned} \tag{3.34}$$

is sc-smooth by Lemma 3.17 so that the map

$$(r, h, k) \mapsto (\sigma(r, h, k), \Gamma_1(r, h))$$

is also sc-smooth. Consequently, the first part of the map in (3.31) is sc-smooth since it can be written as a composition of two sc-smooth maps, namely

$$(r,h,k)\mapsto \widehat{\Gamma}_1(\sigma(r,h,k),\Gamma_1(r,h)).$$

Next we turn to the second part of the map (3.31), namely to

$$(r,h,k)\mapsto \gamma(\cdot - t_0(h)-\tfrac{R'}{2})\cdot (1-\beta(\cdot-\tfrac{R}{2})\cdot k(\cdot - R). \tag{3.35}$$

Since γ has the support contained in $(-\infty,1]$ and since $t_0(h)+\frac{1}{2}R'=\frac{1}{2}[R+t_1(k)+t_0(h)]$ and since the maps $h\mapsto t_0(h)$ and $k\mapsto t_1(k)$ are bounded, the support of the function $s\mapsto \gamma(s-t_0(h)-\frac{R'}{2})$ is contained in $(-\infty,\frac{R}{2}+C+1]$ for some constant $C>1$. Because we also have that $\beta(s-\frac{R}{2}-2C)=1$ for all $s\leq \frac{R}{2}+2C$, the map in (3.35) can be rewritten as

$$s\mapsto \gamma(s-t_0(h)-\tfrac{R'}{2})\cdot\beta(s-\tfrac{R}{2}-2C)\cdot(1-\beta(s-\tfrac{R}{2})\cdot k(s-R). \tag{3.36}$$

Introduce the function $g(s):=\beta(s-2C)\cdot(1-\beta(s))$. It has compact support so that, in view of Theorem 2.17, the map

$$\Gamma_2:(r,k)\mapsto g(\cdot-\tfrac{R}{2})\cdot k(\cdot - R)$$

is sc-smooth. Recalling the sc-maps $(r,h,k)\mapsto\sigma(r,h,k)$ and $(r,h)\mapsto \widehat{\Gamma}_1(r,h)$ from (3.34) and (3.32) we see that

$$(r,h,k)\mapsto \widehat{\Gamma}_1(\sigma(r,h,k),\Gamma_2(r,k))$$

is sc-smooth as a composition of sc-smooth maps. But this map is precisely the one in (3.35). Consequently, the map in (3.31) is sc-smooth. Finally, we consider the two other summands in (3.30), namely,

$$(r,h,k)\mapsto \frac{\tau'}{\alpha'}\cdot\bigl[t_0(h)*\oplus_R(u,v)-\oplus_{R'}(u',v')\bigr]. \tag{3.37}$$

To see that this map is also sc-smooth we recall that u,v,u',v' are in E^∞ and are of the form $u=\phi+h_0$, $v=\psi+k_0$, $u'=\phi+h_0'$ and $v'=\psi+k_0'$ with the smooth path ϕ connecting a with b and the smooth path ψ connecting b with c. Both ϕ and ψ are constant outside of compact subsets of $\mathbb{R}$.

The previous discussion shows that

$$(r,h,k)\mapsto \frac{\tau'}{\alpha'}\cdot\bigl[t_0(h)*\oplus_R(h_0,k_0)-\oplus_{R'}(h_0',k_0')\bigr] \tag{3.38}$$

for fixed h_0, h_0', k_0, and k_0' is an sc- smooth map. Next we consider the map

$$(r,h,k) \mapsto \frac{\tau'}{\alpha'} \cdot \bigl[t_0(h) * \oplus_R(\phi,\psi) - \oplus_{R'}(\phi,\psi)\bigr]. \tag{3.39}$$

LEMMA 3.16. *The map* (3.39) *is sc-smooth.*

PROOF. The map is sc-smooth for r away from 0. So, we only consider the case when r is close 0, i.e, when R is large. Recalling that $\tau_R(s) = \beta(s - \frac{R}{2})$ and $\tau' = \tau_{R'}$ where $R' = R + [t_1(k) - t_0(h)]$, and abbreviating $t_0 = t_0(h)$ and $t_1 = t_1(k)$, we compute

$$\begin{aligned}
&[t_0 * \oplus_R(\phi,\psi) - \oplus_{R'}(\phi,\psi)](s)\\
&\quad = \tau_R(s+t_0)\cdot\phi(s+t_0) + (1-\tau_R(s+t_0))\cdot\psi(s+t_0-R)\\
&\qquad - \tau_{R'}(s)\phi(s) - (1-\tau_{R'}(s))\cdot\psi(s-R')\\
&\quad = \tau_R(s+t_0)\cdot[\phi(s+t_0)-\phi(s)]\\
&\qquad + (1-\tau_R(s+t_0))\cdot[\psi(s+t_0-R)-\psi(s-R')]\\
&\qquad + [\tau_R(s+t_0)-\tau_{R'}(s)]\cdot[\phi(s)-\psi(s-R')]
\end{aligned} \tag{3.40}$$

Since the maps $h \mapsto t_0(h)$ and $k \mapsto t_1(k)$ are bounded, the support of the function $s \mapsto \tau_R(s+t_0(h)) - \tau_{R'}(s) = \beta(s+t_0(h)-\frac{R}{2}) - \beta(s-\frac{1}{2}[R+t_1(k)-t_0(h)])$ is contained in the interval $[-\frac{R}{2}-C-1, \frac{R}{2}+C+1]$ for some positive constant C. If s belongs to this interval, then $\phi(s)-\psi(s-R') = 0$ because $\phi(s) = b$ for $s \geq 1$ and $\psi(s) = b$ for $s \leq -1$. Thus the last line of (3.40) is equal to 0 and the map under consideration is the sum of two maps, namely

$$\begin{aligned}
&\gamma_{R'}(s)\cdot[t_0 * \oplus_R(\phi,\psi)(s) - \oplus_R(\phi,\psi)(s)]\\
&\quad = \gamma_{R'}(s)\cdot\tau_R(s+t_0)\cdot[\phi(s+t_0)-\phi(s)]\\
&\qquad + \gamma_{R'}(s)\cdot(1-\tau_R(s+t_0))\cdot[\psi(s+t_0-R)-\psi(s-R)],
\end{aligned} \tag{3.41}$$

where we have set $\gamma_{R'}(s) = \gamma(s - \frac{R'}{2}) = \frac{\tau'(s)}{\alpha'(s)}$. We consider the first summand and shall represent it as a composition of sc-smooth maps. Define $\delta : E \to E$ by

$$\delta(h)(s) = \phi(s+t_0(h)) - \phi(s).$$

Then $\delta : E \to E$ is sc-smooth. In view of Lemma 3.17, the function

$$\sigma_1(r,h) := \varphi^{-1}(\varphi(r) - 2t_0(h)) \tag{3.42}$$

where $R=\varphi(r)$ is the gluing profile, is sc-smooth. Recalling the sc-map map Γ_1 defined in (3.32), the composition

$$\begin{aligned}\Gamma_1(\sigma_1(r,h),\delta(h))(s)&=\beta\big(s-\frac{\varphi(\sigma_1(r,h))}{2}\big)\cdot\delta(h)(s)\\&=\beta(s+t_0(h)-\tfrac{R}{2})\cdot[\phi(s+t_0(h))-\phi(s)]\end{aligned}\tag{3.43}$$

is sc-smooth. With the function

$$\begin{aligned}\sigma_2(r,h,k)&=\varphi^{-1}(\varphi(r)+t_1(k)-t_0(h))\\&=\varphi^{-1}(R+t_1(k)-t_0(h))=\varphi^{-1}(R')\end{aligned}\tag{3.44}$$

which is sc-smooth by Lemma 3.17 and with the map $\widehat{\Gamma}_1$ defined in (3.33), the composition

$$\begin{aligned}&\widehat{\Gamma}_1(\sigma_2(r,h,k),\Gamma_1(\sigma_1(r,h),\delta(h)))(s)\\&\quad=\gamma\big(s-\varphi(\sigma_2(r,h,k))\big)\cdot\Gamma_1(\sigma_1(r,h),\delta(h))(s)\\&\quad=\gamma(s-\tfrac{R'}{2})\cdot\Gamma_1(\sigma_1(r,h),\delta(h))(s)\\&\quad=\gamma_{R'}(s)\cdot\tau_R(s+t_0(h))\cdot[\phi(s+t_0(h))-\phi(s)].\end{aligned}\tag{3.45}$$

is also sc-smooth. This composition is precisely the first summand in (3.41).

To see that the second summand in (3.41) also defines an sc-smooth map we define $\xi:E\oplus E\to E$ by

$$\xi(h,k)(s)=\psi(s-t_0(h))-\psi(s-t_0(h)-t_1(k)).$$

The map ξ is sc-smooth. The support of γ is contained in the interval $(-\infty,1]$. Since $h\mapsto t_0(h)$ and $k\mapsto t_1(k)$ are bounded, $\gamma_{R'}(s)=\gamma(s-\frac{R'}{2})=\gamma(s-\frac{1}{2}[R+t_1(k)-t_0(h)])$ has a support contained in the interval $(-\infty,\frac{R}{2}+C+1]$ for some constant $C>1$. The function $\tau_R(s+t_0(h)-3C)=\beta(s+t_0(h)-\frac{R}{2}-3C)$ is equal to 1 for all $s\leq\frac{R}{2}+3C$. Thus setting $g(s)=(1-\beta(s))\cdot\beta(s-3C)$ so that $g(s+t_0(h)-\frac{R}{2})=(1-\tau_R(s+t_0(h))\cdot\tau_R(s+t_0(h))$ we obtain

$$\begin{aligned}&\gamma_{R'}(s)\cdot(1-\tau_R(s+t_0(h)))\cdot[\psi(s+t_0(h)-R)-\psi(s-R')]\\&=\gamma_{R'}(s)\cdot g(s+t_0(h)-\tfrac{R}{2})\cdot[\psi(s+t_0(h)-R)-\psi(s-R')].\end{aligned}$$

Because the function $g(s)=(1-\beta(s))\cdot\beta(s-3C)$ has compact support, the map

$$\Gamma_3(r,k)=g(\cdot-\tfrac{R}{2})\cdot k(\cdot-R)$$

is sc-smooth. With the sc-smooth function σ_1 from (3.42),

$$\begin{aligned}\xi(h,k)(s-\varphi(\sigma_1(h,k))) &= \psi(s+t_0(h)-R)-\psi(s+t_0(h)-t_1(k))\\ &= \psi(s+t_0(h)-R)-\psi(s-R'),\end{aligned}$$

recalling $R' = R + t_1(k) - t_0(h)$. So, the composition

$$\begin{aligned}&\Gamma_3(\sigma_1(r,h,k),\xi(h,k))(s)\\ &\qquad= g(s-\tfrac{1}{2}\varphi(\sigma_1(r,h,k)))\cdot\xi(h,k)(s-\varphi(\sigma_1(r,h,k))\\ &\qquad= g(s+t_0(h)-\tfrac{R}{2})\cdot[\psi(s+t_0(h)-R)-\psi(s-R')]\end{aligned}$$

is sc-smooth. Recalling the map $\widehat{\Gamma}_1$ from (3.33) and σ_2 from (3.44), the second summand in (3.41) is precisely the composition

$$\widehat{\Gamma}_1(\sigma_2(r,h,k),\Gamma_3(\sigma_1(r,h),\xi(h,k)))$$

and so it is sc-smooth. The proof of the lemma is complete. ∎

Since the map in (3.37) is the sum of the sc-smooth maps (3.38) and (3.39), the map (3.37) is sc-smooth. We have proved that the transition map between two charts of type 2 is sc-smooth. Up to Lemma 3.17 this completes the proof of Theorem 3.12. ∎

It remains to prove the following calculus lemma used in Theorem 3.12.

LEMMA 3.17. *Consider the gluing profile*

$$\varphi(x) = e^{\frac{1}{x}} - e, \quad x \in (0,1].$$

Define the function $B : [0,r') \times [-C,C] \to \mathbb{R}$ *for some* $0 < r' < 1$ *small by*

$$B(x,c) = \begin{cases} \varphi^{-1}[\varphi(x)+c] & \text{if } x \in (0,r') \\ 0 & \text{if } x = 0. \end{cases}$$

Then B *is smooth and*

$$\begin{aligned} D_x B(0,c) &= 1 \\ D^\alpha B(0,c) &= 0 \end{aligned}$$

for all multi-indices $\alpha = (\alpha_1, \alpha_2)$ *with* $\alpha_1 \geq 2$ *and* $\alpha_2 \geq 0$.

PROOF. The function $\varphi : (0,1] \to [0,\infty)$ is a diffeomorphism. Its inverse $\varphi^{-1} : [0,\infty) \to (0,1]$ is the function

$$\varphi^{-1}(y) = \frac{1}{\ln[e+y]}.$$

With B as defined above,

$$B(x,c) = \frac{1}{\ln[e^{1/x} + c]} \quad \text{if } x > 0.$$

To prove our claim we have to show that

$$\begin{gathered} B(x,c) \to 0 \\ D_x B(x,c) \to 1 \quad \text{and} \quad D^{n,m} B(x,c) \to 0 \end{gathered} \tag{3.46}$$

as $x \to 0$ uniformly in c, for all $n \geq 2$. Writing

$$\begin{aligned} \ln[e^{1/x} + c] &= \ln\big[e^{1/x} \cdot \big(1 + c \cdot e^{-1/x}\big)\big] \\ &= \frac{1}{x} + \ln\big[1 + c \cdot e^{-1/x}\big], \end{aligned}$$

we obtain

$$B(x,c) = x \cdot \frac{1}{1 + x \ln\big[1 + c \cdot e^{-1/x}\big]} = x \cdot f(x,c)$$

where

$$f(x,c) = \frac{1}{1 + x \ln\big[1 + c \cdot e^{-1/x}\big]}.$$

Clearly, $f(x,c) \to 1$ as $x \to 0$, uniformly in c. Defining the function g by

$$f(x,c) = \frac{1}{1 + g(x,c)},$$

we see that it suffices to show that $D^\alpha g(x,c) \to 0$ for $|\alpha| \geq 1$ uniformly in c as $x \to 0$. Explicitly, $g(x,c) = x \ln[1 + ce^{-1/x}]$. In order to prove the required properties of g, it suffices to show that the function h defined by

$$h(x,c) = ce^{-1/x}$$

satisfies $D^\alpha h(x,c) \to 0$, uniformly in c, as $x \to 0$. This latter assertion, however, is trivial.

In order to prove the second assertion for B, observe that a derivative of order n of $e^{1/x}$ is a product of $e^{1/x}$ with a polynomial in the variable $1/x$. From this we deduce the desired conclusion. The proof of Lemma 3.17 is complete. ∎

3.3. The level-k curves as an M-Polyfold

To be revised later on

So far we have considered level-1 curves connecting two different points and broken level-2 connecting three different points. We shall now generalize our previous construction to arbitrary level-k curves. We do this first in $\mathbb{R}^n$ since in case of a manifold M dealt with later on the gluing constructions are carried out in local coordinate charts in which the same formulas are used.

Assume that we are given a countable collection $\mathcal{P}$ of distinct points in $\mathbb{R}^n$ so that very compact subset of $\mathbb{R}^n$ contains only finitely many of them. For every point $a \in \mathcal{P}$ we choose an increasing sequence of weights δ_m^a, $m \geq 0$, starting at $m = 0$ with $\delta_0^a = 0$. We denote by $X(a,b)$ for $a, b \in \mathcal{P}$ the collection of equivalence classes $[u]$ of H^2-curves $u : \mathbb{R} \to \mathbb{R}^n$ connecting the point a at $-\infty$ with the point $b \in \mathbb{R}^n$ at ∞. The level m consists of the equivalence classes $[u]$ which are represented by elements u in H^{2+m}_{loc} and which are of class $(m+2, \delta_m^a)$ near $-\infty$ and of class $(m+2, \delta_m^b)$ near ∞. For a finite sequence $\widehat{a} = (a_0, a_1, \ldots, a_k)$ of mutually distinct points $a_j \in \mathcal{P}$ we introduce the space

$$X(\widehat{a}) = X(a_0, a_1) \times X(a_1, a_2) \times \cdots \times X(a_{k-1}, a_k)$$

consisting of level-k curves connecting a_0 with a_k.

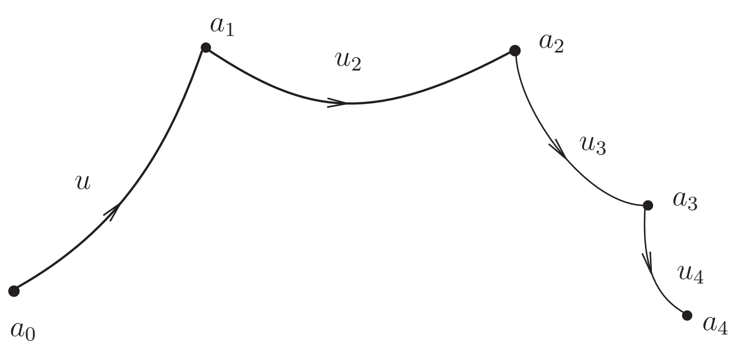


Figure 3.7. Level-4 curve

Let $\widehat{\mathcal{P}}$ be the collection of all such finite sequences of lenght $k \geq 1$. Take the disjoint union over $k \geq 1$,

$$X = \coprod_{\widehat{a} \in \widehat{\mathcal{P}}} X(\widehat{a}).$$

For our general gluing procedure it is helpful to first reconsider the procedure for level-2 curves again.

We choose representatives $u : L \to \mathbb{R}^n$ and $v : N \to \mathbb{R}^m$ of curves connecting a with b and b with c, where $L = \mathbb{R}$ and $N = \mathbb{R}$. For each line we have chosen two half lines, a negative one and a positive one. So far our choice was always $(-\infty, 0]$ and $[0, \infty)$. But we can make

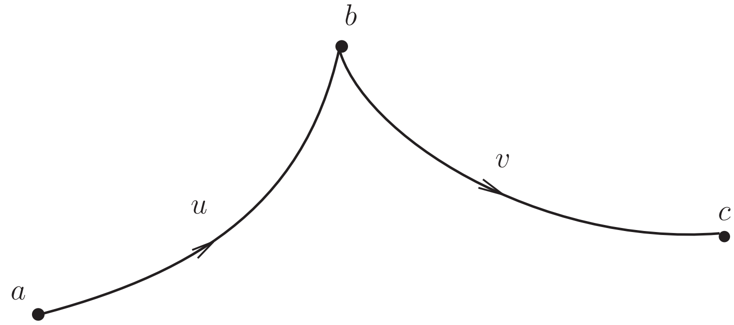


FIGURE 3.8
$([u],[v]) \in X(a,b) \times X(b,c)$

different choices and define the negative resp. positive regions of L and N by pairs of real numbers $s_0 \leq t_0$ and $s_1 \leq t_1$ as follows.

- $-$ region on L: $(-\infty, s_0]$
- $+$ region on L: $[t_0, \infty)$
- $-$ region on N: $(-\infty, s_1]$
- $+$ region on N: $[t_1, \infty)$.

In order to reduce the gluing later to the old gluing corresponding to the choice $s_0 = t_0 = 0$ and $s_1 = t_1 = 0$ we introduce the coordinates

$$\begin{aligned} &\varphi : [0,\infty) \to [t_0,\infty), && s \mapsto s + t_0 \\ &\psi : (-\infty, 0] \to (-\infty, s_1], && s' \mapsto s' + s_1 \end{aligned} \tag{3.47}$$

for the + region of L and the $-$region of N. If R is the gluing parameter we define the gluing line

$$L \oplus_R N = \big((-\infty, t_0 + R] \coprod [s_1 - R, \infty)\big)/_{\varphi(s)\sim\psi(s')}.$$

with the equivalence relation $\varphi(s) \sim \psi(s')$ if $s \in [0,R]$ and $-R+s = s'$. See Figure 3.9.

Hence $t_0 + s \sim s_1 + s'$ if and only if $s \in [0,R]$ and $-R + s = s'$. The manifold $L \oplus_R N$ is diffeomorphic to $\mathbb{R}$ and we choose the special global coordinates

$$\mathbb{R} \to L \oplus_R N, \quad s \mapsto [s]$$

induced by

$$s \mapsto \begin{cases} s \in (-\infty, t_0 + R] & s \leq t_0 + R \\ s_1 - R + s - t_0 \in [s_1 - R, \infty) & s \geq t_0. \end{cases}$$

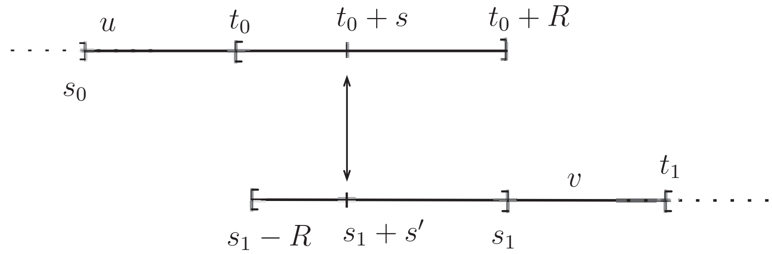


FIGURE 3.9

Define the glued curve connecting a with c by

$$(u\oplus_R v)([s]) = \beta(s-t_0-\tfrac{R}{2})\cdot u(s)+(1-\beta(s-t_0-\tfrac{R}{2}))\cdot v(s_1-R+s-t_0)$$

for all $s\in\mathbb{R}$, where the cut-off function β is defined in (2.6). Identifying $L\oplus_R N$ with $\mathbb{R}$ is such a way that $L\subset\mathbb{R}$ is the inclusion, the glued curve is, in the global coordinates, given by the formula,

$$\begin{aligned}(u\oplus_R v)(s) &= \beta(s-t_0-\tfrac{R}{2})\cdot u(s)\\ &\quad + (1-\beta(s-t_0-\tfrac{R}{2}))\cdot v(s_1-R+s-t_0)\end{aligned} \tag{3.48}$$

for all $s\in\mathbb{R}$. The negative and positive regions of the glued curve $u\oplus_R v$ become

- $-$ region of $u\oplus_R v$: $(-\infty, s_0]$
- $+$ region $u\oplus_R v$: $[t_0+R+t_1-s_1,\infty)$

From the properties of the cuf-off function β read off,

$$(u\oplus_R v)(s) = \begin{cases} u(s) & s\geq t_0+\frac{R}{2}{}_1\\ v(s_1-R+s-t_0) & s\geq t_0+\frac{R}{2}+1.\end{cases}$$

Denote our old gluing defined in (3.1) by $\oplus_R^0$. It is based on the negative regions $(-\infty,0]$ and the positive regions $[0,\infty)$. Introducing the elements in E^+ and E^-

$$\begin{aligned} u\circ\varphi &: [0,\infty)\to\mathbb{R}^n\\ v\circ\psi &: (-\infty,0]\to\mathbb{R}^n\end{aligned}$$

we observe that

$$(u\oplus_R v)(\varphi(s)) = \oplus_R^0(u\circ\varphi, v\circ\psi)(s) \tag{3.49}$$

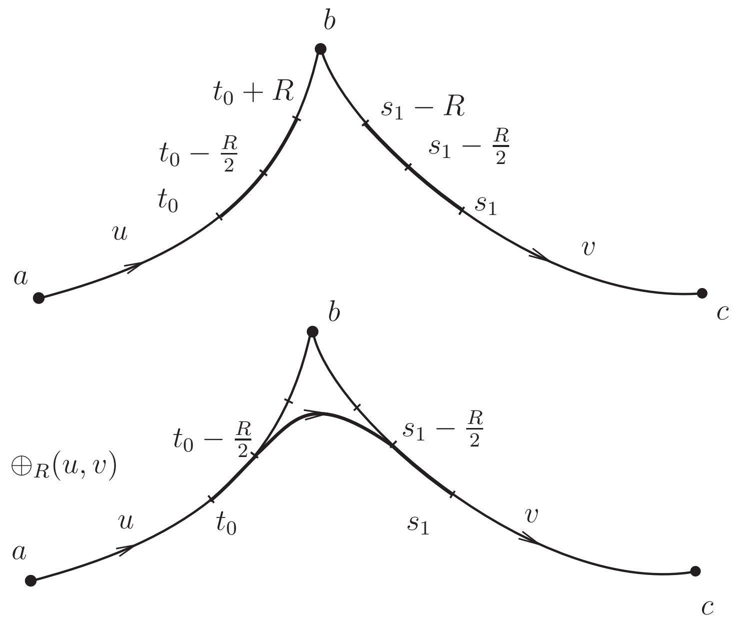


FIGURE 3.10

as the following argument shows,

$$\begin{aligned}
&\oplus_R^0(u \circ \varphi, v \circ \psi)(s) \\
&\qquad = \beta(s - \tfrac{R}{2}) \cdot u \circ \varphi(s) + (1 - \beta(s - \tfrac{R}{2})) \cdot v \circ \psi(s - R) \\
&\qquad = \beta(s - \tfrac{R}{2}) \cdot u(s + t_0) + (1 - \beta(s - \tfrac{R}{2})) \cdot v(s + s_1 - R) \\
&\qquad = (u \oplus_R v)(\varphi(s)).
\end{aligned}$$

Next we consider a level-3 curve.

The negative and postive regions of u, v and w are characterized by the pairs $s_0 \leq t_0$, $s_1 \leq t_1$ and $s_2 \leq t_2$. Recall that the negative and positive regions of the glued curve $u \oplus_R v$ connecting a with c is characterized by the pair $s_0 \leq \tau$, where

$$\tau = t_0 + R + t_1 - s_1. \tag{3.50}$$

LEMMA 3.18. *For gluing parameters $R, R' \geq 2$*

$$(u \oplus_R v) \oplus_{R'} w = u \oplus_R (v \oplus_{R'} w).$$

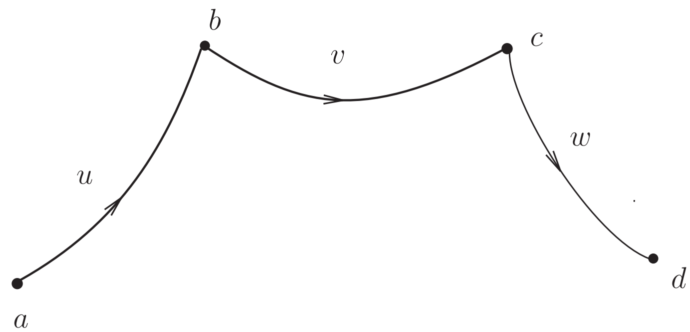


FIGURE 3.11

PROOF. We first verify the associativity of the gluing by a computation and give afterwards argument which such computations superfluous. Recall that the cut-off function β satisfies $\beta(s) = 0$ for $s \geq 1$ and $\beta(s) = 1$ for $s \leq -1$. Therefore, if R and $R' \geq 2$,

$$\beta(s - \tau - \tfrac{R'}{2}) \cdot \beta(s - t_0 - \tfrac{R}{2}) = \beta(s - t_0 - \tfrac{R}{2})$$

and hence

$$(1 - \beta(s - t_0 - \tfrac{R}{2})) \cdot (1 - \beta(s - \tau - \tfrac{R'}{2})) = 1 - \beta(s - \tau - \tfrac{R'}{2}),$$

where $\tau = t_0 + R + t_1 - s_1$. Using this we compute applying the gluing formula (3.48) twice,

$$\begin{aligned}
&\big((u \oplus_R v) \oplus_{R'} w\big)(s) \\
&= \beta(s - \tau - \tfrac{R'}{2})(u \oplus_R v)(s) + (1 - \beta(s - \tau - \tfrac{R'}{2})) \cdot w(s + s_2 - \tau - R') \\
&= \beta(s - \tau - \tfrac{R'}{2}) \cdot \beta(s - t_0 - \tfrac{R}{2}) \cdot u(s) \\
&\quad + \beta(s - \tau - \tfrac{R'}{2})) \cdot (1 - \beta(s - t_0 - \tfrac{R}{2})) \cdot v(s + s_1 - t_0 - R) \\
&\quad + (1 - \beta(s - \tau - \tfrac{R'}{2})) \cdot w(s + s_2 - \tau - R') \\
&= \beta(s - t_0 - \tfrac{R}{2}) \cdot u(s) + (1 - \beta(s - t_0 - \tfrac{R}{2})) \cdot \big[\beta(s - \tau - \tfrac{R'}{2}) \cdot v(s + s_1 - t_0 - R) \\
&\quad +(1 - \beta(s - \tau - \tfrac{R'}{2})) \cdot w(s + s_2 - \tau - R')\big] \\
&= \beta(s - t_0 - \tfrac{R}{2})u(s) + (1 - \beta(s - t_0 - \tfrac{R}{2})) \cdot (v \oplus_R w)(s + s_1 - t_0 - R) \\
&= \big(u \oplus_R (v \oplus_{R'} w)\big)(s).
\end{aligned}$$

■

Actually, the associativity of the gluing follows without any computations immediately from the following schematics procedure

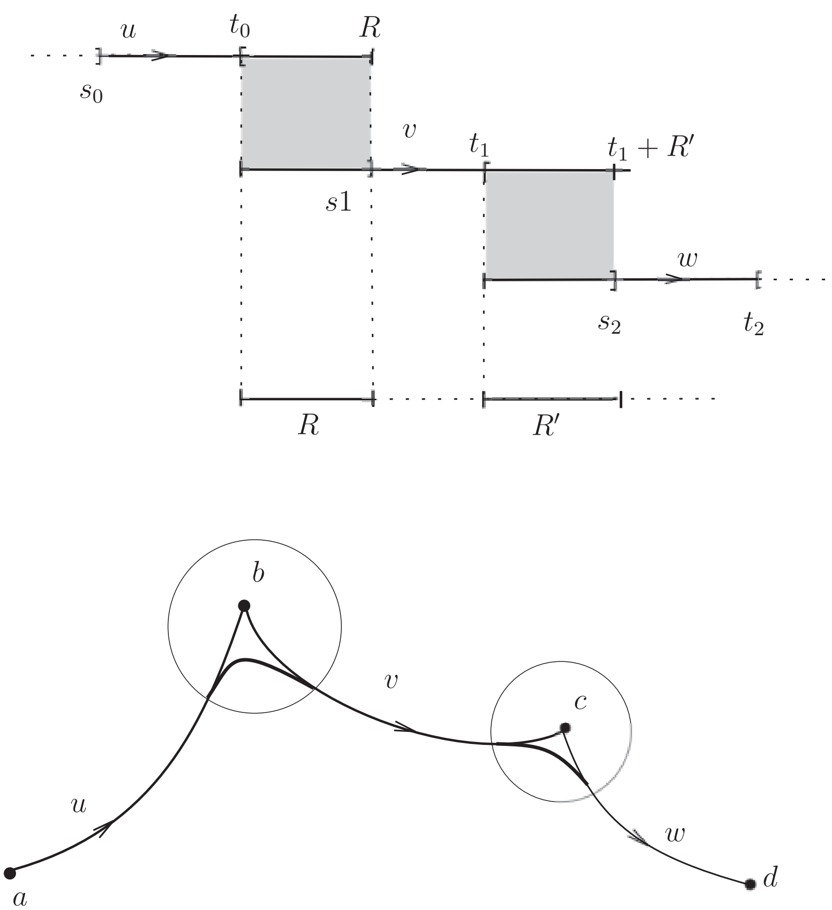


FIGURE 3.12

Using our earlier notation and replacing the gluing parameter R by r related by the gluing profile

$$R = e^{\frac{1}{r}} - e$$

we shall in the following often write $u \oplus_r v$ instead of $u \oplus_R v$ and introduce the notation

$$\oplus_{(r_1,r_2)}(u, v, w) = (u \oplus_{r_1} v) \oplus_{r_2} w = u \oplus_{r_2} (v \oplus_{r_2} w).$$

Consequently, for a level-k curve

$$([u_1], \dots, [u_k]) \in X(a_0, a_1) \times \cdots \times X(a_{k-1}, a_k)$$

we define the successively by

(3.51)

$$\oplus_{(r_1,\ldots,r_k)}(u_1,\ldots,u_k) = \big(\ldots\big(((u_1 \oplus_{r_1} u)2) \oplus_{r_2} u_3) \oplus_{r_3} u_4) \ldots\big) \oplus_{r_k} u_k.$$

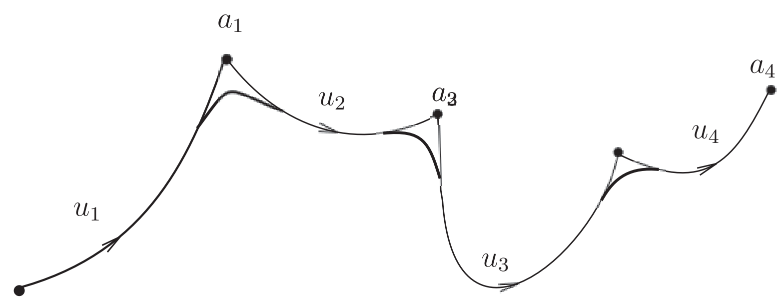


FIGURE 3.13

In view of the above scheme the gluing is obviously associative and the following general formula holds

$$\begin{aligned}&\oplus_{(r_1,\ldots,r_k)}(u_1,\ldots,u_k)\\ &\qquad = \big[\oplus_{(r_1,\ldots,r_{i-1})}(u_1,\ldots,u_i)\big] \oplus_{r_i} \big[\oplus_{(r_{i+1},\ldots,r_{k-1})}(u_i,\ldots,u_k)\big].\end{aligned}$$

In order to define the gluing of vector fields along curves we take $u \in \widehat{X}(a,b)$ and $v \in \widehat{X}(b,c)$ whose $\pm$ regions are characterized by the pairs $s_0 \leq t_0$ and $s_1 \leq t_1$. Take a vector field h along u and a vector field k along v,

$$h(s) \in T_{u(s)}\mathbb{R}^n - \mathbb{R}^n$$
$$k(s') \in T_{v(s)}\mathbb{R}^n = \mathbb{R}^n$$

for all $s \in \mathbb{R}$ and $s' \in \mathbb{R}$. In order to define the gluing and anti-gluing near the corner b we take the coordinates φ and ψ in (3.47), so that $u \circ \varphi : \mathbb{R}^+ \to \mathbb{R}^n$ and $v \circ \psi : \mathbb{R}^- \to \mathbb{R}^n$, and consider the vector fields

$$h^+(s) = h \circ \varphi(s) \in T_{u\circ\varphi(s)} = \mathbb{R}^n$$
$$k^-(s') = k \circ \psi(s') \in T_{v\circ\psi(s')} = \mathbb{R}^n$$
$$.$$

Denoting by $\oplus^0_R(h^+,k^-)$ the old gluing formula from Section 3.3 we define the gluing of vector fields h and k by

$$(h \oplus_R k)(\varphi(s)) := \oplus^0_R(h \circ \varphi, k \circ \psi)(s)$$

which leads to the formula

$$(h \oplus_R k)(\varphi(s)) := \beta(s - t_0 - \tfrac{R}{2}) \cdot h(s) \\ + (1 - \beta(s - t_0 - \tfrac{R}{2}) \cdot k(s_1 - R + s - t_0) \tag{3.52}$$

for all $s \in \mathbb{R}$. The gluing intervals are chosen as in Figure 3.14

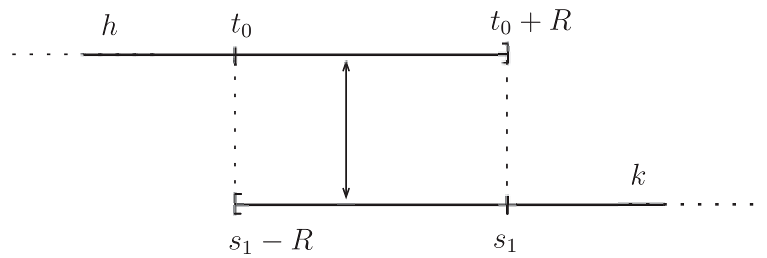


FIGURE 3.14

From the properties of the cut-off function β one reads off,

$$(h \oplus_R k)(\varphi(s)) := \begin{cases} h(s) & s \leq t_0 + \frac{R}{2} - 1 \\ k(s_1 - R + s - t_0) & s \geq t_0 + \frac{R}{2} + 1. \end{cases}$$

After the gluing, the information about $h(s)$ for $s \geq t_0 + \frac{R}{2} - 1$ and $k(s')$ for $s' \leq s_1 - \frac{R}{2} - 1$ has disappeared. It will be recovered by the **anti-gluing formula** for vector fields h and k, defined by

$$\ominus_R(h, k)(\varphi(s)) = \oplus_R^0(h \circ \varphi, k \circ \psi)(s)$$

for all $s \in \mathbb{R}$, which leads to the formula

$$\ominus_R(h, k)(s) = -(1 - \beta(s - t_0 - \tfrac{R}{2}) \cdot h(s) \\ + \beta(s - t_0 - \tfrac{R}{2}) \cdot k(s_1 + R + s - t_0) \tag{3.53}$$

for all $s \in \mathbb{R}$. This time we have glued as in Figure 3.15.

From (3.53) we read off

$$(h \ominus_R k)(\varphi(s)) := \begin{cases} k(s_1 - R + s - t_0) & s \leq t_0 + \frac{R}{2} - 1 \\ -h(s) & s \geq t_0 + \frac{R}{2} + 1. \end{cases}$$

Hence the anti-gluing contains the informations about the vector fields outside of the gluing intervals towards the corner b. In coordinates in which $b = 0$, the situation is illustrated by Figure 3.16.

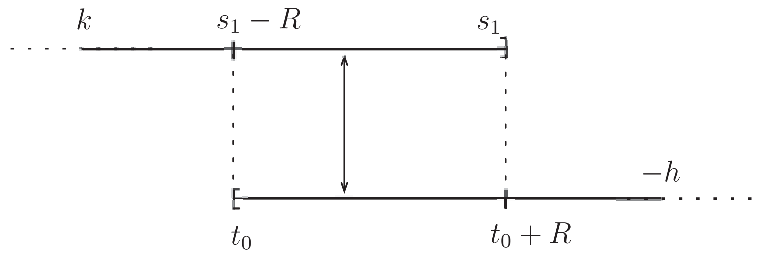


FIGURE 3.15

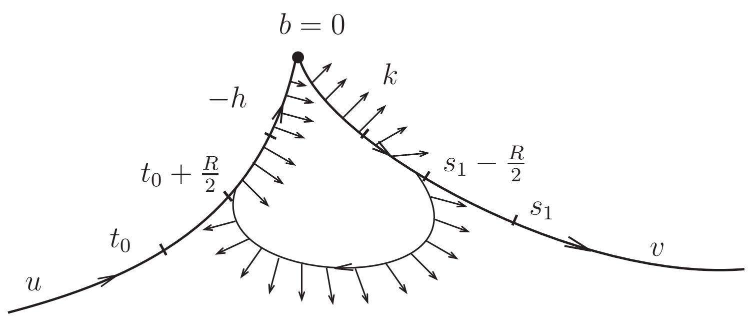


FIGURE 3.16

Denoting by exp the exponential map associated with the Euclidean matric on $\mathbb{R}^n$ we can define curves near u and b by

$$\begin{aligned}\exp_u(h)(s) &= \exp_{u(s)}(h(s)) = u(s) + h(s)\\ \exp_v(k)(s') &= \exp_{v(s')}(k(s')) = v(s') + k(s')\end{aligned}$$

for all $s \in \mathbb{R}$ and $s' \in \mathbb{R}$. From the gluing formul (3.48) fro curves and (3.53) for vector fields we read off the following results.

LEMMA 3.19. *If h and k are sections along the curves $u \in \widehat{X}(a,b)$ and $v \in \widehat{X}(b,c)$, then*

$$\exp_u(h) \oplus_R \exp_v(k) = \exp_{u \oplus_R v}(h \oplus_R k).$$

The effect of the anti-gluing is illustrated by the next statement.

PROPOSITION 3.20. *Let $(u,v) \in \widehat{X}(a,b) \times \widehat{X}(b,c)$. If $w \in \widehat{X}(a,c)$ is sufficiently close to $u \oplus_R v$, then there exist unique vector fields h along u and k along v, so that*

$$\begin{aligned}\exp_{\oplus_r(u,v)}(\oplus_r(h,k))(s) &= w(s)\\ \oplus_r(h,k)(s) &= 0\end{aligned}$$

The two vector fields h and k are solutions of the two equations,

$$\begin{aligned}\oplus_R(u,v)(s) + \oplus_R(h,k)(s) &= w(s)\\ \ominus_R(h,k)(s) &= 0\end{aligned}$$

Consequently, by the gluing formulas (3.48), (3.52), and (3.53), abbreviating $\tau(s) = \beta(s - t_0 - \frac{R}{2})$ and $\alpha = \tau^2 + (1-\tau)^2$,

$$\begin{bmatrix} \tau & 1-\tau \\ \tau - 1 & \tau \end{bmatrix} \cdot \begin{bmatrix} h(s) \\ k(s_1 - R + s - t_0) \end{bmatrix} = \begin{bmatrix} w(s) - \oplus_R(u,v)(s) \\ 0 \end{bmatrix}$$

The matrix is invertible. Hence

$$\begin{bmatrix} h(s) \\ k(s_1 - R + s - t_0) \end{bmatrix} = \tfrac{R}{\alpha} \cdot \begin{bmatrix} \tau & \tau - 1 \\ 1-\tau & \tau \end{bmatrix} \cdot \begin{bmatrix} w(s) - \oplus_R(u,v)(s) \\ 0 \end{bmatrix},$$

so that the required solution is equal to

$$\begin{aligned}h(s) &= \tfrac{\tau}{\alpha} \cdot [w(s) - \oplus_R(u,v)(s)]\\ k(s_1 - R + s - t_0) &= \tfrac{1-\tau}{\alpha} \cdot [w(s) - \oplus_R(u,v)(s)].\end{aligned}$$

PROOF. We observe that the vector fields h along u and v along v satisfy, in particular,

$$h(s) = \begin{cases} w(s) - \oplus_R(u,v)(s) & s \le t_0 + \frac{R}{2} - 1 \\ 0 & s \ge t_0 + \frac{R}{2} + 1 \end{cases}$$

$$k(s') = \begin{cases} 0 & s' \le -\frac{R}{2} - 1 \\ w(s') - \oplus_R(u,v)(s') & s' \ge -\frac{R}{2} + 1 \end{cases}$$

Using the gluing construction one defines as in Section 3.1 a second countable topology on the set X which induces the original topology on its corresponding parts.

In order to define a polyfold structure on the space X we look at a smooth representative

$$(u_1, \ldots, u_k) \in \widehat{X}(a_0, a_1) \times \cdots \times \widehat{X}(a_{k-1}, a_k)$$

of a level-k curve connecting a_0 with a_k. For every curve $u_j \in \widehat{X}(a_{j-1}, a_j)$ we fix a distinguished negative region $(-\infty, s_j]$ and the positive region $[t_j, \infty)$ so that $u_j(s_j)$ is close to a_{j-1} and $u_j(t_j)$ is close to a_j. Since

$a_{j-1} \neq a_j$ we find $\tau_j \in (s_j, t_j)$ such that the tangent vector $\dot{u}_j(\tau_j)$ does not vanish. Following the Recipe 1.37 we now find an sc-Banach space F_{u_j} of vector fields along the smooth curve u_j and an open neighborhood $\mathcal{O}_{u_j} \subset F_{u_j}$ of the origin, such that the map

$$\begin{aligned} A_{u_j} : &\mathcal{O}_{u_j} \to X(a_{j-1}, a_j) \\ &h \mapsto [\exp_{u_j}(h)] \end{aligned}$$

defines a homeomorphism onto some open neighborhood $\mathcal{U}_j$ of $[u_j]$ in the quotient space $X(a_{j-1}, a_j)$. Next we shall define the sc-smooth splicing projection π_r on the sc-smooth Banach space

$$E = F_{u_1} \oplus F_{u_2} \oplus \cdots \oplus F_{u_k} \tag{3.54}$$

for the gluing parameter $r = (r_1, \ldots, r_{k-1}) \in [0,1)^{k-1}$. For simplicity we look at a level-3 curve again.

■

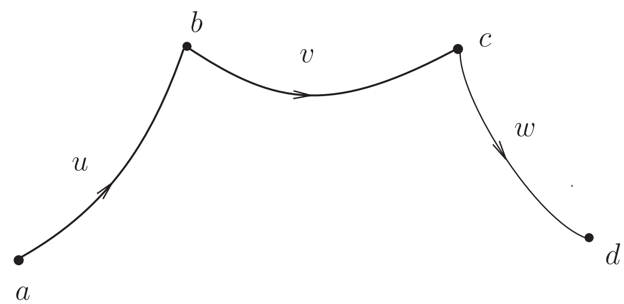


FIGURE 3.17

Associated with u, v and w we have the positive regions defined by the pairs $s_0 \leq t_0$, $s_1 \leq t_1$ and $s_2 \leq t_2$. We first focus in a neighborhood of the point b. If h is vector field along h and k a vector field along v we consider, in coordinates φ and ψ from (3.47), the vector fields

$$\begin{aligned} h \circ \varphi &: [0, \infty) \to \mathbb{R}^n \\ k \circ \psi &: (-\infty, 0] \to \mathbb{R}^n. \end{aligned}$$

In the notation of Section 2.2 we have $h \circ \varphi \in E^+$ and $k \circ \psi \in E^-$. We denote the old splicing from (2.8) by

$$\pi_r^0(h \circ \varphi, k \circ \psi) = (\widehat{h \circ \varphi}, \widehat{k \circ \psi}).$$

Then the splicing for the two vector fields h and k along the curves u and v are the two vector fields $\widehat{h}$ and $\widehat{k}$ along u and v defined by

$$\pi_r(h,k) = (\widehat{h}, \widehat{k}) \tag{3.55}$$

where $(\widehat{h}, \widehat{k})$ is given by

$$(\widehat{h}\circ\varphi, \widehat{k}\circ\psi) = \pi_r^0(h\circ\varphi, k\circ\psi).$$

Therefore, explicitly, with β denoting the cut-off function and $\alpha = \beta^2 + (1-\beta)^2$,

$$\begin{aligned}
\widehat{h}(s) &= \frac{\beta(s-t_0-\frac{R}{2})}{\alpha(s-t_0-\frac{R}{2})}\cdot\bigl[\beta(s-t_0-\tfrac{R}{2})\cdot h(s)\\
&\qquad + (1-\beta(s-t_0-\tfrac{R}{2}))\cdot k(s-R+s_1-t_2)\bigr]\\
\widehat{k}('s) &= \frac{\beta(-s'+s-\frac{R}{2})}{\alpha(-s'+s-\frac{R}{2})}\cdot\bigl[(1-\beta(-s'+s-\tfrac{R}{2}))\cdot h(s'+s_1-\tfrac{R}{2})\\
&\qquad + \beta(-s'+s-\tfrac{R}{2}\cdot k(s')\bigr]
\end{aligned}$$

for all $s, s' \in \mathbb{R}$.

In view of the properties of the cuf-off function β we obtain, in particular,

$$\widehat{h}(s) = \begin{cases} h(s) & s \le t_0 + \frac{R}{2} - 1\\ 0 & s \ge t_0 + \frac{R}{2} + 1\end{cases}$$

$$\widehat{k}(s') = \begin{cases} 0 & s' \le s_0 - \frac{R}{2} - 1\\ k(s') & s' \ge s_1 - \frac{R}{2} + 1\end{cases}$$

We see that outside of the gluing intervals the vector fields vanish towards the corner b but remain unchanged towards the other side. This defines $\widehat{h}$ completely. For k, however, there is second condition comming from the other end of the curve v, namely, form the point c. Denoting by p a vector field along the curve w, we apply the definition (3.55) to the vector fields k and p along the curves v and w and obtain

$$\pi_{r'}(k,p) = (\widehat{k}, \widehat{p}), \tag{3.56}$$

where

$$\begin{aligned}
\widehat{k}(s') &= \frac{\beta(s'-t_1-\frac{R'}{2})}{\alpha(s'-t_1-\frac{R'}{2})}\cdot\bigl[\beta(s'-t_1-\tfrac{R'}{2})\cdot k(s')\\
&\qquad + (1-\beta s'-t_1-\tfrac{R'}{2})\cdot p(s'-R'+s_2-t_1)\bigr].
\end{aligned}$$

In particular,

$$\widehat{k}(s') = \begin{cases} k(s') & s' \leq t_1 + \frac{R'}{2} - 1 \\ 0 & s' \geq t_1 + \frac{R'}{2} + 1. \end{cases}$$

Summarizing, the vector field $\widehat{k}$ along the curve v connecting two points where gluing takes place is determined by the two conditions $\pi_r(h,k) = (\widehat{h}, \widehat{k})$ and $\pi_{r'}(k,p) = (\widehat{k}, \widehat{p})$, and satisfies, in particular,

$$\widehat{k}(s') = \begin{cases} 0 & s' \leq s_1 - \frac{R}{2} - 1 \\ k(s) & s_1 - \frac{R}{2} + 1 \leq s' \leq t_1 + \frac{R'}{2} - 1 \\ 0 & t_1 + \frac{R'}{2} + 1 \leq s'. \end{cases}$$

The situation is illustrated by Figure 3.18.

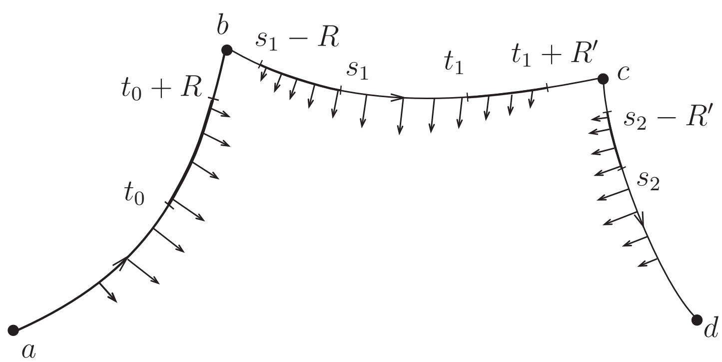


FIGURE 3.18

Iterating this construction we obtain the sc-smooth projection map $\pi_r : E \to E$ satisfying $\pi_r \circ \pi_r = \pi_r$. One finds $r_0 \in (0,1)$ such that for $r = (r_1, \ldots, r_{k-1}) \in [0,1)^{k-1}$ the map

$$[0,\infty)^{-k-1} \oplus U_1 \oplus U_1 \oplus \cdots \oplus U_k \to X$$
$$(r, h_1, \cdots, h_k) \mapsto [\oplus_r(u_1 + h_1, \ldots, u_k + h_k)]$$

if restricted to the splicing core, i.e., to the open set

$$\mathcal{O} := ([o, r_0)^{k-1} \oplus U_1 \oplus \cdots \oplus U_k) \cap K^{\mathcal{S}},$$

is a homeomorphism onto the open neighborhood $\mathcal{U}$ of $([u_1], \ldots, [u_k])$ in X. The inverse of the map defines a polyfold chart for X. The transition maps between these charts are sc-smooth and we have constructed a polyfold structure for X.

3.3.0.1. *The Manifold Version.* Assume that we are given a manifold M together with a countable collection of distinct points, say P, so that every compact subset of M only contains finitely many. For every $a \in P$ fix a sequence if weights δ_m^a starting at 0 and which is increasing. We denote by $X(a,b)$ the collection of equivalence classes of H^2-curves $\mathbb{R} \to M$, connecting at $-\infty$ the point a with the point b at $+\infty$. We define the level m to consist of equivalence class $[u]$ represented by elements u which are in H^{2+m}_{loc} and which are near $-\infty$ of class $(m+2, \delta_m^a)$ and of class $(m+2, \delta_m^b)$ near $+\infty$. For a finite sequence $\widehat{a}$ of mutually different points $\widehat{a} = (a_0, a_1, .., a_k)$ in P consider

$$X(\widehat{a}) = X(a_0, a_1) \times .. \times X(a_{k_1}, a_k).$$

We call k the length. Let $\widehat{P}$ the collection of all such sequence of finite length $k \geq 1$. Define

$$X = \coprod_{a \in \widehat{P}} X(\widehat{a}).$$

If we consider a smooth level-k-map represented by $(u_1, ..., u_k)$ we fix for every u_i a distinguished negative interval $(-\infty, t_i]$ (if $i > 0$) and a distinguished positive interval $[s_, \infty)$ (if $i < k$)). Let us denote the sequence of asymptotic limits by $a_0, ..., a_k$ so that $[u_i] \in X(a_{i-1}, a_i)$. Fix a chart φ_i for $i = 1, .., k-1$ around a_i so that $\varphi_i(a_i) = 0$. We assume that the closures of their domains are disjoint and that their images contain the closed 2-balls around $0 \in \mathbb{R}^n$. We may assume that the image of $(-\infty, s_i]$ under $\varphi_{i-1} \circ u_{i-1}$ is contained in the $\frac{1}{2}$-ball and that the same holds for $[t_i, \infty)$ under the map $\varphi_i \circ u_i$. Since the image of the u_i connects two different points we can find a number $\tau_i \in (s_i, t_i)$ where $\dot{u}_i(\tau_i) \neq 0$. Here we put $s_0 = -\infty$ and $t_k = +\infty$. We can pick charts around the t_i mapping τ_i to 0 and having disjoint domains from the other charts so that the image again contains the closed 2-ball around 0. Pull-back the standard Riemannian metric on the 2-ball. Then we obtain a partially defined flat metric on M which we can extend smoothly. Following Recipe 1.37 we can define charts around the individual u_i of the form

$$h_i \to [\exp_{u_i}(h_i)],$$

where $h_i(\tau_i) \in H_i \subset T_{u_i(\tau_i)}M$ is the constrained transversal to the derivative of u_i at that point. Here h_i lies in some O_{u_i}. We assume that all the properties hold as stated in Recipe 1.37.

Let us restrict for simplicity of notation to the case of a level-2-curve. The following procedure can be carried at every a_i with $0 < i < k$. Let $u = u_i$, $v = u_{i+1}$ and $\psi = \phi_i$. The (nonlinear) gluing is defined as follows:

$$\psi(\oplus_R(u',v')) = \oplus_R^0(\psi\circ u', \psi\circ v')$$

for u' close to u and v' close to v. Here $\oplus_R^0$ denotes the $\mathbb{R}^n$-gluing, i.e. the model gluing. Since the exponential map comes from a metric flat near $a = a_i$ the following holds. Take H^2 sections h and k along u and v which are sufficiently small so that the following formula make sense. Denoting by $d\psi$ the map $pr_2 \circ T\psi$, i.e. the principal part we see

$$\begin{aligned}
& \psi(\oplus_R(\exp_u(h), \exp_v(k)) \\
= \ & \oplus_R(\psi(\exp_u(h)), \psi(\exp_v(k))) \\
= \ & \oplus_R^0(\psi(u), \psi(v)) + \oplus_R^0(d\psi(h), d\psi(v)k) \\
= \ & \psi(\oplus_R(u,v)) + \oplus_R^0(d\psi(u)h, d\psi(v)k) \\
= \ & \exp^0_{\psi(\oplus_R(u,v))}(\oplus_R^0(d\psi(u)h, d\psi(v)k)) \\
= \ & \psi(\exp_{\oplus_R(u,v)}(T\psi(\oplus_R(u,v)))^{-1}(\oplus_R^0(d\psi(u)h, d\psi(v)k))).
\end{aligned}$$

Let us define linear gluing as follows

$$\oplus_R(h,k) = T\psi(\oplus_R(u,v)))^{-1}(\oplus_R^0(d\psi(u)h, d\psi(v)k)).$$

Then $\oplus_R(h,k)$ is a section along $\oplus_R(u,v)$. Using these definitions we have the following nice formula

LEMMA 3.21. *For h and k being sufficiently small sections along u and v we have*

$$\oplus_r(\exp_u(h), \exp_v(k)) = \exp_{\oplus_r(u,v)}(\oplus_r(h,k)). \tag{3.57}$$

Let us define next the anti-gluing $\ominus_R(h,k)$. This will be a map

$$\mathbb{R} \to T_aM$$

defined by

$$\ominus_R(h,k) := T\psi(a)^{-1}(\ominus_R^0(d\psi(u)h, d\psi(v)k)).$$

LEMMA 3.22. *Assume that u and v are as described above. Consider $\oplus_r(u,v)$ and assume that w is sufficiently close. Then there exists a unique pair of sections (h,k) of u^*TM and v^*TM so that*

$$\oplus_r(h,k) = 0 \text{ and } \exp_{\oplus_r(u,v)}(\oplus_r(h,k)) = 0.$$

PROOF. This is a system of two equations. Using the definitions it is equivalent to

$$\begin{gathered}
\ominus_R^0(d\psi(u)h, d\psi(u)k) = 0 \\
\oplus_R^0(d\psi(u)h, d\psi(u)k) = \psi(w) - \Psi(\oplus_r(u,v))
\end{gathered}$$

By the properties of the $\mathbb{R}^n$-gluing and anti-gluing we see that $d\psi(u)h$ and $d\psi(v)k$ are uniquely determined. Hence the same is true for (h,k). ■

Next we have a look at the construction of charts. We have a suitable neighborhood O_u around zero in F_u and similarly O_v around $0 \in F_v$. The data is assumed to have the properties described by Recipe 1.37. Then define for a sufficiently small $r_0 > 0$

$$A : [0, r_0) \oplus O_u \oplus O_v \to X$$

by

$$A(r,h,k) = [\oplus_r(\exp_u(h), \exp_v(k))].$$

Observe that if $A(r,h,k) = A(r',h',k')$ then there exists t with

$$\oplus_{r'}(\exp_u(h'), \exp_v(k')) = t * \oplus_r(\exp_u(h), \exp_v(k)).$$

By the properties of the charts for $X(a,b)$ and $X(b,c)$ given by the Recipe, we see that $t = 0$ and $R = R'$ as before. If we know in addition that $\ominus_r(h,k) = \ominus_r(h',k')$ it follows that the two pairs are equal. Let us define the associated splicing projection π_r via the already studied model splicing which we denote by π_r^0. The obvious definition is given by

$$\pi_r(h,k) = (\widehat{h}, \widehat{k}),$$

with $(h,k), (\widehat{h}, \widehat{k}) \in F_u \oplus F_v$, where $(\widehat{h}, \widehat{k})$ are defined by

$$(d\psi(h)\widehat{h}, d\psi(v)\widehat{k}) = \pi_r(d\psi(u)h, d\psi(v)k).$$

From this we deduce that

$$\widehat{h}(s) = \frac{\tau(s)^2}{\alpha(s)} h(s) + \frac{\tau(s)(1-\tau(s))}{\alpha(s)} d\psi(u(s)^{-1} d\psi(v(s-R))k(s-R).$$

There is a similar formula for $\widehat{k}$. It follows immediately from the definition that π_r is sc-smooth. In fact, the map

$$F_u \oplus F_v \to E : (h,k) \to (d\psi(u)h, d\psi(v)k)$$

is a linear sc-isomorphism and therefore sc-smooth. Then π_r^0 is a sc-smooth splicing projection on E. Then this is followed by the inverse of the linear sc-operator. Hence $(\pi_r, F_u \oplus F_v, [0, r_0))$ is a sc-smooth splicing. Then the restriction to the splicing core gives a chart and all these charts are smoothly compatible. Hence we have proved:

THEOREM 3.23. *The space of level-k-curves X in the manifold M has for given sequence $\delta^a = (\delta^a_m)$ for every $a \in P$ a natural M-polyfold structure where the charts have the form*

$$(r_1, .., r_{k-1}, h_1, .., h_k) \to [\exp_{\oplus_{(r_1,..,r_{k-1})}(u_1,..,u_k)}(\oplus_{(r_1,..,r_{k-1})}(h_1, ..., h_k))].$$

CHAPTER 4

M-Polyfold Bundles

4.1. Local Strong sc-Bundles

Prompted by constructions of pull back bundles later on we continue with the conceptual framework and consider two Banach spaces E and F equipped with the sc-smooth structures defined by the filtrations E_m, resp. F_m for $m \geq 0$. Their $\triangleleft$-product

$$E \triangleleft F$$

is the Banach space $E \oplus F$ equipped with the double filtration

$$(E \triangleleft F)_{m,k} = E_m \oplus F_k$$

where $m \geq 0$ and $0 \leq k \leq m+1$.

Similarly, if $U \subset E$ is an open subset, then $U \triangleleft F$ is the open set $U \oplus F$ equipped with the double filtration

$$(U \triangleleft F)_{m,k} = U_m \oplus F_k$$

for all $m \geq 0$ and $0 \leq k \leq m+1$. Here $U_m = U \cap E_m$, as introduced in Section 1.1. With the canonical projection map

$$U \triangleleft F \to U$$

we obtain a bundle, called a **local strong sc-bundle** . The **$\triangleleft$−tangent bundle** of the product $U \triangleleft F$ is defined as

$$T_{\triangleleft}(U \triangleleft F) = (TU) \triangleleft (TF).$$

Recalling the tangent bundle from Section 1.1, the $\triangleleft$-tangent bundle is equipped with the double filtration,

$$\begin{aligned} T_{\triangleleft}(U \triangleleft F)_{m,k} &= (TU)_m \oplus (TF)_k \\ &= (U_{m+1} \oplus E_m) \oplus (F_{k+1} \oplus F_k). \end{aligned}$$

for all $m \geq 0$ and $0 \leq k \leq m+1$.

Recall the notation $F_0 = F$. Associated with the product $U \triangleleft F$ one has the derived sc-spaces

$$U^n \oplus F \quad \text{and} \quad U^n \oplus F^1$$

for every $n \geq 0$. They are equipped with the standard (simple) filtrations

$$\begin{aligned}(U^n \oplus F)_m &= U_{n+m} \oplus F_m, \qquad m \geq 0\\ (U^n \oplus F^1)_m &= U_{n+m} \oplus F_{m+1}, \quad m \geq 0.\end{aligned}$$

Definition 4.1. *If $U \triangleleft F \to U$ and $V \triangleleft G \to V$ are two local strong sc-bundles, then a* $\mathbf{sc^0_\triangleleft}$**−map** *is a map*

$$f : U \triangleleft F \to V \triangleleft G$$

of the form

$$f(u, h) = (a(u), b(u, h))$$

which induces C^0-maps

$$U_m \oplus F_k \to V_m \oplus F_k$$

for all $m \geq 0$ and $0 \leq k \leq m + 1$.

Recalling Definition 1.3 of a sc^0-map we note that the map f of class sc$^0_\triangleleft$ induces sc^0-maps between the derived sc-spaces

$$U^n \oplus F^i \to V^n \oplus G^i$$

for every $n \geq 0$ and $i = 0, 1$. Recalling Definition 1.6 of an sc^1-map we introduce the next smoothness concept.

Definition 4.2. *The sc$^0_\triangleleft$-map $f : U \triangleleft F \to V \triangleleft G$ is called of* **class** $\mathbf{sc^1_\triangleleft}$ *if it induces sc^1-maps between the derived spaces*

$$U^n \oplus F^i \to V^n \oplus G^i$$

for every $n \geq 0$ and $i = 0, 1$.

The **tangent map** associated with the sc$^1_\triangleleft$-map f is the sc$^0_\triangleleft$-map

$$T_\triangleleft f : T_\triangleleft(U \triangleleft F) \to T_\triangleleft(V \triangleleft G)$$

defined by

$$(T_\triangleleft f)(u, h, v, k) = (a(u), Da(u)h, b(u, v), Db(u, v)[h, k])$$

where $(u, h, v, k) \in U_{m+1} \oplus E_m \oplus F_{k+1} \oplus F_k$ with $0 \leq k \leq m + 1$. From the chain rule for sc^1-maps (Theorem 1.13) one easily deduces the following chain rule for sc$^1_\triangleleft$-maps.

Theorem 4.3. *Assume the maps $f : U \triangleleft F \to R \triangleleft G$ and $g : V \triangleleft G \to W \triangleleft H$ are of class sc$^1_\triangleleft$, where $U \subset E$ and $V \subset R$ are open subsets equipped with the induced sc-structures. Assume also that*

$g(U \triangleleft F) \subset (V \triangleleft G)$. *Then the composition* $g \circ f$ *is also of class* $sc^1_\triangleleft$ *and the tangent maps satisfy*

$$T_\triangleleft(g \circ f) = (T_\triangleleft g) \circ (T_\triangleleft f).$$

Moreover, $T_\triangleleft(g \circ f)$ *is* $sc^0_\triangleleft$.

An $\mathbf{sc}^{\boldsymbol{k}}_{\triangleleft}$**−vector bundle map** is an $\mathrm{sc}^k_\triangleleft$-map $\Phi : U \triangleleft E \to V \triangleleft F$ of the form

$$\Phi(x, h) = (a(x), b(x, h))$$

which is linear in h. The map Φ is called an $\mathrm{sc}_\triangleleft$-vector bundle isomorphism, if it is $\mathrm{sc}_\triangleleft$-smooth and if the same also holds for the inverse map.

We distinguish between two different classes of sc-smooth sections of the local strong sc-bundle $U \triangleleft F \to U$.

Definition 4.4. *An* **sc−smooth section** f *is a map of the form*

$$u \to (u, \overline{f}(u)) \in U \oplus F$$

so that the principal part $\overline{f} : U \to F$ *is sc-smooth. An sc-section* f *is called an* $\mathbf{sc}^+$ **− smooth** *section if its principal part induces an sc-smooth map* $\overline{f} : U \to F^1$.

The tangent map Tf of a sc^+-section f is a sc^+-section of the $\triangleleft$-tangent bundle $TU \triangleleft TF \to TU$. Let us denote the space of sc-smooth sections by $\Gamma(U \triangleleft F)$ and that of sc^+-sections by $\Gamma^+(U \triangleleft F)$. Assume that $\Phi : U \triangleleft F \to V \triangleleft G$ is an $\mathrm{sc}_\triangleleft$-vector bundle isomorphism. Then the pull back maps induce linear isomorphisms

$$\Gamma(V \triangleleft G) \to \Gamma(U \triangleleft G)$$

and

$$\Gamma^+(V \triangleleft G) \to \Gamma^+(U \triangleleft G).$$

The same, of course, holds true for the push-forward.

4.2. Strong sc-Vector Bundles

We use the discussion in Section 4.1 to define a useful class of sc-bundles over an sc-manifold. We consider two sc-manifolds X and Y sc-manifolds together with a surjective sc-smooth map

$$p : Y \to X.$$

We assume that every fiber $p^{-1}(x) \subset Y$ has the structure of a Banach space. To define an additional structure we now assume that we can cover Y with sc$_\triangleleft$-charts. An **sc$_\triangleleft$ − chart** is a tuple

$$(p^{-1}(O), \Phi, U \triangleleft F)$$

where $O \subset X$ is an open subset and

$$\Phi : p^{-1}(O) \to U \oplus F$$

is a homeomorphism which is fiber-wise linear and covers an sc-smooth chart

$$\varphi : O \to U$$

so that $\mathrm{pr}_1 \circ \Phi = \varphi \circ p$. In addition, we assume that all transition maps between our sc$_\triangleleft$-charts

$$\Psi \circ \Phi^{-1} : \varphi(O \cap O') \triangleleft F \to \psi(O \cap O') \triangleleft G$$

are sc$_\triangleleft$-smooth vector bundle isomorphisms. Recall that this requires the following. The transition map $\Gamma = \Psi \circ \Phi^{-1}$ necessarily has the form

$$\Gamma(u, h) = (\psi \circ \varphi^{-1}(u), b(u, h))$$

and is linear in h. In addition,

$$\Gamma : \varphi(O \cap O')^n \oplus F^i \to \psi(O \cap O')^n \oplus G^i$$

is sc-smooth for all the derived spaces with $n \geq 0$ and $i = 0, 1$. Summarizing, we have constructed a strong sc-vector bundle structure for $p : Y \to X$ according to the following definition.

Definition 4.5. *A* **strong sc-vector bundle structure** *for a surjective sc-smooth map $p : Y \to X$ between sc-manifolds consists of the following data. Every fiber has the structure of a Banach space. There is a maximal atlas of smoothly sc$_\triangleleft$-compatible sc$_\triangleleft$ charts.*

We point out that the postulated compatible sc$_\triangleleft$-charts define a double filtration $Y_{m,k}$ on Y where $m \geq 0$ and $0 \leq k \leq m + 1$. The map $p : Y \to X$ maps $Y_{m,k}$ onto X_m. An element $x \in X$ contained in X_m is called of regularity m. The points $y \in p^{-1}(x) \subset Y_{m,k}$ have fiber regularity k, where k is restricted to $0 \leq k \leq m + 1$.

We remark that the tangent bundle $p : TX \to X^1$ is in general not! a strong sc-vector bundle for its natural structure (at least not in a natural way).

Clearly p is sc-smooth and surjective and the fibers have a Banach space structure. The transition maps for sc-charts $T\varphi$ and $T\psi$ have the form

$$T(\psi \circ \phi^{-1}) : \phi(O \cap O')^1 \oplus E \to \psi(O \cap O')^1 \oplus F.$$

Let us write this for convenience shortly as

$$\Gamma : W^1 \oplus E \to V^1 \oplus F.$$

It is straightforward to verify that the sc-smoothness implies that the induced maps

$$\Gamma : W^{1+n} \oplus E \to V^{1+n} \oplus F$$

are sc-smooth. In general the maps do not induce sc-smooth maps

$$\Gamma : W^{1+n} \oplus E^1 \to V^{1+n} \oplus F^1.$$

In fact this will, in general, not even be the case for $n = 0$. However, if we restrict the tangent bundle TX to X^2, then $TX|X^2 \to X^2$ admits a natural $\mathrm{sc}_\triangleleft$-structure.

4.3. Example of a Strong Bundle over $X(a,b)$

Let $X(a,b)$ be the sc-manifold consisting of the equivalence classes of maps $\mathbb{R} \to M$ connecting the points a and b as introduced in Section 1.5. We will construct a strong sc-vector bundle $Y(a,b)$ over $X(a,b)$. Choose $[u] \in X(a,b)$. Then take a representative u and consider the pull back bundle $u^*TM \to \mathbb{R}$. If u is on level m, that is of class H^{m+2,δ_m}, it makes sense to talk about H^{m+2}_{loc}-sections along u. By using trivialisations near a and b coming from charts we can define for every $\varepsilon \geq 0$ sections which in local coordinates near the ends have weak derivatives up to order m which if weighted by $e^{\varepsilon|s|}$ belong to L^2. If $0 \leq k \leq m+1$, the space $H^{k+1,\delta_k}(u^*TM)$ is well-defined. Now consider pairs (u,h) of level (m,k) which by definition consist of a path $u \in \hat{X}(a,b)$ of class m and $h \in H^{k+1,\delta_k}(u^*TM)$. Two such pairs are equivalent if they lie in the same orbit of the obvious $\mathbb{R}$-action. Let us denote the whole collection by $Y(a,b)$. Clearly we have a canonical map

$$Y(a,b) \to X(a,b).$$

Also observe that at this point we have the double filtration $Y(a,b)_{m,k} \to X(a,b)_m$ with $0 \leq k \leq m+1$. Let us define next strong sc-vector bundle charts. They will be constructed out of our charts in the way as we are familiar from the usual constructions of manifold of maps, see [**12**]. We give the inverses of the charts

$$(h,\ell) \to [(\exp_u(h), \nabla_2 \exp_u(h)\ell)].$$

Here h belongs to the usual neighborhoods of $0 \in F_u$ and $\ell \in H^1$. The transition maps are clearly sc-smooth.

4.4. Strong M-Polyfold Bundles

4.4.1. Spliced Fibered Scales. Next we introduce a spliced version of our fibered scales. We take an open subset V of a partial cone and two sc-smooth Banach spaces E and H, and consider two sc-smooth splicings $\mathcal{S}_0 = (\pi, E, V)$ and $\mathcal{S}_1 = (\sigma, H, V)$ as defined in Section 2.1. The parameter set V is the same for both splicings.

In the following we denote by

$$[k, m]$$

a pair of integers satisfying $0 \leq k \leq m+1$.

Definition 4.6. *A spliced sc-fibered Banach scale is the triplet*

$$\mathcal{S}_{\triangleleft} = (\Pi, E \triangleleft H, V)$$

where $\Pi_v = (\pi_v, \sigma_v) : E \triangleleft H \to E \triangleleft H$ *is the family of projections parametrized by* $v \in V$.

Taking two splicing cores $K_0 = K^{\mathcal{S}_0} = \{(v,e) \in V \oplus E |\ \pi_v(e) = e\}$ and $K_1 = K^{\mathcal{S}_1} = \{(v,h) \in V \oplus H |\ \sigma_v(h) = h\}$ we can construct their $\triangleleft$-product

$$K^{\mathcal{S}_{\triangleleft}} := K_0 \triangleleft_V K_1$$

having the double filtration defined by

$$(K_0 \triangleleft_V K_1)_{[k,m]} = \{(v,e,h) \in V \oplus E_m \oplus H_k \mid \pi_v(e) = e \text{ and } \sigma_v(h) = h\}.$$

The canonical projection

$$(V \oplus E) \triangleleft H \to V \oplus E$$

induces the natural sc-smooth projection

$$\mathrm{pr}_1 : K^{\mathcal{S}_{\triangleleft}} \to K^{\mathcal{S}_0}.$$

As before we can define a tangent splicing $TS_{\triangleleft}$ and the associated splicing core $K^{T\mathcal{S}_{\triangleleft}}$, so that $T\,\mathrm{pr}_1$ induces the projection

$$T\,\mathrm{pr}_1 : TK^{\mathcal{S}_{\triangleleft}} \to TK^{\mathcal{S}_0}.$$

We are interested in pairs $(K^{\mathcal{S}_{\triangleleft}}|O, \mathcal{S}_{\triangleleft})$, where $\mathcal{S}_{\triangleleft}$ is a spliced sc-fibered Banach scale $(\Pi, E \triangleleft H, V)$ and where

$$K^{\mathcal{S}_{\triangleleft}}|O$$

stands for the the preimage under the canonical projection

$$K^{\mathcal{S}_{\triangleleft}} \to K^{\mathcal{S}_0}$$

of an open subset O of $K^{\mathcal{S}_0} \subset V \oplus E$. A smooth morphism

$$(K^{\mathcal{S}_\triangleleft}|O, \mathcal{S}_\triangleleft) \to (K^{\mathcal{S}'_\triangleleft}|O', \mathcal{S}'_\triangleleft)$$

is an sc-smooth map

$$K^{\mathcal{S}_\triangleleft}|O \to K^{\mathcal{S}'_\triangleleft}|O'$$

of the form

$$(v, e, h) \mapsto (\varphi(v, \pi_v(e)), \Phi(v, \pi_v(e), \sigma_v(h)))$$

where $(v, e, h) \in \widehat{O} \oplus H$ and $\varphi(v, \pi_v(e)) \in O' \subset V' \oplus E'$ and where $\Phi(v, \pi_v(e), \sigma_v(e)) \in H'$. Moreover, the fiber map Φ is linear in the last argument. Recall from Section 2.3 that the subset $\widehat{O}$ is open in $V \oplus E$ and defined by $\widehat{O} = \{(v, e) \in V \oplus E \mid (v, \pi_v(e)) \in O\}$. Similarly we can define the sc-sections and sc$^+$-sections $\Gamma(\mathcal{S}_\triangleleft, K^{\mathcal{S}_\triangleleft}|O)$ and $\Gamma^+(\mathcal{S}_\triangleleft, K^{\mathcal{S}_\triangleleft}|O)$. In future, if irrelevant, we might suppress the $\mathcal{S}_\triangleleft$ in the notation and simply write for example $\Gamma(K^{\mathcal{S}_\triangleleft}|O)$. We also consider sc$_\triangleleft$-smooth vector bundle morphisms which are linear on the fibers.

4.4.2. M-Polyfold Bundles. Let Y and X be M-polyfolds and $p : Y \to X$ a surjective sc-smooth map so that the preimage of every point carries the structure of a Banach space.

DEFINITION 4.7. *A* **M-polybundle chart** *for $p : Y \to X$ is a triple $(U, \Phi, (K^{\mathcal{S}_\triangleleft}|O, \mathcal{S}_\triangleleft))$, where U is an open set in X and Φ is a homeomorphism*

$$\Phi : p^{-1}(U) \to K^{\mathcal{S}_\triangleleft}|O$$

which is linear on the fibers and is covering a homeomorphism

$$\varphi : U \to O \subset K^{\mathcal{S}_0}$$

so that $pr_1 \circ \Phi = \varphi \circ p$. Moreover, Φ and φ are smoothly compatible with the M-polyfold structures on Y and X. Two M-polybundle charts are called **sc$_\triangleleft$ − compatible** *provided the transition maps are sc$_\triangleleft$-smooth local polyfold-bundle morphisms. A* **M-polybundle atlas** *consists of a family of M-polybundle charts so that the underlying open sets U cover X and so that the transition maps are sc$_\triangleleft$-smooth. A maximal smooth atlas of M-polybundle charts is called a* **M-polyfold bundle structure.**

The situation of an M-polyfold bundle chart is illustrated by the following diagram.

$$\begin{array}{ccc} Y \supset p^{-1}(U) & \xrightarrow{\Phi} & K^{\mathcal{S}_\triangleleft}|O \subset K^{\mathcal{S}_\triangleleft} \\ \Big\downarrow p & & \Big\downarrow \mathrm{pr}_1 \\ X \ \supset \ U & \xrightarrow[\varphi]{} & O \subset K^{\mathcal{S}_0} \end{array}$$

Here O is an open subset of the splicing core

$$K^{\mathcal{S}_0} = \{(v,e) \in V \oplus E \mid \pi_e(v) = e\},$$

where V is a partial cone in a finite dimensional vector space W and E is an sc-Banach space.

$$K^{\mathcal{S}_\triangleleft}|O = \{(v,e,f) \in V \oplus E \oplus F \mid (v,e) \in O \text{ and } \sigma_v(f) = f\},$$

where F is an sc-Banach space. The fiber over $(v,e) \in O$ is the Banach space

$$\mathrm{pr}_1^{-1}(v,e) = (v,e) \times \{f \in F | \, \sigma_v(f) = f\}.$$

4.4.3. A Strong M-Polyfold Bundle over X. We have previously introduced the strong sc-vector bundle $Y(a,b) \to X(a,b)$. Let us first study the case where the underlying manifold is $\mathbb{R}^n$ and consider the M-polyfold of level-k-curves. We define $Y \to X$ in the obvious way and will show that it carries the structure of a strong M-polyfold bundle.The local coordinates are constructed using a splicing core associated to the strong local bundle

$$(F_{u_1} \oplus .. \oplus F_{u_k}) \triangleleft (H^1(u_1^*TM) \oplus .. \oplus H^1(u_\ell^*TM) \to F_{u_1} \oplus .. \oplus F_{u_k}$$

with sc$_\triangleleft$-structure given by quality $((m+2,\delta_m),(k+1,\delta_k))$. In local coordinates the splicing projections for fiber and basis are given by the same formulas as before. If this splicing is denoted by $\mathcal{S}_\triangleleft$ with underlying splicing $\mathcal{S}_0$ we see that the inverse of charts have the form

$$(r_1,..,r_{k-1},h_1,..,h_k,b_1,..,b_k) \to [\oplus_r(\nabla_2 \exp_{u_1}(h_1)b_1,..,\nabla_2 \exp_{u_k}(h_k)b_k)],$$

where the data ranges in $K^{\mathcal{S}_\triangleleft}|O$ and O is ithe mage of the chart in $K^{\mathcal{S}_\triangleleft}$, and where ∇_2 denotes the fiber derivative of the exponential map.

CHAPTER 5

Local sc-Fredholm Theory

We begin with a version of the implicit function theorem in our "sc-context".

5.1. An Infinitesimal sc-Implicit Function Theorem

In order to study the local behavior of functions near a point we make use of the concept of germs. If E is a Banach space with the sc-smooth structure

$$E = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_m \supset \cdots$$

we denote by $\mathcal{O}(E, 0)$ an sc-germ of neighborhoods of 0 consisting of a decreasing sequence

$$U_0 \supset U_1 \supset U_2 \supset \cdots \supset U_m \supset \cdots$$

where U_m is an open neighborhood of 0 in E_m for every $m \geq 0$. We point out that in contrast to the notation in Chapter 1, it is not required that U_m is induced from U_0 as $U_m = U_0 \cap E_m$.

If F is a second Banach space with the sc-structure $F = F_0 \supset F_1 \supset F_2 \supset \cdots \supset F_m \supset \cdots$, then an **$\mathbf{sc^0}$- germ**

$$f : \mathcal{O}(E, 0) \to (F, 0)$$

is a continuous map $f : U_0 \to F_0$ satisfying $f(0) = 0$ such that

$$f : U_m \to F_m$$

is continuous for every $m \geq 0$. The **sc-tangent germ** $\mathcal{O}(E, 0)$ associated with the neighborhoods U_m belonging to $\mathcal{O}(E, 0)$ consists of the decreasing sequence

$$U_{m+1} \oplus E_m, \quad m \geq 0$$

of open subsets of $E_{m+1} \oplus E_m$. An **$\mathbf{sc^1}$- germ** $F : \mathcal{O}(E, 0) \to (F, 0)$ is an sc^0-germ which, in addition, is of class sc^1 in the sense of Definition 1.6 or Proposition 1.7, however, this time with respect to the nested sequence U_m of the sc-neighborhood germ $\mathcal{O}(E, 0)$.

Definition 5.1. *Consider a finite dimensional space V and an sc-Banach space E. Then an sc^0-germ $f : \mathcal{O}(V \times E, 0) \to (E, 0)$ is called an* $\mathbf{sc^0}$ **− contraction germ** *if it has the form*

$$f(v, u) = u - B(v, u)$$

so that the following holds. For every $m \geq 0$ there is a constant $0 < \rho_m < 1$ such that

$$\|B(v, u) - B(v, u')\|_m \leq \rho_m \cdot \|u - u'\|_m$$

for (v, u) and (v, u') close to 0 in $V \times E_m$. Here the notion of close depends on m.

We start with the following trivial consequence of a parameter dependent version of Banach's fixed point theorem.

Theorem 5.2. *Let $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ be an sc^0-contraction germ. Then there exists a uniquely determined sc^0-germ $\delta : \mathcal{O}(V, 0) \to (E, 0)$ so that the associated graph germ $gr(\delta) : V \to V \oplus E$, $v \mapsto (v, \delta(v))$ satisfies*

$$f \circ gr(\delta) = 0.$$

Our main concern now is the regularity of the solution δ.

Theorem 5.3. *If the sc^0-contraction germ $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ is of class sc^1, then the solution germ $\delta : \mathcal{O}(V, 0) \to (E, 0)$ in Theorem 5.2 is also of class sc^1.*

Proof. We fix $m \geq 0$ and show first that the set

$$\frac{1}{|b|}\|\delta(v + b) - \delta(v)\|_m \tag{5.1}$$

is bounded for v and $b \neq 0$ belonging to a small ball around zero in V whose radius depends on m. Since the map B is of class sc^1, there exists at $(v, u) \in U_{m+1}$ a bounded linear map $DB(u, v) \in \mathcal{L}(V \oplus E_m, F_m)$, and we introduce the following notation,

$$\begin{aligned} DB(v, u)[\widehat{v}, \widehat{u}] &= DB(v, u)[\widehat{v}, 0] + DB(v, u)[0, \widehat{u}] \\ &= D_1B(v, u)[\widehat{v}] + D_2B(v, u)[\widehat{u}]. \end{aligned}$$

Since $v \mapsto \delta(v)$ is a continuous map into E_{m+1} and since the map B is of class C^1 as a map from an open neighborhood of 0 in $V \oplus E_{m+1}$ into

E_m, we have the identity

$$B(v+b,\delta(v+b)) - B(v,\delta(v+b)) = \left[\int_0^1 D_1B(v+sb,\delta(v+b))ds\right][b].$$

As a consequence,

$$\begin{aligned}\frac{1}{|b|}&\|B(v+b,\delta(v+b)) - B(v,\delta(v+b))\|_m \\ &\leq \int_0^1 \|D_1B(v+sb,\delta(v+b))\|ds \leq C_m.\end{aligned} \tag{5.2}$$

Recalling $\delta(v) = B(v,\delta(v))$ and $\delta(v+b) = B(v+b,\delta(v+b))$ we have the identity,

$$\begin{aligned}\delta(v+b) - \delta(v) &- [B(v,\delta(v+b)) - B(v,\delta(v))] \\ &= B(v+b,\delta(v+b)) - B(v,\delta(v+b)).\end{aligned} \tag{5.3}$$

From the contraction property of B in the second variable one concludes

$$\|B(v,\delta(v+b)) - B(v,\delta(v))\|_m \leq \rho_m\|\delta(v+b) - \delta(v)\|_m. \tag{5.4}$$

Now, using $0 < \rho_m < 1$ one derives from (5.3) using (5.2) and (5.4) the estimate

$$\begin{aligned}\tfrac{1}{|b|}&\|\delta(v+b) - \delta(v)\|_m \\ &\leq \frac{1}{1-\rho_m}\cdot\frac{1}{|b|}\|B(v+b,\delta(v+b)) - B(v,\delta(v+b))\|_m \\ &\leq \frac{1}{1-\rho_m}\cdot C_m\end{aligned}$$

as claimed in (5.1). Since B is of class C^1 from $V \oplus E_{m+1}$ into E_m and since $\|\delta(v+b) - \delta(v)\|_m \leq C'_m \cdot |b|$ by (5.1), the estimate

$$\delta(v+b) - \delta(v) - DB(v,\delta(v))\cdot[b,\delta(v+b) - \delta(v)] = o_m(b) \tag{5.5}$$

holds true, where $o_m(b) \in E_m$ is a function satisfying $\frac{1}{|b|}o_m(b) \to 0$ in E_m as $b \to 0$ in V. We next prove

$$\|D_2B(v,\delta(v))h\|_m \leq \rho_m \cdot \|h\|_m \tag{5.6}$$

for all $h \in E_{m+1}$. Fixing $h \in E_{m+1}$, we can estimate

$$
\begin{aligned}
&\|D_2B(v,\delta(v))h\|_m\\
\le\ &\frac{1}{|t|}\cdot\|B(v,\delta(v)+th)-B(v,\delta(v))-D_2B(v,\delta(v))[th]\|_m\\
&+\frac{1}{|t|}\cdot\|B(v,\delta(v)+th)-B(v,\delta(v))\|_m.
\end{aligned}
$$

In view of the postulated contraction property of B, the second term is bounded by $\rho_m\cdot\|h\|_m$, while the first term tends to 0 as $t\to 0$ because B is of class C^1. Hence the claim (5.6) follows. Using (5.6) and the fact that E_{m+1} is dense in E_m, we derive for the continuous linear operator $D_2B(v,\delta(v)):E_m\to E_m$ the bound

$$
\|D_2B(v,\delta(v))h\|_m\le\rho_m\cdot\|h\|_m \tag{5.7}
$$

for all $h\in E_m$. Thus, in view of $\rho_m<1$, the continuous linear map

$$
\begin{aligned}
L(v)&:E_m\to E_m\\
L(v)&:=\mathrm{Id}-D_2B(v,\delta(v))
\end{aligned}
$$

is an isomorphism and we conclude from (5.5) the estimate in E_m,

$$
\delta(v+b)-\delta(v)-L(v)^{-1}D_1B(v,\delta(v))b=o_m(b).
$$

Therefore, the map $v\mapsto\delta(v)$ into E_m is differentiable and its derivative $\delta'(v)\in\mathcal{L}(V,E_m)$ is given by the formula

$$
\delta'(v)=L(v)^{-1}D_1B(v,\delta(v)). \tag{5.8}
$$

It remains to show that $v\mapsto\delta'(v)\in\mathcal{L}(V,E_m)$ is continuous. To see this we define the map $F:(V\oplus V)\oplus E_m\to E_m$ by setting

$$
F(v,b,h)=D_1B(v,\delta(v))[b]+D_2B(v,\delta(v))[h].
$$

The map F is continuous and, in view of (5.7), it is a contraction in h. Applying a parameter dependent version of Banach's fixed point theorem to F we find a continuous function $(v,b)\mapsto h(v,b)$ from a small neighborhood of 0 in $V\oplus V$ into E_m satisfying $F(v,b,h(v,b))=h(v,b)$. Since we also have $F(v,b,\delta'(v)b)=\delta'(v)b$, it follows from the uniqueness that $h(v,b)=\delta'(v)b$ and so the map $(v,b)\mapsto\delta'(v)b$ is continuous. Using now the fact that V is a finite dimensional space we conclude that $v\mapsto\delta'(v)\in\mathcal{L}(V,E_m)$ is a continuous map. The proof of the theorem is complete. ■

Theorem 5.3 shows that the sc^0-contraction germ f which is also of class sc^1 has a solution germ δ satisfying $f(v,\delta(v))=0$ which is also of class sc^1. We shall show next, that if f is of class sc^2, then δ is also of class sc^2. To do so we define the sc^0-germ $f^{(1)}$ by

$$\begin{aligned} f^{(1)} &: \mathcal{O}(TV \oplus TE, 0) \to TE \\ f^{(1)}(v, b, u, w) &= (u - B(v, u), w - DB(v, \delta(v))[b, w]) \\ &= (u, w) - B^{(1)}(v, b, u, w), \end{aligned}$$

where the last line defines the map $B^{(1)}$. For v small, the map $B^{(1)}$ has the contraction property with respect to (u, w). Indeed, on the m-level of $(TE)_m = E_{m+1} \oplus E_m$, i.e., for $(u, w) \in E_{m+1} \oplus E_m$ and for (v, b) small we can estimate, using (5.6),

$$\begin{aligned} &\|B^{(1)}(v, b, u', w') - B^{(1)}(v, b, u, w)\|_m \\ &\qquad = \|B(v, u') - B(v, u)\|_{m+1} \\ &\qquad\qquad + \|DB(v, \delta(v))[b, w'] - DB(v, \delta(v))[b, w]\|_m \\ &\qquad \leq \rho_{m+1}\|u' - u\|_{m+1} + \|D_2B(v, \delta(v))[w' - w]\|_m \\ &\qquad \leq \max\{\rho_{m+1}, \rho_m\} \cdot \big(\|u' - u\|_{m+1} + \|w' - w\|_m\big) \\ &\qquad = \max\{\rho_{m+1}, \rho_m\} \cdot \|(u', w') - (u, w)\|_m. \end{aligned}$$

Consequently, the germ $f^{(1)}$ is an sc^0-contraction germ. If now f is of class sc^2, then the germ $f^{(1)}$ is of class sc^1, as one verifies by comparing the tangent map Tf with the map $f^{(1)}$ and using the fact that the solution δ is of class sc^1. Hence we deduce from Theorem 5.3 that the solution germ $\delta^{(1)}$ of $f^{(1)}$ is of class sc^1. It solves the equation

$$f^{(1)}(v, b, \delta^{(1)}(v, b)) = 0. \tag{5.9}$$

But also the tangent germ $T\delta$ defined by $T\delta(v, b) = (\delta(v), \delta'(v)b)$ is a solution of (5.9). From the uniqueness we conclude $\delta^{(1)} = T\delta$ so that $T\delta$ is of class sc^1, and hence δ of class sc^2, as claimed. Proceeding inductively one verifes the following result.

THEOREM 5.4. *If $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ is a contraction germ which is, in addition, of class sc^k, then the solution germ*

$$\delta : \mathcal{O}(V, 0) \to (E, 0)$$

satisfying

$$f(v, \delta(v)) = 0$$

is of class sc^k.

PROOF. The proof is a consequence of the statements (k) below which we prove by induction. In the following we denote by $\delta^j(v_1) : V \oplus \cdots \oplus V \to E$ the j-th derivative of $\delta : V \to E$ at the point v_1. It is a multilinear map $(v_2, \cdots, v_{j+1}) \mapsto \delta^j(v_1)[v_2, \cdots, v_{j+1}]$. Moreover, $D_{1,j}B(x_1, \cdots, x_j)$ denotes the linearization of the function B with respect to the first variable x_1 and the last variable x_j. .

(k) Let $f : \mathcal{O}(V \oplus E, 0) \to (E, 0)$ be a contraction germ of class sc^k and let $\delta : \mathcal{O}(V, 0) \to (E, 0)$ be a solution germ. Then δ is of class sc^k. Set $B^{(1)} = B$ and define $f^{(1)}(v_1, h) = h - B^{(1)}(v_1, h)$ for $(v_1, h) \in V \oplus E$. Then for every $2 \leq j \leq k$ there exists a map

$$f^{(j)} : V \oplus V \oplus \cdots \oplus V \oplus E \to E \tag{5.10}$$

of the form

$$f^{(j)}(v_1, \cdots, v_j, h) = h - B^{(j)}(v_1, \cdots, v_j, h) \tag{5.11}$$

where

$$\begin{aligned} &B^{(j)}(v_1, \cdots, v_j, h) \\ &\quad = D_{1,j}B^{(j-1)}\big(v_1, \cdots, v_{j-1}, \delta^{j-2}(v_1)[v_2, \cdots, v_{j-1}]\big)[v_j, h] \end{aligned} \tag{5.12}$$

having the following properties. If $1 \leq j \leq k$, the germ $(v_1, \cdots, v_j) \mapsto \delta^{j-1}(v_1)[v_2, \cdots, v_j]$ solves the equation

$$f^{(j)}(v_1, \cdots, v_j, \delta^{j-1}(v_1)[v_2, \cdots, v_j]) = 0. \tag{5.13}$$

Moreover, every map $f^{(j)}$ is multilinear with respect to $(v_2, v_3, \cdots, v_j, h)$ and is a contraction germ with respect to h.

The statement (k) for $k = 1$ is just the conclusion of Theorem 5.3. Now assume the statement to hold true for $k \geq 1$. We shall prove that it holds true also for $k + 1$. Let f be of class sc^{k+1}. By assumption (k), the solution germ $v \to \delta(v)$ from V to E is of class sc^k. Since f is of class sc^k, the map $B^{(k)}$ is of class sc^2 so that the map $(v_1, \cdots, v_{k+1}, h_1, h) \mapsto D_{1,k+1}B^{(k)}(v_1, \cdots, v_k, h_1)[v_{k+1}, h]$, where $(v_1, \cdots, v_{k+1}, h_1, h) \in V \oplus V \cdots V \oplus E^1 \oplus E$, is of class sc^1. Composing this map with the sc^1-map

$$(v_1, \cdots, v_{k+1}, h) \mapsto (v_1, \cdots, v_{k+1}, \delta^{k-1}(v_1)[v_2, \cdots, v_k], h)$$

we define the sc^1-map

$$\begin{aligned} &B^{(k+1)}(v_1, \cdots, v_{k+1}, h) \\ &\quad = D_{1,k+1}B^{(k)}(v_1, \cdots, v_k, \delta^{k-1}(v_1)[v_2, \cdots, v_k])[v_{k+1}, h] \end{aligned} \tag{5.14}$$

and introduce

$$f^{(k+1)}(v_1, \cdots, v_k, h) := h - B^{(k+1)}(v_1, \cdots, v_{k+1}, h). \tag{5.15}$$

Abbreviating for the moment $w = (v_1, \cdots, v_k, \delta^{k-1}(v_1)[v_2, \cdots, v_k])$ and using the fact that $B^{(k)}$ is linear with respect to h we obtain for the linearization in (5.14),

$$\begin{aligned} D_{1,k+1}B^{(k)}(w)[v_{k+1}, h] &= D_1 B^{(k)}(w)[v_{k+1}] + D_2 B^{(k)}(w)[h] \\ &= D_1 B^{(k)}(w)[v_{k+1}] + B^{(k)}(v_1, \cdots, v_k, h). \end{aligned}$$

This shows that $f^{(k+1)}$ is a contraction germ because, by assumption, $B^{(k)}(v_1, \cdots, v_k, h)$ is a contraction with respect h. In addition, we see that $f^{(k+1)}$ is linear with respect to the variables $(v_2, \cdots, v_{k+1}, h)$. Applying Theorem 5.3 to the map $f^{(k+1)}$, we find a unique map

$$(v_1, v_2, \cdots, v_{k+1}) \to h(v_1, \cdots, v_{k+1})$$

from $V \oplus V \oplus \cdots \oplus V$ to E which is of class sc^1, solves the equation

$$h(v_1, \cdots, v_{k+1}) = B^{(k+1)}(w, h(v_1, \cdots, v_{k+1})),$$

and satisfies $h(0, \cdots, 0) = 0$. On the other hand, by assumption (k),

$$\delta^{k-1}(v_1)[v_2, \cdots, v_k] = B^{(k)}(v_1, \cdots, v_k, \delta^{k-1}(v_1)[v_2, \cdots, v_k])$$

for $(v_1, v_2, \cdots v_k) \in V \oplus V \oplus \cdots \oplus V$. Differentiating this identity with respect to v_1 we find

$$\begin{aligned} \delta^k(v_1)[v_2, \cdots, v_{k+1}] &= D_1 B^{(k)}(v_1, \cdots, v_k, \delta^{k-1}(v_1)[v_2, \cdots, v_k])[v_{k+1}] \\ &\quad + D_{k+1} B^{(k)}(v_1, \cdots, v_k, \delta^{k-1}(v_1)[v_2, \cdots, v_k])[\delta^k(v_1)[v_2, \cdots, v_{k+1}]] \\ &= B^{(k+1)}(v_1, \cdots, v_{k+1}, \delta^k(v_1)[v_2, \cdots, v_{k+1}]). \end{aligned}$$

Hence, by uniqueness, $\delta^k(v_1)[v_2, \cdots, v_{k+1}] = h(v_1, \cdots, v_{k+1})$. Since $(v_1, \cdots, v_{k+1}) \to h(v_1, \cdots, v_{k+1})$ is of class sc^1, the map $(v_1, \cdots, v_{k+1}) \to \delta^k(v_1)[v_2, \cdots, v_{k+1}]$ is also of class sc^1. This implies that the map $v \mapsto \delta(v)$ is of class sc^{k+1} as claimed. The proof of the theorem is complete.

■

5.2. sc-Fredholm Germs and Perturbations

After some preparation we shall introduce the crucial concept of a germ of an sc-Fredholm section.

5.2.1. Linearized sc-Fredholm Germs. Consider two sc-smooth Banach spaces E and F and a strong sc-bundle

$$b : U \triangleleft F \to U$$

where U is an open subset of E. If q is a smooth point, i.e., if

$$q \in U_\infty = U \cap U_\infty$$

then we denote by $[b, q]$ or $[U \triangleleft F, q]$ the germ of b around the point q. Similarly, $[f, q]$ denotes the germ of an sc- smooth section of the bundle $[b, q]$ around the point q. If

$$b' : U' \triangleleft F' \to U'$$

is a second strong sc-bundle with the open set U' of E', we also choose a smooth point $q' \in U'$, take the germ $[b', q']$ around the point q' and consider a germ of an sc-smooth section $[g, q']$ of the bundle $[b', q']$. Assume that

$$[\Phi, q] : [b, q] \to [b', q']$$

is a germ of a strong sc-vector bundle isomorphism covering the germ $[\sigma, q]$ of the sc-diffeomorphism satisfying $\sigma(q) = q'$. It is of the form

$$\Phi(x, h) = (\sigma(x), \varphi(x)[h])$$

where our notation indicates that the fiber maps $h \to \varphi(x)[h]$ from F to F' are linear sc-isomorphisms. Now define the germ $[g, q']$ of a smooth sc-section of $[b', q']$ as the push-forward of the germ $[f, q]$ of section of the bundle $[b, q]$, as usual, by

$$g = \Phi_* f := \Phi \circ f \circ \sigma^{-1}.$$

If $\overline{f} : U \to F$ is the principal part of the section f and $\overline{g} : U' \to F'$ the principal part of the section g, then

$$\overline{g}(x) = pr_2 \circ \Phi \circ f \circ \sigma^{-1}(x) =: \varphi(x)[\overline{f} \circ \sigma^{-1}(x)].$$

Linearizing $\overline{g}$ at the smooth point $q' = \sigma(q)$ we obtain by the chain rule the formula

$$D\overline{g}(q')h = \varphi(q')[D\overline{f}(q) \circ D\sigma(q)^{-1}h] + A(q')[h],$$

where $A(q')[h]$ is the linearization of the map

$$x \to \varphi(x)[\overline{f}(q)] \in (F')^1 \tag{5.16}$$

at the smooth point q' in the direction $h \in E^1$. Now, since Φ is a strong sc-vector bundle map, and since q and hence also $\overline{f}(q)$ are smooth points, the map (5.16) is an sc-smooth map $U' \to (F')^1$. Therefore, the linear map $A(q') : E' \to F'$ is an sc$^+$-map in the sense of Definition 1.20. The linear map

$$h \mapsto \varphi(q')[D\overline{f}(q) \circ D\sigma(q)^{-1} \cdot h]$$

from E' to F' is sc-Fredholm in the sense of Definition 1.18 if and only if $D\overline{f}(q) : E \to F$ is an sc-Fredholm map. If this is the case, then we conclude from the stability of the Fredholm property in Proposition 1.21 that the linear map $D\overline{g}(q') : E' \to F'$ is also an sc-Fredholm map.

As a consequence we can define the following intrinsic property of the germ $[f, q]$.

Definition 5.5. *If q is a smooth point, then an sc-smooth section germ $[f, q]$ of the strong sc-germ $[b, q]$ is called* **linearized sc−Fredholm**, *if the linearization $D\overline{f}(q)$ of its principal part at q is sc-Fredholm in the sense of Definition 1.18.*

In view of the above discussion, the property of being linearized sc-Fredholm is invariant under (germs) of coordinate changes for strong sc-bundles. We summarize this fact as a proposition.

Proposition 5.6. *If q is a smooth point, and if $[f, q]$ is an sc-smooth germ of a section of a strong sc-vector bundle and $[g, p]$ is the push-forward by a strong sc-vector bundle isomorphism germ Φ, then $[f, q]$ is linearized sc-Fredholm if and only if this holds true for $[g, p]$.*

From the stability of the Fredholm property under perturbations, formulated in Proposition 1.21, we deduce the following result.

Proposition 5.7. *Let U be open in the sc-Banach space E and $U \triangleleft F \rightarrow U$ be a strong sc-vector bundle. Assume that $q \in U$ is smooth and $[f, q]$ is a sc-smooth germ. Assume that $[s, q]$ is a sc$^+$-smooth germ. If $[f, q]$ is linearized sc-Fredholm, then so is $[f + s, q]$.*

Proof. Taking the principal parts $\overline{f}$ and $\overline{s}$ we know that $D\overline{s}(q)$ is an sc$^+$-operator in the sense of Definition 1.20. Hence if $D\bar{f}(q)$ is an sc-Fredholm operator so is $D\bar{f}(q) + D\bar{s}(q)$ by Proposition 1.21 . ■

5.2.2. sc-Fredholm Germs. Given the strong sc-smooth bundle

$$b : U \triangleleft F \rightarrow U,$$

where $U \subset E$ is open, and given the smooth point $q \in U$, we consider the germ $[f, q]$ of an sc-smooth section around q.

Definition 5.8. *The sc-smooth germ $[f, q]$ of the strong sc-bundle germ $[b, q]$ is an* **sc−Fredholm germ**, *if the following two properties hold true.*

1) *Regularizing property: for every* $m \geq 0$*, there is a smooth open neighborhood* $O_m \subset U_m$ *of* q *so that if* $p \in O_m$ *and* $f(p) \in (U \triangleleft F)_{m,m+1}$*, then* $p \in U_{m+1}$*.*
2) *Contraction property: there exists a partial cone* V *in* $\mathbb{R}^n$*, an sc-smooth Banach space* W*, and a germ of a strong sc-vector bundle isomorphism*

$$\Psi : \mathcal{O}(U \triangleleft F, q) \to \mathcal{O}((V \oplus W) \triangleleft (\mathbb{R}^N \oplus W), 0)$$

which covers the sc-diffeomorphism germ $\sigma : \mathcal{O}(U, q) \to (V \oplus W, 0)$ *satisfying* $\sigma(q) = 0$ *and which has the following property. If* $pr_2 : (V \oplus W) \triangleleft (\mathbb{R}^N \oplus W) \to \mathbb{R}^N \oplus W$ *and* $P : \mathbb{R}^N \oplus W \to W$ *denote the canonical projections, then the sc-smooth germ*

$$\Psi(f) : \mathcal{O}(V \oplus W, 0) \to (W, 0)$$

$$\Psi(f)(v, w) := P \circ pr_2 \circ [\Psi \circ f \circ \sigma^{-1}(v, w) - \Psi(f(q))]$$

is an sc^0*-contraction germ in the sense of Definition 5.1.*

Proposition 5.9. *The sc-Fredholm germ* $[f, q]$ *is a linearized sc-Fredholm section in the sense of Definition 5.5.*

Proof. By assumption, the sc-smooth germ $\Psi(f)$ is of the form $\Psi(f)(v, w) = w - B(v, w)$. Linearizing at the point $(v, w) = 0$ we find

$$D\Psi(f)(0)[\widehat{v}, \widehat{w}] = \widehat{w} - D_2B(0)\widehat{w} - D_1B(0)\widehat{v}.$$

As in the proof of Theorem 5.3 one verifies that the linear operator $D_2B(0) : W_m \to W_m$ is a contraction for every $m \geq 0$, i.e., $\|D_2B(0)\|_m < 1$. Consequently,

$$\mathrm{Id} - D_2B(0) : W \to W$$

is an sc-isomorphism. Since V is finite dimensional and since P projects onto a subspace of finite codimension, it follows that the linearization of the principal part of the push-forward of the section $[f, q]$,

$$(v, w) \to \Psi \circ f \circ \sigma^{-1}(v, w),$$

at the point $(0, 0)$ is an sc-Fredholm operator. From Proposition 5.6 we conclude that $[f, q]$ is a linearized sc-Fredholm section, as claimed in Proposition 5.9. ■

Clearly, the property of being sc-Fredholm is also invariant under coordinate changes of strong sc-bundles.

Proposition 5.10. *Let* f *be an sc-smooth section of the bundle* $U \triangleleft F \to U$*. Assume that* $\Phi : \mathcal{O}(U \triangleleft F, q) \to \mathcal{O}(U' \triangleleft F', p)$ *is a germ of an sc-smooth vector bundle isomorphism. Then the section* $[f, q]$

is an sc-Fredholm germ if and only if the push-foward $[\Phi_* f, p]$ *is an sc-Fredholm germ.*

Finally, we are able to introduce sc-Fredholm sections.

Definition 5.11. *An sc-smooth section f of the strong sc-vector bundle $U \triangleleft F \to F$ is called* **sc− Fredholm** *if at every smooth point $q \in U$, the germ $[f, q]$ is a an sc-Fredholm germ.*

In view of the invariance under coordinate changes of strong sc-bundles, the definition immediately generalizes to strong sc-bundles over sc-manifolds.

5.3. Fillers and Local M-Polyfold Fredholm Germs

Recalling the cumulating notations first we consider the two splicings $\mathcal{S}_0 = (\pi, E, V)$ and $\mathcal{S}_1 = (\sigma, F, V)$ having the same parameter set V which is an open subset of a partial cone in $\mathbb{R}^n$. The spaces E and F are sc-smooth Banach spaces and

$$\pi_v : E \to E \quad \text{and} \quad \sigma_v : F \to F$$

are the projections parametrized by $v \in V$ as introduced in Section 2.1. For fixed v we shall abbreviate the image of the projection π_v in E by

$$K_{0,v} = \{e \in E |\, \pi_v(e) = e\}$$

and its complement in E by

$$K^c_{0,v} = \{e \in E |\, (\mathrm{Id} - \pi_v)(e) = e\}$$

so that for every $v \in V$

$$E = K_{0,v} \oplus K^c_{0,v}. \tag{5.17}$$

Every $e \in E$ splits accordingly to the splitting (5.17) into the sum

$$e = e^1_v \oplus e^2_v.$$

Similarly, the projection $\sigma_v : F \to F$ defines the splitting

$$F = K_{1,v} \oplus K^c_{1,v}. \tag{5.18}$$

Associated with the splicings $\mathcal{S}_0$ and $\mathcal{S}_1$ the spliced sc-fibered Banach scale

$$\mathcal{S}_{\triangleleft} = ((\pi, \sigma), E \triangleleft F, V)$$

has been introduced in Section 4.4. The $\triangleleft$-product $E \triangleleft F$ is the Banach space $E \oplus F$ equipped with the double filtration $(E \oplus F)_{m,k} = E_m \oplus F_k$

for $m \geq 0$ and $0 \leq k \leq m+1$. The splicing cores associated with $\mathcal{S}_0$ and $\mathcal{S}_1$ are the image bundles of the projections,

$$K_0 = K^{\mathcal{S}_0} = \{(v,e) \in V \oplus E|\ \pi_v(e) = e\}$$
$$K_1 = K^{\mathcal{S}_1} = \{(v,f) \in V \oplus F|\ \sigma_v(f) = f\}.$$

The total splicing core $K^{\mathcal{S}_\triangleleft}$ is their $\triangleleft$-product

$$K^{\mathcal{S}_\triangleleft} = K_0 \triangleleft K_1$$

which as a set is equal to $\{(v,e,f) \in |(V \oplus E) \triangleleft F|\ \pi_v(e) = e$ and $\sigma_v(f) = f\}$. The projection

$$(V \oplus E) \triangleleft F \to V \oplus E$$

induces the sc-smooth projection

$$K^{\mathcal{S}_\triangleleft} \to K^{\mathcal{S}_0}.$$

If $O \subset K^{\mathcal{S}_0}$ be open subset of the splicing core, we shall study the bundle

$$b : K^{\mathcal{S}_\triangleleft}|O \to O \tag{5.19}$$

which is the local model for a strong M-polyfold bundle according to Definition 4.7. A section

$$f : O \to K^{\mathcal{S}_\triangleleft}|O \tag{5.20}$$

is of the form

$$(v, e_v^1) \mapsto ((v, e_v^1), f(v, e_v^1))$$

where $f(v, e_v^1) \in K_{1,v} \subset F$. We use the same letter for the section and the principal part of the section. According to Definition 2.8, the section f is called sc-smooth if the extension

$$\begin{aligned} \bar{f} &: \widehat{O} \subset V \oplus E \to F \\ \bar{f}(v,e) &:= f(v, \pi_v(e)) \end{aligned} \tag{5.21}$$

is sc-smooth on the open subset $\widehat{O}$ of $V \oplus E$ defined by $\widehat{O} = \{(v,e) \in V \oplus E|\ (v, \pi_v(e)) \in O\}$. In the following we shall use the same letter f also for the extended function $\overline{f}$. An sc-smooth section germ $[f,q]$ of the bundle (b,q) consists of a smooth $q \in O$ and an sc-smooth germ f of section of the bundle $b : K^{\mathcal{S}_\triangleleft}|O \to O$.

Definition 5.12. *A* **filler for the sc-smooth section germ** $[f,q]$ *near a smooth point* $q \in V \oplus E$ *of the bundle* (b,q) *consists of an sc-smooth section germ* $[\widehat{f}, q]$ *of the bundle* $(V \oplus E) \triangleleft F \to V \oplus E$ *whose principal part has the form*

$$\widehat{f}(v,e) = f(v,e) + f^c(v,e). \tag{5.22}$$

Moreover, f^c is defined on an open neighborhood of q in $V\oplus E$, mapping its points (v,e) into $K^c_{1,v}$ in such a way that the mapping

$$\begin{aligned} K^c_{0,v} &\to K^c_{1,v} \\ e^2_v &\mapsto f^c(v, e^1_v \oplus e^2_v) \end{aligned} \tag{5.23}$$

is a linear sc-isomorphism for every v and e^1_v, where $e = e^1_v \oplus e^2_v$ is the decomposition with respect to the splitting $K_{0,v} \oplus K^c_{0,v} = E$.

By definition, $f(v,e) \in K_{1,v}$ and $f^c(v,e) \in K^c_{1,v}$. Hence one concludes from

$$f(v,e) + f^c(v,e) = 0$$

that $f(v,e) = 0$ and $f^c(v,e) = 0$. Consequently, $e^2_v = 0$ so that $e = \pi_v(e) = e^1_v$ and we see that the solution sets of the sections $[f,q]$ and $[\widehat{f},q]$ are naturally the same.

Definition 5.13. *The sc-smooth germ $[f,q]$ of a section of the local M-polyfold bundle (b,q) is called a* **polyfold Fredholm germ** *if it possesses a filler $\widehat{f}$ having the property that $[\widehat{f},p]$ is an sc-Fredholm germ in the sense of Definition 5.8 at every smooth point p near q.*

We point out that in dealing with germs around a smooth point q we may always assume by means of a germ of an sc-diffeomorphism that $q = 0$.

Setting $q = 0$ we consider the sc-smooth section germ f of the bundle $(b,0)$ defined in (5.19) and assume that its principal part satisfies $f(0) = 0$. At $v = 0$ we have the sc-smooth Banach spaces

$$\begin{aligned} K_{0,0} &= \{e \in E \mid \pi_0(e) = e\} \\ K_{1,0} &= \{f \in F \mid \sigma_0(f) = f\}. \end{aligned}$$

Recalling the splitting $E = K_{0,0} \oplus K^c_{0,0}$ and the corresponding decomposition $e = e^1_0 + e^2_0$ we observe that $f(0, e^2_0) = 0$ for all $e^2_0 \in K^c_{0,0}$. Therefore, the linearization of f at the point $0 = (v,e)$ is the sc-linear map

$$\begin{aligned} Df(0) : T_0V \oplus E &\to K_{1,0} \\ (\widehat{v}, e) &\mapsto Df(0)[\widehat{v}, e^1_0], \end{aligned}$$

where $T_0V = \mathbb{R}^n$. Now assume that the section f possesses the filler $\widehat{f}$. Then the linearization of $\widehat{f}$ at the point 0 is the sc-linear mapping

$$
\begin{aligned}
&D\widehat{f}(0) : T_0V \oplus E \to F \\
&D\widehat{f}(0)[\widehat{v}, e)] = Df(0)[\widehat{v}, e] + f^c(0, e_0^2)
\end{aligned}
\tag{5.24}
$$

where again $e = e_0^1 + e_0^2$ corresponds to the splitting $E = K_{0,0} \oplus K_{0,0}^c$. Since $f^c(0, e_0^2) \in K_{1,0}^c$ we see that $Df(0)$ is surjective if and only if the linearized filler $D\widehat{f}(0)$ is surjective. Because $f^c(0, e_0^2) = 0$ if and only if $e_0^2 = 0$, we conclude

$$
\ker\, D\widehat{f}(0) \subset T_0V \oplus K_{0,0}
$$

and

$$
\ker(Df(0)|T_0V \oplus K_{0,0}) = \ker D\widehat{f}(0).
$$

5.4. Local solutions of polyfolds Fredholm germs

We shall demonstrate that the solution set of $f = 0$ of a polyfold Fredholm germ near a transversal smooth point carries in a natural way a smooth manifold structure. Using the infinitesimal implicit function theorem from Section 5.1, we shall reduce the problem by means of a Lyapuonov-Schmidt type construction to a finite dimensional problem. We distinguish between the interior and boundary case.

5.4.1. Interior Case. We consider the polyfold Fredholm section germ $[f, 0]$ according to Definition 5.13. Its principal part

$$
f(v, e) = f(v, \pi_v(e)) \tag{5.25}
$$

is sc-smooth in a neighborhood of 0 in $V \oplus E$. We assume

$$
f(0) = 0
$$

and first study the special case in which the parameter set V of the splicing is an open neighborhood of the origin in $\mathbb{R}^k$. By the definition of a polyfold Fredholm germ there exists a filler $\widehat{f}$ of f which is a smooth section

$$
\widehat{f} : \mathcal{O}(V \oplus E, 0) \to \mathcal{O}((V \oplus E) \triangleleft F, 0)
$$

whose principal part is of the form

$$
\widehat{f}(v, e) = f(v, e_v^1) + f^c(v, e_v^1 + e_v^2),
$$

the sum $e_v^1 + e_v^2$ corresponding to the splitting $E = K_{0,v} \oplus K_{0,v}^c$ introduced in (5.17). We shall assume now that the point $0 = (v, e)$ is a transversal point of the section f requiring the linearisation of f at 0,

$$
Df(0) : T_0V \oplus E \to F
$$

to be surjective. Note that $T_0V = \mathbb{R}^k$. Equivalently, the linearisation of the filler $\widehat{f}$ at 0,

$$D\widehat{f}(0) : T_0V \oplus E \to F$$

is surjective. Since $[\widehat{f}, 0]$ is, by assumption, an sc-Fredholm germ there exists a germ of a strong sc-vector bundle diffeomorphism

$$\Phi : \mathcal{O}((V \oplus E) \triangleleft F, 0) \to \mathcal{O}((\mathbb{R}^n \oplus W) \triangleleft (\mathbb{R}^N \oplus W), 0)$$

so that the push forward section $g = \Phi_* \widehat{f}$, if composed with the projection pr_2 onto its principal part and the natural projection $P : \mathbb{R}^N \oplus W \to W$, is a contraction germ according the Definition 5.1. Hence, for $(a, u) \in \mathbb{R}^n \oplus W$ near the origin,

$$P \circ \mathrm{pr}_2 \circ g(a, u) = u - B(a, u). \tag{5.26}$$

Explicitly, if

$$\Phi(v, e, h) = (\sigma(v, e), \varphi(v, e) \cdot h) \tag{5.27}$$

with a diffeomorphism germ $\sigma : \mathcal{O}(V \oplus E, 0) \to \mathcal{O}(\mathbb{R}^n \oplus W, 0)$ satisfying $\sigma(0) = 0$, the section has the principal part $g : \mathcal{O}(\mathbb{R}^n \oplus W, 0) \to \mathcal{O}(\mathbb{R}^N \oplus W, 0)$ denoted by the same letter and defined by

$$g(a, u) = \varphi(\sigma^{-1}(a, u)) \cdot \widehat{f}(\sigma^{-1}(a, u)).$$

It satisfies $g(0) = 0$. Abbreviating

$$\psi(a, u) = \varphi(\sigma^{-1}(a, u))$$
$$\tau(a, u) = \sigma^{-1}(a, u),$$

the principal part becomes

$$g(a, u) = \psi(a, u) \cdot \widehat{f}(\tau(a, u)). \tag{5.28}$$

Using $f(0) = 0$ the linerization of g at $0 = (a, u)$ is the sc-linear mapping

$$Dg(0)[b, h] = \psi(0) \cdot D\widehat{f}(0) \circ D\tau(0)[b, h]$$

for $(b, h) \in \mathbb{R}^n \oplus W$. By the assumption of transversality, the linear operator $Df(0)$ and hence also $D\widehat{f}(0)$ is surjective. Since $D\tau(0)$ and $\psi(0)$ are linear isomorphisms, the map

$$Dg(0) : \mathbb{R}^n \oplus W \to \mathbb{R}^N \oplus W$$

is also surjective. The solution set for g solves the equation

$$g(a, u) = 0$$

or, equivalently, the two equations

$$\begin{aligned} Pg(a,u) &= 0 \\ (\mathrm{Id}-P)g(a,u) &= 0. \end{aligned} \tag{5.29}$$

Since $Pg(a,u) = u - B(a,u)$, the solutions of the first equation near 0, are, in view of Theorem 5.3, represented as the graph of a function $\delta : \mathbb{R}^n \to W$ so that

$$Pg(a,\delta(a)) = 0. \tag{5.30}$$

In fact these are all the solutions of the first equation locally near $0 = (a,u)$. The function δ satisfies $\delta(0) = 0$ and possesses, in view of Theorem 5.4, the following regularity properties. Given any level m and any integer $j \geq 0$, there exists an open neighborhood $O = O(m,j) \subset \mathbb{R}^n$ of the origin so that if $a \in O$, then $\delta(a) \in W_m$ and the map $\delta : O \to W_m$ is of class C^j. It remains to solve the second equation in (5.29) which becomes

$$(\mathrm{Id}-P)g(a,\delta(a)) = 0$$

and which is to be solved for a near the origin in $\mathbb{R}^n$. We define the mapping

$$G : \mathbb{R}^n \to \mathbb{R}^N$$

near $0 \in \mathbb{R}^n$ by

$$G(a) = (\mathrm{Id}-P)g(a,\delta(a)). \tag{5.31}$$

Given an integer $j \geq 1$, there exists an open neighborhood O of the origin in $\mathbb{R}^n$ on which $G \in C^j(O,\mathbb{R}^N)$.

LEMMA 5.14. *The map $DG(0) \in \mathcal{L}(\mathbb{R}^n,\mathbb{R}^N)$ is surjective.*

PROOF. The linearization of G at the point 0 is equal to

$$DG(0)[b] = (\mathrm{Id}-P)Dg(0)\cdot[b,\delta'(0)b] \tag{5.32}$$

for $b \in \mathbb{R}^n$. Given $(r,0) \in \mathbb{R}^N \oplus W$ we can solve the equation

$$(r,0) = Dg(0)[b,h]$$

for $(b,h) \in \mathbb{R}^n \oplus W$, in view of the surjectivity of $Dg(0)$. Equivalently, there exists $(b,h) \in \mathbb{R}^n \oplus W$ solving the two equations

$$\begin{aligned} r &= (\mathrm{Id}-P)Dg(0)[b,h] \\ 0 &= PDg(0)[b,h]. \end{aligned}$$

Explicitly, the second equation is the following equation

$$0 = -D_1B(0)\cdot b + [\mathrm{Id} - D_2B(0)]\cdot h \tag{5.33}$$

Recalling the proof of Theorem 5.3, the operator $\mathrm{Id}-D_2B(0): W\to W$ is a linear isomorphism. Therefore, given $b\in\mathbb{R}^n$ the solution h of the equation (5.33) is uniquely determined. On the other hand, linearizing $Pg(a,\delta(a))=0$ at the point $a=0$ leads to $PDg(0)[b,\delta'(0)b]=0$ for all $b\in\mathbb{R}^n$. Hence, by uniqueness $h=\delta'(0)b$, so that the linear map $DG(0)$ in (5.32) is indeed surjective. ■

We denote the kernel of $DG(0)$ by C and take its orthogonal complement $C^\perp$ in $\mathbb{R}^n$, so that

$$C\oplus C^\perp=\mathbb{R}^n.$$

Then $G: C\oplus C^\perp\to\mathbb{R}^N$ becomes a function of two variables, $G(c_1,c_2)=G(c_1+c_2)$ for which $D_1G(0)=0$ while $D_2G(0)\in\mathcal{L}(C^\perp,\mathbb{R}^N)$ is an isomorphism. By the implicit function theorem there exists a unique map $c: C\mapsto C^\perp$ solving

$$G(r+c(r))=0 \tag{5.34}$$

for r near 0 and satisfying

$$c(0)=0,\quad Dc(0)=0.$$

Moreover, given any $j\geq 1$, there is an open neighborhood U of the origin in C such that $c\in C^j(U,C^\perp)$. Summarizing we have demonstrated so far that all solutions $(a,u)\in\mathbb{R}^n\oplus W$ of $g(a,u)=0$ near the origin are represented by

$$g(r+c(r),\delta(r+c(r)))=0$$

for r in an open neighborhood of 0 in $C=\ker DG(0)\subset\mathbb{R}^n$. Consequently, in view of formula (5.28), all solutions $(v,e)\in V\oplus E$ of $\widehat{f}(v,e)=0$ near the origin are represented by

$$\widehat{f}(\beta(r))=0$$

where $\beta: C\to\mathbb{R}^k\oplus E$ is defined near 0 by

$$\beta(r)=\tau(r+c(r),\delta(r+c(r))).$$

The function β satisfies $\beta(0)=0$ and, of course, for every level m and every integer $j\geq 1$, there exists an open neighborhood U of $0\in C$ such that if $r\in U$, then $\beta(r)\in\mathbb{R}^k\oplus E_m$ and $\beta\in C^j(U,\mathbb{R}^k\oplus E_m)$. From the definition of the filler it follows that $\beta(r)\in K_0$. From the regularizing property in Definition 5.8 of an sc-Fredhom germ we conclude $\beta(r)\in\mathbb{R}^k\oplus E_\infty$ so that

$$\beta(r)\in(\mathbb{R}^k\oplus E_\infty)\cap K_0.$$

Since the solution set of the filler $\widehat{f}$ and of f are the same, all the solutions (v,e) of $f(v,e)=0$ near the origin are also represented by

$$f(\beta(r))=0$$

for r near 0 in C. In order to represent the solution set as a graph over the kernel of the linearized equation at 0 we introduce

$$\begin{aligned} N :&= \ker D\widehat{f}(0) \\ &= \ker Df(0)|T_0V \oplus K_{0,0}. \end{aligned} \tag{5.35}$$

In view of Proposition 5.9, the linear operator $D\widehat{f}(0) \in \mathcal{L}(\mathbb{R}^k \oplus E, \mathbb{R}^N \oplus W)$ is an sc-Fredholm operator in the sense of Definition 1.18. Hence dim $N<\infty$ and $N \subset \mathbb{R}^k \oplus E_\infty$ and we have an sc-splitting

$$N \oplus N^c = \mathbb{R}^k \oplus E.$$

By construction, the image of the linearization $D\beta(0) \in \mathcal{L}(C, \mathbb{R}^k \oplus E)$ is equal to the kernel of $D\widehat{f}(0)$. Moreover, from $D\beta(0)\cdot\widehat{r} = d\tau(0)[\widehat{r}, \delta'(0)\widehat{r}]$ it follows that $D\beta(0)$ is also injective, so that $D\beta(0): C\to N$ is a linear isomorphism. We define the map $\alpha: N\to \mathbb{R}^k\oplus E$ near $0\in N$ by

$$\alpha(n) = \beta(D\beta(0)^{-1}\cdot n).$$

The solution set is now parametrized by $f(\alpha(n))=0$ for n near $0\in N$. Let $Q: N\oplus N^c \to N$ be the natural projection and consider the map $Q\circ\alpha : N\to N$ near 0. Since $D(Q\circ\alpha)(0)=\mathrm{Id}$, it is a local diffeomorphism leaving the origin fixed. We denote by γ the inverse of this local diffeomorphism satisfying $Q\circ\alpha(\gamma(n))=n$ for all small n. Then

$$\begin{aligned} \alpha(\gamma(n)) &= Q\circ\alpha(\gamma(n)) + (\mathrm{Id}-Q)\alpha(\gamma(n)) \\ &= n + (\mathrm{Id}-Q)\alpha(\gamma(n)). \end{aligned}$$

The map

$$A: N\to N^c$$

near the origin in N is defined by $A(n) = (\mathrm{Id}-Q)\alpha(\gamma(n))$. It satisfies $A(0)=0$ and $DA(0)=0$ and $\alpha(\gamma(n)) = n + A(n)$ so that

$$f(n+A(n))=0.$$

In addition, given any level m and any integer j there exists an open neighborhood U of 0 in N such that if $n\in U$, then $A(n)\in N^c_m$ and $A\in C^j(U, N^c_m)$. We have demonstrated that the solution set of $f(v,e)=0$ near the transversal point 0 is represented as a graph over the kernel of the linearized map $Df(0)$ of a function which is smooth at the point 0.

Proposition 5.15. *Let $V \subset \mathbb{R}^k$ be an open neighborhood of the origin and let $[f,0]$ be a polyfold Fredholm section germ according to Definition 5.13 satisfying $f(0)=0$. Let N be the kernel of $Df(0)|T_0V \oplus K_{0,0}$. Then dim $N < \infty$ and $N \subset \mathbb{R}^k \oplus E_\infty$ and there is an sc-complement N^c so that*

$$N \oplus N^c = \mathbb{R}^k \oplus E.$$

If $Df(0)$ is surjective, then there exists a germ $A : \mathcal{O}(N,0) \to \mathcal{O}(N^c,0)$ near $0 \in N$ satisfying $A(0)=0$ and $DA(0)=0$ such that for n near 0,

$$f(n + A(n)) = 0$$
$$n + A(n) \in (\mathbb{R}^k \oplus E_\infty) \cap K_0.$$

All solutions of $f(v,e)=0$ in a sufficiently small neighborhood of the origin in $\mathbb{R}^k \oplus E$ are in the image of the map $n \mapsto n + A(n)$. In addition. given any level m and any integer r j, there exists an open neighborhood $U = U(m,j) \subset N$ of the origin on which $A \in C^j(U, N^c_m)$.

So far we have only used the fact that the filler $\widehat{f}$ induces an sc-Fredholm germ $[\widehat{f},0]$ in the sense of Definition 5.8 at the point 0. We next make use of the assumption that $[\widehat{f},q]$ is an sc-Fredholm germ at every smooth point q in an open neighborhood of 0 in order to show that the map $n \mapsto n + A(n)$ is smooth on an open neighborhood of the origin in N. We fix a level m and a sufficiently small neighborhood O of the origin in N on which the map $A : O \to N^c_m$ is of class C^1. Choose a point $n_0 \in N$ near 0 and set

$$q_0 = n_0 + A(n_0).$$

Then q_0 is a smooth point and we shall apply Proposition 5.15 to the point q_0 replacing the point $q=0$ in there. This is possible in view of the next lemma.

Lemma 5.16. *If n_0 is small enough, the linearization $D\widehat{f}(q_0)$: $\mathbb{R}^k \oplus E \to F$ is surjective and Fredholm. Moreover, dim $[kerD\widehat{f}(q_0)] =$ dim $[ker\ D\widehat{f}(0)] =$ dim N. Setting $N' = kerD\widehat{f}(q_0)$ the natural projection $p : N \oplus N^c = \mathbb{R}^k \oplus E \to N$ induces an isomorphism $p|N' : N' \to N$.*

Proof. We use the notation in the proof of Proposition 5.15 and consider equivalently the section $g(a,u)$ defined in (5.28) and its linearization

$$Dg(q_0) : \mathbb{R}^n \oplus W \to \mathbb{R}^N \oplus W \tag{5.36}$$

at the point $q_0 = (a, \delta(a))$. In order to prove surjectivity we consider the equation $Dg(q_0)[h, b] = [w, r]$ which in matrix notation according to the splitting in (5.36) becomes

$$\begin{bmatrix} A & A_1 \\ A_2 & A_3 \end{bmatrix} \cdot \begin{bmatrix} h \\ b \end{bmatrix} = \begin{bmatrix} w \\ r \end{bmatrix} \tag{5.37}$$

with the linear operators

$$\begin{aligned} Ah &= [\mathrm{Id} - D_2B(a, \delta(a)] \cdot h \\ A_1 b &= D_1 B(a, \delta(a)) \cdot b \\ A_2 h &= (\mathrm{Id} - P) D_2 g(a, \delta(a)) \cdot h \\ A_3 b &= (\mathrm{Id} - P) D_1 g(a, \delta(a)) \cdot b \end{aligned}$$

where $(b, h) \in \mathbb{R}^n \oplus W$ and $(r, w) \in \mathbb{R}^N \oplus W$. In view of the proof of Theorem 5.3, the linear operator $A : W \to W$ is an sc-isomorphism. Hence the equation (5.37) for (h, b) becomes

$$\begin{aligned} h &= A^{-1}w - A^{-1}A_1 b \\ [A_2A^{-1}A_1 - A_3]b &= A_2 A^{-1} w + r. \end{aligned} \tag{5.38}$$

We abbreviate the continuous family of matrices

$$M(a) := [A_2 A^{-1} A_1 - A_3] \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^N).$$

By assumption, $Dg(0)$ is surjective. Therefore, if $a = 0$, then the matrix $M(0)$ is surjective. Consequently, for small a the matrix $M(a)$ is also surjective and so is the linear operator $Dg(a, \delta(a))$. Choosing $w = 0$ and $r = 0$ in (5.38), the kernel of $Dg(q_0)$ is determined by the two equations

$$\begin{aligned} h &= A^{-1}A_1 b \\ [A_2 A^{-1} A_1 - A_3]b &= 0. \end{aligned}$$

Consequently, the kernel is determined by the kernel of the matrix $M(a)$. Setting $K' = \ker Dg(q_0)$ and $K = \ker Dg(0)$ we conclude from the surjectivity of $M(a)$ for small a that $\dim K' = \dim K$. In addition, the natural projection $p : K \oplus K^c = \mathbb{R}^k \oplus E \to K$ induces the isomorphism $p|K' : K' \to K$ and the lemma follows. ■

Introducing

$$N' = \ker D\widehat{f}(q_0)$$

we have $\dim N' = \dim N$ and Proposition 5.15 guarantees a map $n' \mapsto n' + A(n') : N' \to N' \oplus (N')^c$ defined for n' near 0, satisfying $A'(0) = 0$ and $DA'(0) = 0$, and solving

$$f(q_0 + n' + A'(n')) = 0.$$

Moreover, given integers m and j there is an open neighborhood U of the origin in N' on which $A' : U \to (N')^c_m$ is of class C^j. Denote by

$$p : N \oplus N^c \to N$$

the natural sc-projection. Then $p|N' : N' \to N$ is an isomorphism if q_0 is sufficiently close to 0, and we define the map $\alpha : N' \to N$ near the origin by

$$\alpha(n') = p[q_0 + n' + A'(n')].$$

Then $\alpha(0) = n_0$ and, by the uniqueness of the solution set of $f = 0$ near the origin, we have $\alpha(n') + A(\alpha(n')) = q_0 + n' + A'(n')$. In addition, $D\alpha(0)h = p(h)$ for $h \in N'$. Consequently, $D\alpha(0) \in \mathcal{L}(N', N)$ is an isomorphism and hence the map α is a local diffeomorphism. In addition, given any integer j there is an open neighborhood U of $0 \in N'$ on which $\alpha \in C^j(U, N)$. From the identity

$$n + A(n) = p[q_0 + \alpha^{-1}(n) + A'(\alpha^{-1}(n))]$$

for n near n_0 we conclude for given integers m and j that there is an open neighborhood $U(n_0)$ of $n_0 \in N$ on which the map $n \mapsto n + A(n)$ belongs to $C^j(U(n_0), \mathbb{R}^k \oplus E_m)$. In particular, at the point n_0, the map is smooth. The arguments apply to every n_0 in a neighborhood of the origin in N. Therefore, we have demonstrated the following result, where in abuse of the notation we simply write

$$Df(0) = Df(0)|T_0V \oplus K_{0,0}$$

and use the same letter f for the section and for its principal part.

THEOREM 5.17. *Let $[f, 0]$ be a polyfold Fredholm germ as defined in Definition 5.13. Assume $f(0) = 0$ and denote by N the kernel of the linearization $Df(0)$. Then dim $N < \infty$ and $N \subset \mathbb{R}^k \oplus E_\infty$ and there exists an sc-smooth complement N^c so that*

$$N \oplus N^c = \mathbb{R}^k \oplus E.$$

If $Df(0)$ is surjective, there exists an open neighborhood O of the origin in N and a smooth map $A : O \to N^c$ satisfying $A(0) = 0$ and $DA(0) = 0$ such that

$$f(n + A(n)) = 0$$
$$n + A(n) \in (\mathbb{R}^k \oplus E_\infty) \cap K_0$$

if $n \in O$. Moreover, $A \in C^\infty(O, N^c_m)$ for every integer m. All solutions of $f(v, e) = 0$ in a sufficiently small neighborhood of the origin in $\mathbb{R}^k \oplus E$ are in the image of the map $n \mapsto n + A(n)$.

5.4.2. Boundary Case. So far, the parameter set V has been an open neighborhood of the origin in $\mathbb{R}^n$. In this subsection we consider the case in which V is an open neighborhood of the origin in the partial cone

$$C = [0, \infty)^k \times \mathbb{R}^{n-k}$$

in $\mathbb{R}^n$. There are many possible concepts for a submanifold in the presence of boundaries or, more generally, of boundary with corners. Later on we shall use a relatively strong concept called neat submanifolds. It has the advantage of a high degree of compatibility with the corner structure.

We consider the polyfold-Fredholm germ $[f, 0]$ of the bundle $\mathcal{O}(K_0 \triangleleft_V K_1, 0)$ satisfying $f(0) = 0$, and abbreviate the linearization at 0 by

$$Df(0) = Df(0)|\mathbb{R}^n \oplus K_{0,0}.$$

Note that $\mathbb{R}^n = T_0 V$ is the tangent space of the parameter set V at the origin. The kernel

$$N = \ker\, Df(0)$$

is a finite dimensional subspace of $\mathbb{R}^n \oplus K_{0,0}$. We denote by

$$p : \mathbb{R}^n \oplus K_{0,0} \to \mathbb{R}^n$$

the canonical projection.

Definition 5.18. *The polyfold-Fredholm germ* $[f, 0]$ *is called* **neat***, if the kernel* N *of the linearization* $Df(0)$ *has the property*

$$\mathbb{R}^k \times \{0\} \subset p(N).$$

We note that the subspace $\mathbb{R}^k \times \{0\}$ is the tangent space at 0 of the part of V containing the corner.

Definition 5.19. *If* $[f, 0]$ *is neat and* N *the kernel of* $Df(0)$ *in* $\mathbb{R}^n \oplus K_{0,0}$*, then an sc-complement* R *of* N*,*

$$\mathbb{R}^n \oplus K_{0,0} = N \oplus R,$$

is called **neat***, if*

$$R \subset (\{0\} \times \mathbb{R}^{n-k}) \oplus K_{0,0}.$$

Such a neat sc-complement does always exist. Indeed, take a k-dimensional subspace $Z \subset N$ satisfying $p(Z) = \mathbb{R}^k \times \{0\}$. Then the subspace $Z^c \subset N$, determined by $Z^c = N \cap \big((\{0\} \times \mathbb{R}^{n-k}) \oplus K_{0,0}\big)$ has dimension $\dim\, N - k$ and satisfies $Z \oplus Z^c = N$. Let R be an sc-complement of Z^c satisfying $Z^c \oplus R = (\{0\} \times \mathbb{R}^{n-k}) \oplus K_{0,0}$. Then $\mathbb{R}^n \oplus K_{0,0} = N \oplus R$ as claimed.

THEOREM 5.20. *Let $[f, 0]$ be a neat polyfold Fredholm section germ of a strong local M-polyfold bundle satisfying $f(0) = 0$. If N is the kernel of the linearization $Df(0)$ in $\mathbb{R}^n \oplus K_{0,0}$, then dim $N < \infty$ and $N \subset \mathbb{R}^n \oplus E_\infty$ and there exists a neat sc-complement R such that*

$$N \oplus R = \mathbb{R}^n \oplus K_{0,0}.$$

If $Df(0)$ is surjective, then there exists an open neighborhood U of the origin in $N \cap (C \oplus K_{0,0})$ and an sc-smooth map

$$A : U \to R$$

satisfying $A(0) = 0$ and $DA(0) = 0$ such that

$$f(n + A(n)) = 0$$
$$n + A(n) \in K_0 \cap (\mathbb{R}^n \oplus E_\infty)$$

for all $n \in N$. Moreover, $A \in C^\infty(U, R_m)$ for every level m. All solutions of $f(v, e) = 0$ in a small neighborhood of the origin in $C \oplus E$ are in the image of the map $n \mapsto n + A(n)$.

PROOF. In view of the neatness requirement and recalling the linearization concept in the presence of corners from Remark **??**, the proof follows the lines of the proof of Theorem 5.17 with minor modifications. We keep the same notation as in the proof Theorem 5.17. Since this time we are interested in parametrizing solutions $(v, e) \in V \oplus E$ of $f(v, e) = 0$ where (v, e) is near the corner 0, we have to make sure that the maps defined in the proof of Theorem 5.17 are defined on open neighborhoods of 0 whose projections onto $\mathbb{R}^k$ are open neighborhoods of 0 in $[0, \infty)^k$. Then the partial derivatives of these maps in the direction of $e_1, \cdots, e_k$ of the standard basis for $\mathbb{R}^k$ are defined, and so these maps have well-defined linerizations at 0.

This time the diffeomorphism germ $\tau : \mathcal{O}(\mathbb{R}^n \oplus W, 0) \to (V \oplus E, 0)$ which enters the definition of the sc-strong bundle diffeomorphism Φ (5.27) and the definition of the map g in (5.28), is a map

$$\tau : [0, \infty)^l \times \mathbb{R}^{m-l} \oplus W \to [0, \infty)^k \times \mathbb{R}^{n-k} \oplus E$$

satisfying $\tau(0) = 0$. Since τ is a diffeomorphism, it follows that $k = l$. We abbreviate $C = [0, \infty)^k \times \mathbb{R}^{n-k}$ and $C_0 = [0, \infty)^k \times \mathbb{R}^{m-k}$. Setting $\tau = (\tau_1, \tau_2, \tau_3)$ where τ_1, τ_2, and τ_3 take their values in $[0, \infty)^k$, $\mathbb{R}^{n-k}$ and E respectively, the derivative of the map $\tau_1 : [0, \infty)^k \times \mathbb{R}^{m-k} \oplus W$ satisfies $D_2\tau_1(0) = 0$ and $D_3\tau_1(0) = 0$. Moreover, the derivative $D_1\tau_1(0) : \mathbb{R}^k \to \mathbb{R}^k$ is an isomorphism. In addition, since $\tau_1 \geq 0$, we conclude that $D_1\tau_1(0)[0, \infty)^k = [0, \infty)^k$ so that

$$D\tau(0)(C_0 \oplus W) = C \oplus E.$$

We are interested in solutions (a, u) of the equation $g(a, u) = 0$ for $(a, u) \in C \oplus W$ close to 0. As before $g(a, u) = 0$ is equivalent to the two equations

$$\begin{aligned} Pg(a, u) &= 0 \\ (1 - P)g(a, u) &= 0 \end{aligned}$$

where $P : \mathbb{R}^N \oplus W \to W$ is the canonical projection. In view of Theorem 5.3, the solutions of the first equation are of the form $(a, \delta(a))$ with $\delta : C_0 \to W$. The map δ has, by Theorem 5.4, the following smoothness property. For all integers m and j there is a neighborhood O depending on j and the level m such that δ is of class $C^j(O, W_m)$.

We introduce the map $G : C_0 = [0, \infty)^k \times \mathbb{R}^{m-k} \to \mathbb{R}^N$ defined by

$$G(a) = (\mathrm{I} - P)g(a, \delta(a))$$

for $a \in C_0$ and close to 0. As in the proof of Lemma 5.14 the linear map $DG(0) : R^k \times \mathbb{R}^{m-k} \to \mathbb{R}^N$ is surjective. Its kernel $K \subset \mathbb{R}^m$ has dimension equal to $N - m$ and is isomorphic to the kernel N of the linearization $D\widehat{f}(0) : \mathbb{R}^n \oplus E \to \mathbb{R}^N \oplus W$. The isomorphism between K and N is provided by the linear map $h \mapsto D\tau(0)[h, \delta'(0)h]$ for $h \in K$. The linear map

$$DG(0) : \mathbb{R}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^N$$

is a surjection and so there are N columns $j_1 < j_2 < \ldots, j_N$ of $DG(0)$ which are linearly independent. The vectors $D_1\tau_1(0)e_1, \cdots D_1\tau_1(0)e_k$, where $\{e_1, \cdots, e_k\}$ is the standard basis of $\mathbb{R}^k$ span $\mathbb{R}^k$, and we conclude, in view of the neatness assumption, that $j_1 > k$. Write the set $\{1, \ldots, m\}$ as the disjoint union of $J = \{j_1, \ldots, j_N\}$ and $I = \{i_1, \ldots i_{m-N}\}$ with $i_1 < i_2 < \cdots < i_{m-N}$. Since $j_1 > k$, we have $i_1 = 1, i_2 = 2, \cdots, i_k = k$. Define $\mathbb{R}^I = A_1 \times \cdots \times A_m$ and $\mathbb{R}^J = B_1 \times \cdots B_m$ where $A_i = \mathbb{R}$ and $B_i = \{0\}$ if $i \in I$ and $A_i = \{0\}$ and $B_i = \mathbb{R}$ if $i \notin I$. Consider the map $\widehat{G} : (\mathbb{R}^I \cap C_0) \times \mathbb{R}^J \to \mathbb{R}^N$ defined by

$$\widehat{G}(c_1, c_2) = G(c_1 + c_2).$$

The linearization of G at the point $(c_1, c_2) = (0, 0)$ with respect to the variable c_2, $D_2\widehat{G}(0) : \mathbb{R}^J \to \mathbb{R}^N$, is given by $D\widehat{G}(0)h = DG(0)[0, h])$ for all $h \in R^J$, and so it is an isomorphism. Applying the implicit function theorem, we find open sets $U_0 \subset \mathbb{R}^I \cap C_0$ containing 0 and $V_0 \subset \mathbb{R}^J$ containing 0, and a map $c : U_0 \to V_0$ such that $c(0) = 0$ and $\widehat{G}(r, c(r)) = G(r + c(r)) = 0$. Hence all the solutions $(v, e) \in C \oplus E$ of $\widehat{f}(v, e) = 0$ near the origin are of the form

$$\beta(r) = \tau(r + c(r), \delta(r + c(r))$$

with $r \in U_0 \subset R^I \cap C$. Since the zero set of the filler $\widehat{f}$ coincides with the zero set of f we see that

$$f(\beta(r)) = 0$$

for all $r \in U$. Now the linearization $D\beta(0) : \mathbb{R}^I \to C \oplus E$ is clearly injective, and maps $\mathbb{R}^I$ onto the kernel N of the linearization $D\widehat{f}(0)$. Hence $D\beta(0) : \mathbb{R}^I \to N$ is an isomorphism. In addition, if $\overline{p} : \mathbb{R}^k \times \mathbb{R}^{n-k} \oplus E \to \mathbb{R}^k$ is the canonical projection, then $\overline{p}(D\beta(0)C_0) = [0, \infty)^k$. Hence setting $U_1 = D\beta(0)U_0 \subset N$ we see that $\overline{p}(U_1)$ is an open neighborhood of 0 in $[0, \infty)^k$. Then we introduce the map $\alpha : U_1 \to C \oplus E$ by

$$\alpha(n) = \beta(D\beta(0)^{-1}n).$$

So that the zero set of f is parametrized by $\alpha(n)$ for $n \in U_1$.

In view of the neatness assumption there is a neat complementary subspace $R \in \mathbb{R}^n \oplus K_{0,0}$ so that $N \oplus R = \mathbb{R}^n \oplus K_{0,0}$. With the canonical projection $Q : N \oplus R \to N$ we have $Q \circ \alpha(n) = n$ for $n \in U_1$ and $D(Q \circ \alpha)(0) = \mathrm{Id}$. Hence there is an open neighborhood U of 0 in $C \oplus K_{0,0}$ and map $\gamma : U \cap N \to U_1$ such that $(Q \circ \alpha)(\gamma(n)) = n$ for all $n \in U \cap N$. The map $A : N \to R$ takes the form

$$A(n) = (\mathrm{I} - Q)(\alpha \circ \gamma)(n)$$

defined for $n \in U \cap N$. The solution of $f = 0$ near the corner 0 are of the form $n + A(n)$ so that

$$f(n + A(n)) = 0$$

for $n \in U \cap N$. Clearly, $A(0) = 0$ and $DA(0) = 0$. Moreover, the map A has the following smoothness property. For every integers j and m there is an open neigborhood O (depending on j and m) of 0 in $C \oplus K_{0,0}$ such that $A \in C^j(O \cap N, R_m)$. In particular, A is smooth at the corner 0. Summing up, we have shown that the solution set $(v, e) \in C \oplus K_{0,0}$ of $f = 0$ near the corner 0 is represented as a graph of a map A defined on an open set in $U_2 \cap N$ and taking values in R. The set U_2 is an open neighborhood of 0 in $C \oplus K_{0,0}$. Moreover, the map A is smooth at 0.

To finish the proof it remains to show that the map A is smooth at any other point of $U \cap N$ which is close to 0. Fix a level m. Without loss of generality we may assume that A is of class C^1 on $U_2 \cap N$. If $q \in C \oplus E$ is close to the origin and $f(q) = 0$, then $q = n + A(n)$ for the unique $n \in U_2 \cap N$. There is $a \in C_0$ so that $q = \tau(a, \delta(a))$. Abbreviate by $K(a)$ be the kernel of $DG(a)$ and let $N(a)$ be the kernel of $D\widehat{f}(q)$ where $q = n + A(n) = \tau(a, \delta(a))$. We already know that the map $h \mapsto D\tau(a)[h, \delta'(0)h]$ is an isomorphism between $K(a)$ and $N(a)$. We will show that if $a \in C$ is close to 0, then $N \oplus R = N(a) \oplus R$ and

$\mathbb{R}^k \times \{0\} \subset p(N(a))$ where $p : \mathbb{R}^k \times \mathbb{R}^{n-k} \oplus K_{0,0} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$. Indeed, the columns $j_1 < \cdots < j_N$ with $k < j_1$ of the matrix $DG(0)$ are linearly independent. So the matrix $M(0)$ formed by these columns has non zero determinant. In view of the continuity of $DG(a)$ with respect to a, the matrix $M(a)$ formed by the same columns of the matrix $DG(a)$ has non-zero determinant for a in C_0 and close to 0. Hence the linearlization $DG(a) : \mathbb{R}^n \to \mathbb{R}^N$ is surjective for $a \in C$ and close to 0. Moreover, if $c_1, \ldots, c_{m-N}$ is a basis for $K = K(0)$, then using the implicit function theorem we find a basis $c_1(a), \ldots, c_{n-N}(a)$ for $K(a)$ depending continuously on a and such that $c_i = c_i(0)$ for $1 \leq i \leq m - N$. The corresponding basis $n_1(a), \ldots, n_{m-N}(a)$ for $N(a)$ depends continuously on a. Arguing now as in the proof of Lemma 1.22, we obtain that $N \oplus R = N(a) \oplus R$ for $a \in C$ and close to 0. From this and our neatness assumptions we deduce that $\mathbb{R}^k \times \{0\} \subset p(N(a))$ where $p : \mathbb{R}^n \oplus K_{0,0} \to \mathbb{R}^n$ is the canonical projection. In addition, the fact that $N \oplus R = N(a) \oplus R$ implies that if $\overline{p} : N \oplus R \to N$ is the canonical projection, then $\overline{p}|N(a) : N(a) \to N$ is an isomorphism.

Fix a point $n_0 \in U_2 \cap N$ and let $q_0 = n_0 + A(n_0)$. We will show that A is smooth at n_0. There is a point $a_0 \in C$ satisfying $q_0 = \tau(a_0, \delta(a_0))$. We start with the case that some of the k first coordinates of $p(q_0)$ are equal to 0 where $p : \mathbb{R}^n \oplus K_{0,0} = N \oplus R \to R^n$ is the canonical projection. Without loss of generality we may assume that the first j coordinates of $p(q_0)$ are equal to 0. Then the first j coordinates of $p(n_0)$ are also equal to 0 in view of the neatness assumption. Hence $q_\in [0, \infty)^j \oplus \mathbb{R}^{n-j} \oplus E$ and $q_0 = \tau(a_0, \delta(a_0))$. Since τ is a diffeomorphism, the first j coordinates of a_0 are equal to 0 so that $a_0 \in [0, \infty)^j \times \mathbb{R}^{m-j}$. Abbreviate by $K' = K(a_0)$ the kernel of $Dg(a_0)$ and by the $N' = N(a_0)$ the kernel of $D\widehat{f}(q_0)$ which is the same as the kernel of $Df(q_0)|\mathbb{R}^n \oplus K_{0,0}$. The above discussion shows that $\mathbb{R}^k \times \{0\} \subset p(N')$ and since $j \leq k$, we see that the polyfold Fredholm germ $[f, q_0]$ is neat and there exists a complementary neat subspace R' so that $N' \oplus R' = \mathbb{R}^n \oplus K_{0,0}$.

Thus, we may apply the previous arguments and show the existence of an open neighborhood U' of 0 in $C \oplus K_{0,0}$ and a map $A' : U' \cap N' \to R'$ such that all solution q of $f(q) = 0$ near q_0 are of the form

$$q = q_0 + n' + A'(n').$$

Moreover, the map A' satisfies $A'(0) = 0$, $DA'(0) = 0$, and is smooth at the point $n' = 0$.

Making U' smaller if necessary and recalling that the restriction of the projection $p : N \oplus R \to N$ to N', $p : N' \to N$, is an isomorphism, we define the map $\alpha : U' \cap N' \to U \cap N$ by setting

$$\alpha(n') = p[q_0 + n' + A'(n')].$$

We have $\alpha(0) = n_0$ and conclude that

$$\alpha(n') + A(\alpha(n')) = q_0 + n' + A'(n') \tag{5.39}$$

for all $n' \in U' \cap N'$. The linearization $D\alpha(0) : N' \to N$ is well-defined and is an isomorphism. This implies that α is a local diffeomorphism between some open neighborhood of 0 in $U' \cap N'$ and an open neighborhood of n_0 in $U \cap N$. We may assume that these open neighborhoods are equal to $U' \cap N'$ and $U \cap N$. The map α has the following smoothness property. For every integer j there exists an open neighborhood O of 0 in $U' \cap N'$ is of class C^j on O. Since

$$n + A(n) = p[q_0 + \alpha^{-1}(n) + A'(\alpha^{-1}(n))]$$

for $n \in U \cap N$, we conclude that for given integers m and j there exists an open neighborhood of $O = O_{j,m} \subset U \cap N$ of n_0 such that $n \mapsto n + A(n)$ from O into $\mathbb{R}^n \oplus (K_{0,0})_m$ on which is of class C^j. In particular, the map $n \mapsto A(n)$ for $n \in U \cap N$ and is smooth at 0 as claimed.

Finally, if none of the the first coordinate of $p(q_0)$ is equal to 0, then q_0 is an interior point of $C \oplus E$. Then $q_0 = \tau(a_0, \delta(a_0))$ with a_0 being an interior point of C_0. Moreover, if N' is the kernel of $Df(q_0)$, then, in view of Theorem 5.17, there is a map $A' : N' \to R'$ defined on an open neighborhood of 0 in N' such the solutions of $f = 0$ near q_0 are of the form $n' + A'(n')$. Again we we have $\mathbb{R}^k \times \{0\} \subset p(N')$ if q_0 is close to 0 and so the map α in (5.39) is well-defined. The rest of the proof is the same as above. The proof of the theorem is complete. ∎

5.4.3. Stability of local Fredholm sections. We shall demonstrate the useful fact that Fredholm sections are stable under appropriate perturbations. As before we consider the local polyfold bundle

$$b : K^{\mathcal{S}_\triangleleft}|O \to O$$

where $O \subset K_0$ is open. The bundle

$$b^1 : (K^{\mathcal{S}_\triangleleft}|O)^1 \to O$$

is defined over the level 1 open set $O^1 \subset O$, and the set

$$K^{\mathcal{S}_\triangleleft} = \{(v, e, f) \in (V \oplus E) \triangleleft F | \pi_v(e) = e \text{ and } \sigma_v(f) = f\}$$

has the double filtration

$$(V \oplus E)_{m+1} \oplus F_{k+1}$$

for all $m \geq 0$ and $0 \leq k \leq m+1$. If q is a smooth point, then a section $[f, q]$ of the bundle $[b, q]$ induces a germ $[f, q]^1$ of the bundle $[b^1, q]$. From Definition 4.4 we recall that an sc^1-section $[s, q]$ of the bundle $[b, q]$ has the property that the principal part is an sc-smooth map $O \to F^1$. Hence, for all $m \geq 0$ we have $s(v, e) \in F_{m+1}$ if $(v, e) \in (V \oplus E)_m$. The stability result is now as follows.

Proposition 5.21. *Suppose the section $[f, q]$ is a polyfold Fredholm germ of the bundle $[b, q]$ in the sense of Definition 5.13. If $[s, q]$ is an sc^1-section of $[b, q]$, then $[f + s, q]$ is a polyfold Fredholm section germ of the bundle $[b^1, q]$.*

Proof. Denote by $[\widehat{f}, q]$ the filler of $[f, q]$ at the smooth point q. We shall show that $[\widehat{f} + s, q]$ is a filler for the germ $[f + s, q]$ of the bundle $[b^1, q]$. We have to verify for $\widehat{f} + s$ the two properties of an sc-Fredholm germ in Definition 5.8. We first observe that $\widehat{f} + s$ possesses the regularizing property because the section s is sc^+-smooth. In order to verify the contraction property for $\widehat{f} + s$ we use the fact that $\widehat{f}$ is a filler. Hence there exists a partial cone V in $\mathbb{R}^n$, an sc-smooth Banach space W and a germ of a strong vector bundle isomorphism

$$\Psi : \mathcal{O}(U \triangleleft F, q) \to \mathcal{O}((V \oplus W) \triangleleft (\mathbb{R}^N \oplus W), 0)$$

which covers the sc-diffeomorphism germ $\sigma : \mathcal{O}(U, q) \to (V \oplus W, 0)$ satisfying $\sigma(q) = 0$ and which has the property that the map $\Psi(\widehat{f}) : \mathcal{O}(V \oplus W, 0) \to (W, 0)$ defined by

$$\Psi(\widehat{f})(a, w) = P \circ \mathrm{pr}_2 \circ [\Psi \circ \widehat{f} \circ \sigma^{-1}(a, w) - \Psi(\widehat{f}(q))],$$

is an sc^0-contraction germ in the sense of Definition 5.1. Abbreviate

$$g(a, w) = \Psi(\widehat{f})(a, w)$$

we have the contraction representation

$$g(a, w) = w - B(a, w).$$

Replacing $\widehat{f}$ by the section s we define

$$S(a, w) = \Psi(s)(a, w).$$

Then S is an sc^+-section. Abbreviating the linearization

$$A\widehat{w} = D_2 S(0, 0)\widehat{w},$$

$\widehat{w} \in W$, at the point $(a, w) = (0, 0)$ in the direction of the second argument. Then $A : W \to W$ is, on every level $m \geq 0$, a compact operator. Therefore, $\mathrm{Id}+A : W \to W$ is a linear sc-Fredholm operator of index 0. The associated splitting of W,

$$\mathrm{Id}+A : W = C \oplus X \to W = R \oplus Z$$

with $C = \ker\,(\mathrm{Id}+A)$ and $R = \text{range}\,(\mathrm{Id}+A)$ satisfies $\dim\ C = \dim\ Z < \infty$. Since S is of class C^1 on every level $m \geq 1$ we have the representation

$$S(a,w) = Aw + G(a,w)$$

where $D_2G(0,0) = 0$. Hence G is a contraction with respect to the second variable on every level $m \geq 1$ if only a and w are small enough (depending on the level m). Consider the sum

$$\begin{aligned} g(a,w) + S(a,w) &= w - B(a,w) + Aw - G(a,w) \\ &= (\mathrm{Id}+A)w - \overline{B}(a,w), \end{aligned}$$

where

$$\overline{B}(a,w) = B(a,w) + G(a,w).$$

The map $\overline{B}$ is a contraction in the second variable on every level $m \geq 1$ as long as a and w are sufficiently small. Define the canonical projections

$$\begin{aligned} Q &: W = C \oplus X \to X \\ E &: W = R \oplus Z \to R, \end{aligned}$$

and introduce

$$\begin{aligned} \Phi(a,w) &:= E \circ P \circ \mathrm{pr}_2 \circ [\Psi(\widehat{f}+s) \circ \sigma^{-1}(a,w) - \Psi((\widehat{f}+s)(q))] \\ &= E \circ [(\mathrm{id}+A)w - \overline{B}(a,w)] \\ &= E \circ (\mathrm{id}+A) \circ Q(Qw) - E \circ \overline{B}(a, (\mathrm{Id}-Q)w + Qw). \end{aligned}$$

Using the sc-isomorphism $L := (\mathrm{Id}+A)|X : X \to R$ we obtain

$$L^{-1} \circ \Phi(a, (\mathrm{Id}-Q)w + Qw) = Qw + L^{-1} \circ E \circ \overline{B}(v, (\mathrm{Id}-Q)w + Qw).$$

We shall view the set $(a, (\mathrm{Id}-Q)w)$ as the new parameter set and define

$$\widehat{B}((a, (\mathrm{Id}-Q)w), Qw) := L^{-1} \circ E \circ \overline{B}(a, (\mathrm{Id}-Q)w + Qw).$$

Then $\widehat{B}$ possesses the required contraction property with respect to the variable Qw on all levels $m \geq 1$ provided a and w are small enough. It remains to identify the above expression as obtained from an admissible coordinate change. For this we take any linear isomorphism $\tau : Z \to Z$ and define the fiber transformation

$$\widehat{\Psi} = [(L^{-1} \circ E) \oplus \tau \circ (\mathrm{Id} - E) \circ \Psi.$$

With the projection

$$\overline{P} : \mathbb{R}^n \oplus W = \mathbb{R}^n \oplus C \oplus X \to X$$
$$\overline{P}(a, w) = Qw$$

we finally obtain, setting $(a, (\mathrm{Id} - Q)w + Qw) = (a, (\mathrm{Id} - Q)w, Qw)$, the formula

$$\begin{aligned} \overline{P} \circ \mathrm{pr}_2 \circ [\widehat{\Psi} \circ (\widehat{f} + s) \circ \sigma^{-1}(a, (\mathrm{Id} - Q)w, Qw) - \widehat{\Psi} \circ (\widehat{f} + s)(\sigma^{-1}(q))] \\ = Qw - \widehat{B}(a, (\mathrm{Id} - Q)w, Qw). \end{aligned}$$

We have proved that $[\widehat{f}+s, q]$ is a polyfold Fredholm germ of the bundle $[b^1, q]$ in the sense of Definition 5.13. The proof of Proposition 5.21 is complete. ∎

CHAPTER 6

Global sc-Fredholm Theory

This chapter is devoted to the Fredholm theory in M-polyfold bundles.

6.1. Fredholm sections

Given the M-polyfold bundle

$$b : Y \to X$$

according to Definition 4.7 we denote by $\Gamma(b)$ the space of sc-smooth sections and by $\Gamma^+(b)$ the space of sc^+-sections.

DEFINITION 6.1. **Fredholm section.** *A section $f \in \Gamma(b)$ of the M-polyfold bundle $b : Y \to X$ is called* **Fredholm***, if at every smooth point $q \in X$ there exists an M-polyfold bundle chart*

$$\Phi : b^{-1}(U) \to K^{\mathcal{S}_\triangleleft}|O$$

around $q \in U \subset X$ in the sense of Definition 4.7 in which the section germ $[f, q]$ is a local polyfold Fredholm germ according to Definition 5.13. The **collection of all Fredholm sections** *is denoted by $\mathcal{F}(b)$.*

Recall that Definition 5.13 requires that $[f, q]$ possesses a filler $\widehat{f}$ which is an sc-Fredholm germ in the sense of Definition 5.8, and so possesses the regularizing property and the contraction property.

From the local stability result in Proposition 5.21 one deduces immediately the following global version.

PROPOSITION 6.2. *The map*

$$\Gamma(b) \times \Gamma^+(b) \to \Gamma(b)$$
$$(f, s) \mapsto f + s$$

induces a map

$$\mathcal{F}(b) \times \Gamma^+(b) \to \mathcal{F}(b^1).$$

For the deeper study of Fredholm operators it is useful to introduce first some auxiliary concepts.

6.2. Mixed convergence and auxiliary norms

We begin with a notion of convergence in the bundle $b : Y \to X$ called mixed convergence, referring to a mixture of strong convergence in the base space X and weak convergence in the level 1 fibers Y^1, assuming the fibers are reflexive sc-Banach spaces.

6.2.1. Mixed Convergence. We start with the local definition in an M-polyfold bundle chart and consider the M-polyfold bundle

$$K^{\mathcal{S}_{\triangleleft}}|O \to O$$

where O is an open subset of the splicing core

$$K_0 := K^{\mathcal{S}_0} = \{(v, e) \in V \oplus E |\ \pi_v(e) = e\}$$

and where

$$K^{\mathcal{S}_{\triangleleft}} = \{(v, e, y) \in O \oplus F |\ \sigma_v(y) = y\} = O \triangleleft K_1.$$

Abbreviating the elements in O by $x = (v, e)$ we consider a sequence

$$(x_k, y_k) \in O \triangleleft (K_1)_1,$$

where $(K_1)_1$ refers to F_1, the level 1 space of the sc-Banach space F.

Definition 6.3. **Mixed convergence.** *The sequence $(x_k, y_k) \in O \triangleleft (K_1)_1$ is called* **m-convergent** *to $(x, y) \in O \oplus F_1$ if*

$$(1) \quad x_k \to x \text{ in } O$$
$$(2) \quad y_k \rightharpoonup y \text{ in } F_1.$$

Symbolically,

$$(x_k, y_k) \xrightarrow{m} (x, y).$$

Lemma 6.4. *If $(x_k, y_k) \in O \triangleleft (K_1)_1$ and $(x_k, y_k) \xrightarrow{m} (x, y)$, then $(x, y) \in O \triangleleft (K_1)_1$.*

Proof. By assumption, $x_k = (v_k, e_k)$ and $\pi_{v_k}(e_k) = e_k$. Moreover, $\sigma_{v_k}(y_k) = y_k$. Since the inclusion map $I : F_1 \to F_0$ is a compact map, one concludes from the weak convergence of $y_k \rightharpoonup y$ in F_1 that $y_k \to y$ in F_0. With $x = (v, e)$ it follows from the continuity of the projections $(v, y) \mapsto \sigma_v(y)$ in F that $\sigma_v(y) = y$ in F_0. Since $y \in F_1$ one concludes $\sigma_v(y) = y$ in F_1 so that $(x, y) \in O \triangleleft (K_1)_1$ as claimed in the lemma.

■

We must demonstrate that the concept of m-convergence is compatible with M-polybundle maps provided the fibers are reflexive spaces according to the following definition.

Definition 6.5. *An sc-Banach space F is called* **reflexive** *if all the spaces F_m, $m \geq 0$, of its filtration are reflexive Banach spaces.*

We consider now an sc-smooth polybundle map

$$\Psi : O \triangleleft K \to O' \triangleleft K'$$

assuming that the underlying sc-Banach spaces F and F' are reflexive. With $x = (v, e)$ the map ψ is of the form

$$\Psi(x, y) = (\sigma(x), \psi(x, y)).$$

The maps $\sigma : O \to O'$ and ψ are sc-smooth and linear in y. Consider a sequence $(x_k, y_k) \in O \triangleleft (K_1)_1$. If $(x_k, y_k) \xrightarrow{m} (x, y)$, then $(x, y) \in O \triangleleft (K_1)_1$ in view of Lemma 6.4, and $\sigma(x_k) \to \sigma(x)$. From $(x_k, y_k) \to (x, y)$ in $O \triangleleft (K_1)_0$ it follows that $\psi(x_k, y_k) \to \psi(x, y)$ in $O' \triangleleft (K_1')_0$, so that

$$\Psi(x_k, y_k) \to \Psi(x, y) \quad \text{in } O' \triangleleft (K_1)_0'.$$

By Definition 4.2 of sc-smooth bundle maps, the map $\Psi : O \triangleleft (K_1)_1 \to O' \triangleleft (K_1')_1$ is continuous. From the continuity $\lim_{k\to\infty} \psi(x_k, y) = \psi(x, y)$ in $(F')_1$ for all $y \in F_1$, we deduce by means of the uniform boundedness principle that the sequence of bounded linear maps $\psi(x_k) \in \mathcal{L}(F_1, F_1')$ defined by $\psi(x_k) \cdot y = \psi(x_k, y)$ has uniformly bounded operator norms, $|||\psi(x_k)||| \leq C$ for all k. Consequently, $\|\psi(x_k, y_k)\|_1 \leq C \cdot \|y_k\|_1$. From the weak convergence $y_k \rightharpoonup y$ in F_1' we know that $\|y_k\|_1$ is also a bounded sequence. Hence $\|\psi(x_k, y_k)\|_1$ is a bounded sequence. Because F_1' is a reflexive Banach space, every subsequence of the bounded sequence $\psi(x_k, y_k)$ in F_1' possesses a subsequence having a weak limit in F_1'. The limit is necessarily equal to $\psi(x, y)$.

Summarizing we have proved that

$$\sigma(x_k) \to \sigma(x) \quad \text{in } O'$$

and

$$\mathrm{pr}_2 \circ \Psi(x_k, y_k) \rightharpoonup \mathrm{pr}_2 \circ \Psi(x, y) \quad \text{in } (K_1')_1$$

i.e., on level 1. The discussion shows that the mixed convergence is invariant under M-polyfold chart transformations and hence an intrinsic concept for M-polyfold bundles having reflexive fibers so that we can introduce the following definition.

Definition 6.6. *If $b : Y \to X$ is an M-polyfold bundle with reflexive fibers, then a sequence $y_k \in Y^1$ is said to converge in the m-sense to $y \in Y^1$, symbolically*

$$y_k \xrightarrow{m} y \quad \textit{in } Y^1,$$

if the underlying sequence $x_k = b(y_k) \in X$ *converges in* X *to an element* x *and if there exists an* M*-polyfold bundle chart* Ψ *around the point* $x \in X$ *satisfying*

$$pr_2 \circ \Psi(x_k, y_k) \rightharpoonup pr_2 \circ \Psi(x, y)$$

weakly in $(K_1')_1$*, i.e., on level* 1.

As shown above, the definition does not depend on the choice of the local M-polyfold bundle chart.

6.2.2. Auxiliary Norms. For the general perturbation theory we introduce some auxiliary concepts to estimate the size of perturbations.

DEFINITION 6.7. *An* **auxiliary norm** N *for the M-polyfold bundle* $b : Y \to X$ *consists of a continuous map* $N : Y^1 \to [0, \infty)$ *having the following properties.*

- *For every* $x \in X$ *the induced map* $N|Y^1_x \to [0, \infty)$ *on the fiber* $Y^1_x = b^{-1}(x)$ *is a complete norm.*
- *For every m-convergent sequence* $y_k \xrightarrow{m} y$ *we have*
$$N(y) \leq liminf_{k\to\infty} N(y_k).$$
- *If* $N(y_k)$ *is a bounded sequence and the underlying sequence* x_k *converges to* $x \in X$ *then* y_k *has a m-convergent subsequence.*

In view of the open mapping theorem the complete norm $N|Y^1$ is equivalent to the original 1-norm. Using the paracompactness of X we can easily construct auxiliary norms.

PROPOSITION 6.8. *Let* $b : Y \to X$ *be a M-polyfold bundle with reflexive fiber. Then there exists an auxiliary norm.*

PROOF. Construct for every $x \in X$ via local coordinates a norm $N_{U(x)}$ for $Y^1|U(x)$ where $U(x)$ is a small open neighborhood of $x \in X$. This is defined by $N_{U(x)}(y) = \| pr_2 \circ \Psi(y) \|_1$, where Ψ is a local m-polybundle chart and pr_2 is the projection onto the fiber part. Observe that m-convergence of y_k to some y in $Y^1|U(x)$ implies weak convergence of $pr_2 \circ \Psi(y_k)$. Using the convexity of the norm and standard properties of weak convergence we see that

$$N_{U(x)}(y) \leq \mathrm{liminf}_{k\to\infty} N_{U(x)}(y_k).$$

We have at this point local expressions for auxiliary norms which cover X. Using the paracompactness of X we can find a subordinate partition of unity $(\chi_\lambda)_{\lambda\in\Lambda}$ and define

$$N=\sum \chi_\lambda N_{U(x_\lambda)}.$$

If $N(y_k)$ is bounded and the underlying sequence x_k converges to some $x\in X$ it follows in local coordinates that the representative of y_k is bounded on level 1. In view of the reflexivity, the sequence y_k possesses a m-convergent subsequence. ■

The following result will be useful in compactness investigations.

Theorem 6.9 (Local Compactness). *Consider the M-polyfold bundle $b:Y\to X$ having reflexive fibers and let f be a Fredholm section of b . Assume that an auxiliary norm $N:Y^1\to[0,\infty)$ is given. Then there exists at every smooth point $q\in X$ an open neighborhood $U(q)$ in X so that the following holds.*

- *The set Z defined by*

$$Z=\{x\in\overline{U(q)}|\ f(x)\in Y^1 \text{ and } N(f(x))\leq 1\}$$

  is a compact subset of X.
- *Every sequence (x_k) in $\overline{U(x)}$ satisfying*

$$\liminf_{k\to\infty} N(f(x_k))\leq 1$$

  possesses a convergent subsequence.

Proof. The statement is local and, using in local coordinates the translation $u\mapsto u-q$ which is sc-smooth since q is a smooth point, we may assume that the filler $\widehat{f}$ of the section f is defined near 0. Moreover, after a change of coordinates as in the definition of a contraction germ. we may assume that

$$\widehat{f}:V\oplus W\to\mathbb{R}^N\oplus W$$

is defined near 0, where $V\subset\mathbb{R}^n$ is a partial cone. Denoting by $P:\mathbb{R}^N\oplus W\to W$ the canonical projection, the contraction germ is given by

$$P[f(a,w)-f(0)]=w-B(a,w).$$

It is defined on a sufficiently small closed neighborhood $\overline{U}_0$ of the origin on the level 0 on which it has the contraction property. The auxiliary norm N is defined on $\overline{U}_0\oplus(\mathbb{R}^N\oplus W)^1$ and, possibly choosing a smaller neighborhood, we may assume that it is equal to the 1-norm so that $N(x,h)=\|h\|_1$. We now consider the set of $(a,w)\in\overline{U}_0$ satisfying $f(a,w)\in(\mathbb{R}^N\oplus W)^1$ and

$$\|f(a,w)\|_1\leq 1.$$

The section f splits according to the splitting of the tangent space into

$$f(a,w)=\begin{bmatrix}(\mathrm{Id}-P)f(a,w)\\ Pf(a,w)\end{bmatrix}=\begin{bmatrix}(\mathrm{Id}-P)f(a,w)\\ w-B(a,w)+Pf(0)\end{bmatrix}.$$

Using the contraction property one finds for given $u\in W$ close to $Pf(0)$ and given $a\in V$ close to 0 a unique $w(a,u)\in W$ solving the equation $Pf(a,w(a,u))=u$. Moreover, the map $(a,u)\mapsto w(a,u)$ is continuous on the 0-level. Now, given a sequence $(a_k,u_k)\in\overline{U}_0$ such that

$$f(a_k,w_k)=:(b_k,u_k)$$

belongs to $(\mathbb{R}^N\oplus W)^1$ and satisfies $\|f(a_k,w_k)\|_1\leq 1$, we have the equations

$$\begin{aligned}Pf(a_k,w_k)&=u_k\\ (\mathrm{Id}-P)f(a_k,w_k)&=b_k\\ w_k&=w(a_k,u_k).\end{aligned}$$

We shall show that (a_k,w_k) possesses a convergent subsequence in $\overline{U}_0$. By assumption, $(\mathbb{R}^N\oplus W)^1$ is a reflexive Banach space so that going over to a subsequence,

$$\begin{aligned}(b_k,u_k)&\rightharpoonup(b',u')\quad\text{in } (\mathbb{R}^N\oplus W)^1\\ (b_k,u_k)&\to(b',u')\quad\text{in } (\mathbb{R}^N\oplus W)^0,\end{aligned}$$

and $\|(b',u')\|\leq 1$. In $\mathbb{R}^N$ we may assume $a_k\to a'\in V$. Consequently, $w_k=w(a_k,u_k)\to w':=w(a',u')$. Hence $f(a',w')=(b',u')$ and $\|f(a',w')\|_1\leq 1$. The proof of Theorem 6.9 is complete. ■

6.3. Proper Fredholm Sections

In this section we introduce the important class of proper Fredholm sections.

Definition 6.10. *A Fredholm section f of $b:Y\to X$ is called* **proper** *provided $f^{-1}(0)$ is compact in X.*

The first observation is the following.

Theorem 6.11 (∞-Properness). *Assume that f is a proper Fredholm section of the M-polyfold bundle $b:Y\to X$. Then $f^{-1}(0)$ is compact in X^∞.*

Proof. Properness implies by definition that $f^{-1}(0)$ is compact on level 0. Of course, $f^{-1}(0)$ is a subset of X^∞ since f is regularizing.

Assume that x_k is a sequence of solutions of $f(x) = 0$. We have to show that it possesses a subsequence converging to some solution x_0 in X^∞. After taking a subsequence we may assume that $x_k \to x_0$ on level 0. We choose a contraction germ representation for $[f, x_0]$. After a suitable change of coordinates the sequence (a_k, w_k) corresponds to x_k and the point $(0,0)$ to x_0, and

$$w_k = B(a_k, w_k),$$

with $(a_k, w_k) \to (0,0)$ on level 0. Consequently,

$$w_k = \delta(a_k)$$

for the map $a \to \delta(a)$ constructed in Section 5.1 using Banach's fixed point theorem on level 0. We know from Section 5.1 that for every level m there is an open neighborhood O^m of 0 so that $\delta_m : O^m \to W^m$ is continuous and given by Banach's fixed point theorem on the m-level. We may assume that $O^{m+1} \subset O^m$. Fix a level m. For k large enough, we have $a_k \in O^m$. Then $\delta_m(a_k)$ is the same as $\delta(a_k) = w_k$. Hence $a_k \to 0$ implies $w_k \to 0$ on level m. Consequently, $(a_k, w_k) \to 0$ on every level, implying convergence on the ∞-level. Hence $x_k \to x_0$ in X^∞ as claimed. ■

As a consequence of the local Theorem 6.9 we obtain the following result for proper Fredholm sections.

THEOREM 6.12. *Let $b : Y \to X$ be an M-polyfold bundle with reflexive fibers and assume that f is a proper Fredholm section. Assume that N is a given auxiliary norm. Then there exists an open neighborhood U of the compact set $S = f^{-1}(0)$ so that the following holds true.*

- *For every section $s \in \Gamma^+(b)$ having its support in U and satisfying $N(s(x)) \leq 1$, the section $f + s$ is a proper Fredholm section.*
- *Every sequence (x_k) in $\overline{U}$ satisfying $f(x_k) \in Y^1$ and*

$$\liminf_{k\to\infty} N(f(x_k)) \leq 1$$

possesses a convergent subsequence.

PROOF. We know from the local Fredholm theory that $f + s$ is Fredholm if s is a sc$^+$-section s. For every $q \in S = f^{-1}(0)$ there exists an open neighborhood $U(q)$ having properties as in Theorem 6.9. Since S is a compact set we find finitely many q_i so that the open sets $U(q_i)$ cover S. We denote their union by U. Next assume that the support of $s \in \Gamma^+(b)$ is contained in U. If $f(x) + s(x) = 0$, then necessarily $x \in U$

because otherwise $s(x) = 0$ implying $f(x) = 0$ and hence $x \in S \subset U$, a contradiction. Consequently, the set of solutions of $f(x) + s(x) = 0$ belongs to $\{x \in \overline{U} |\, f(x) \in Y^1 \text{ and } N(f(x)) \leq 1\}$ which by construction is contained in a finite union of compact sets. The proof of Theorem 6.12 is complete. ■

6.4. Transversality and Solution Set

From our local considerations in the previous chapter we shall deduce the main result about the solution set of Fredholm sections.

THEOREM 6.13. *Consider a proper Fredholm section f of the M-polyfold bundle $b : Y \to X$. Assume that for every point $q \in f^{-1}(0)$ the linearization $Df(q)$ is neat and surjective. Then the solution set $f^{-1}(0)$ carries in a natural way the structure of a smooth compact manifold with boundary with corners.*

PROOF. Choose a point $q \in X$ satisfying $f(q) = 0$. Then q is a smooth point, and there is a special M-polybundle chart $(U_0, \Phi, K^{\mathcal{S}_\triangleleft}|O)$ as in Definition 4.7, where U_0 is an open neighborhood around q in X and Φ a homeomorphism

$$\Phi : p^{-1}(U_0) \to K^{\mathcal{S}_\triangleleft}|O$$

which is linear on the fibers and which covers a homeomorphism

$$\varphi : U_0 \to O \subset K^{\mathcal{S}_0},$$

so that $\mathrm{pr}_1 \circ \Phi = \varphi \circ b$. Here O is an open subset of the splicing core $K^{\mathcal{S}_0} = \{(v, e) \in V_0 \oplus E |\ \pi_v(e) = e\}$ where $V_0 = [0, \infty)^k \times \mathbb{R}^{n-k}$ is a partial cone, and E is an sc-Banach space. We may assume $\varphi(q) = 0$. Let g be the push-forward of the section f under Φ and let $\widehat{g}$ be its filler. Then the linearization $Dg(0)$ is by assumption surjective and neat. Denote by N the kernel of $Dg(0)$. In view of Theorem 5.20, $\dim N < \infty$ and $N \subset \mathbb{R}^n \oplus E_\infty$. Moreover, there exists a neat sc-complement R such that

$$N \oplus R = \mathbb{R}^n \oplus K_{0,0}.$$

In addition, there exists an open neighborhood V of zero in a partial cone of N and an sc-smooth map $A : V \to R$ satisfying $A(0) = 0$ and $DA(0) = 0$ such that the map

$$\gamma : V \to N \oplus R, \quad \gamma(n) = n + A(n)$$

parametrizes all the solutions of $g = 0$ near the origin,

$$g(n + A(n)) = 0.$$

Further, $n + A(n) \in N \oplus R_\infty \cap K_0$. Composing the map γ with the coordinate map φ we define the map $\beta^{-1} : V \to X$ by

$$\beta^{-1}(n) = \varphi^{-1}(n + A(n)) = \varphi^{-1} \circ \gamma(n).$$

It parametrizes all solutions of $f(x) = 0$ near the point q. Denote by $U \subset X$ the image of β^{-1}, so that

$$U = \beta^{-1}(V).$$

The pair (U, β) will be a coordinate chart for the solution set $f^{-1}(0)$ and we shall show that all coordinate charts constructed this way are smoothly compatible. Assume that (U', β') is a second such chart so that $U \cap U' \neq \emptyset$ and consider

$$\beta' \circ \beta^{-1} : \beta(U \cap U') \to \beta'(U \cap U').$$

By construction β' maps U' onto an open set V' of a partial cone in the kernel N' which has neat sc-complement R' and

$$(\beta')^{-1}(n') = (\varphi')^{-1}(n' + A'(n')).$$

Take $p \in U \cap U'$. Then $\beta(p) = P \circ \varphi(p) = P(n + A(n)) = n$ and $\beta'(p) = P' \circ \varphi'(p) = P'(n' + A'(n')) = n'$ with the canonical projections $P : N \oplus R \to N$ and $P' : N' \oplus R' \to N'$. Consequently,

$$\begin{aligned} n' &= \beta' \circ \beta^{-1}(n) \\ &= P' \circ \varphi' \circ \beta^{-1}(n) \\ &= P' \circ \varphi' \circ \varphi^{-1}(n + A(n)) \\ &= P' \circ (\varphi' \circ \varphi^{-1}) \circ \gamma(n). \end{aligned}$$

Since, by Theorem 5.20, the map γ is sc-smooth, and since the transition map $\varphi' \circ \varphi^{-1}$ is, by definition, an sc-smooth map, the composition $\beta' \circ \beta^{-1}$ is sc-smooth and hence is a smooth map between open sets of partial cones in Euclidean spaces. The proof of Theorem 6.13 is complete. ■

6.5. Perturbations

Let us next prove some perturbation results.

THEOREM 6.14. *Assume that f is a proper Fredholm section of the M-polyfold bundle $b : Y \to X$. Let us assume that $\partial X = \emptyset$. Let N and α be auxiliary norm and bounds, respectively. Assume that U is the open neighborhood of $f^{-1}(0)$ so that for every $s \in \Gamma^+(b)$ with support in U and $N(s(x)) \leq \alpha(x)$ the section $f + s$ is proper. Denote the space of all sections in Γ^+ as just described by $\overline{\mathcal{O}}$. Then there exists for*

$t \in \frac{1}{2}\mathcal{O}$ and $\varepsilon \in (0, \frac{1}{2})$ a section $s \in \mathcal{O}$ with $s - t \in \varepsilon\mathcal{O}$ so that $f + s$ has for every solution of $f(q) + s(q) = 0$ a surjective linearization.

Proof. The set Ξ of all $x \in X$ with $N(f(x)) \leq \alpha(x)$ and $x \in \overline{U}$ is compact in X. Clearly $\Xi \subset X^1$ by the regularizing property. Consider for given t the solution set S of $f + t = 0$. This set is compact in X^∞. For every $x \in S$ consider the linearisation $(f + t)'(x)$ which is sc-Fredholm. We find finitely many sections $s_1, .., s_k$ in $\Gamma^+(b)$ with support in U so that

$$(\lambda_1, .., \lambda_k, h) \to (f + t)'(x)h + \sum_{i=1}^{k} \lambda_i s_i(x)$$

is onto. The same is true for all y close to x in X^∞. By a finite covering argument we can find finitely many section so that for every x with $(f + t)(x) = 0$ the linearization $(f + t)'(x)$ has a cokernel spanned by the $s_1(x), ..., s_k(x)$. We may view Y as a M-polyfold bundle over $\mathbb{R}^k \oplus X$. Then the map

$$(\lambda_1, .., \lambda_k, x) \to (f + t + \sum_{i=1}^{k} \lambda_i s_i)(x)$$

gives a Fredholm section of $Y^1 \to \mathbb{R}^k \times X^1$ which has the property that along S the linearization is surjective. Let L be the solution set near $\{0\} \times S$. This is a smooth manifold and the projection $L \to \mathbb{R}^k$ is smooth. Take a small regular value near 0. Then the associated section s^* has the property that $f + s^*$ is transversal to the zero-section. ■

6.6. An Example of a Fredholm Operator

We begin by proving some estimates which will be useful in showing that the operator $[u] \to [\frac{du}{dt} + f'(u)]$ induces a Fredholm section of the bundle $p : Y \to X$.

6.6.1. An a priori estimate. We start by recalling the definition of the Sobolev weighted spaces $H^{m,\delta}(S, \mathbb{R}^n)$. If δ is a real number and I is an interval in $\mathbb{R}$, then $H^{m,\delta}(I, \mathbb{R}^n)$ consists of functions w whose weak derivatives $D^j w$ up to order m belong to $L^2(I)$ if weighted by $e^{\delta|s|}$. The $H^{m,\delta}(I)$-norm is defined by

$$\|w\|^2_{H^{m,\delta}(I)} = \sum_{k \leq m} \int_I |D^k w|^2 e^{2\delta|s|} ds.$$

In the following $A : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric linear isomorphism having the eigenvalues λ_i, $1 \leq i \leq n$. As in Section 2.2 we work with the sc-smooth Banach spaces $E = E^+ \oplus E^-$ consisting of pairs (u, v) of maps

$$(u, v) \in E^+ \oplus E^- = H^2([0, \infty), \mathbb{R}^n) \oplus H^2((-\infty, 0], \mathbb{R}^n).$$

The sc-structure is defined by the weighted Sobolev spaces $E_m^\pm = H^{m+2,\delta_m}(\mathbb{R}_\pm)$ for all $m \geq 0$, having the norms

$$\|w^\pm\|_m^2 = \sum_{j \leq m+2} \int_{\mathbb{R}_\pm} |D^j w^\pm(s)|^2 e^{2\delta_m |s|} ds.$$

The norm on E_m is defined as

$$\|(u, v)\|_m = \|u\|_m + \|v\|_m.$$

Here δ_m, $m \geq 0$ is a strictly increasing sequence starting at $\delta_0 = 0$ but this time restricted by the condition

$$\delta_m < \min\{|\lambda_i|, 1 \leq i \leq n\}. \tag{6.1}$$

Similarly, $F = F^+ \oplus F^-$ is the sc-smooth space consisting of pairs (h, k) of maps

$$(h, k) \in F^+ \oplus F^- = H^1(\mathbb{R}_+) \oplus H^1(\mathbb{R}_-)$$

with an sc-structure defined by the Sobolev spaces $F_m^\pm = H^{m+1,\delta_m}(\mathbb{R}_\pm)$.

We also recall from Section 2.2 the total gluing operation $\boxplus_r$ associating with (u, v) the pair of functions

$$\boxplus_r(u, v) = (\oplus_r(u, v), \ominus_r(u, v)). \tag{6.2}$$

If $r > 0$, it is explicitly given by

$$\boxplus_r(u, v) = \begin{bmatrix} \tau_R(s) & 1 - \tau_R(s) \\ \tau_R(s) - 1 & \tau_R(s) \end{bmatrix} \cdot \begin{bmatrix} u(s) \\ v(s - R) \end{bmatrix} \tag{6.3}$$

for $0 \leq s \leq R$. Here $\tau_R(s) = \beta(s - \frac{R}{2})$ with the cut-off function β from (2.5) and

$$R = \varphi(r)$$

with the gluing profile φ from (2.6). We next introduce the 1-parameter family of sc-continuous linear operators

$$L_r : E \to F \tag{6.4}$$

defined, if $r = 0$, by

$$L_0(u, v) = (\dot{u} + Au, \dot{v} + Av) \tag{6.5}$$

and if $r \in (0,1)$, by

$$L_r(u,v) = \boxplus_r^{-1}\left[\left(\frac{d}{ds} + A\right)\left(\oplus_r(u,v), \ominus_r(u,v)\right)\right]. \tag{6.6}$$

Explicitly, taking the derivative in s in the definition (6.2), a calculation shows that

$$L_r(u,v) = L_0(u,v) + F_R(u,v) \tag{6.7}$$

where $F_R(u,v) = (F_R^+(u,v), F_R^-(u,v))$ is defined by the formulae

$$F_R^+(u,v)(s) = \frac{\dot\tau(s)}{\alpha(s)} \cdot [(2\tau(s)-1)\cdot u(s) - v(s-R)]$$

for all $s \geq 0$, and

$$F_R^-(u,v)(s') = \frac{\dot\tau(-s')}{\alpha(-s')} \cdot [u(s'+R) + (1-2\tau(-s'))\cdot v(s')]$$

for all $s' \leq 0$. We have abbreviated $\tau(s) = \tau_R(s)$ and $\dot\tau(s) = \frac{d}{ds}\tau(s)$. Moreover, $\alpha(s) = \tau(s)^2 + (1-\tau(s))^2$. Due to the definition of τ, the support of the function $F_R^+(u,v)$ is contained in the interval $[\frac{R}{2}-1, \frac{R}{2}+1] \subset \mathbb{R}_+$ while the support of $F_R^+(u,v)$ is contained in $[-\frac{R}{2}-1, -\frac{R}{2}+1] \subset \mathbb{R}_-$. By Rellich's compactness theorem, the linear operator $F_R : E \to F$ is compact. In view of Theorem 2.17, the maps $(r,u,v) \mapsto F_R^+(u,v)$ resp. $(r,u,v) \mapsto F_R^-(u,v)$ from $[0,\frac{1}{2})\oplus E \to F^+$ resp. from $[0,\frac{1}{2})\oplus E \to F^-$ are sc-smooth. We formulate this fact as a proposition.

Proposition 6.15. *The map*

$$L : [0,1) \oplus E \to F$$

is sc-smooth.

We shall need in the following the well-known Fredholm properties of the linear operator L_0 defined in (6.6).

Lemma 6.16. *The operator L_0 is sc-Fredholm with index 0 and with n-dimensional kernel and cokernel.*

We denote by $\widehat{E}$ the Sobolev space $H^2(\mathbb{R},\mathbb{R}^n)$ of maps defined on all of $\mathbb{R}$. The sc-structure is given by the spaces $H^{m+2,\delta_m}(\mathbb{R})$. By $\widehat{F}$ we denote the sc-smooth space $H^{m+1,\delta_m}(\mathbb{R})$ whose sc-structure is given by the spaces $H^{m+1,\delta_m}(\mathbb{R})$. On these spaces we introduce the sc-linear continuous operator

$$\widehat{L} : \widehat{E} \to \widehat{F}$$
$$\widehat{L}w = \dot{w} + Aw.$$

We will also need the sc-smooth space $\overline{E} = H^2(\mathbb{R}, \mathbb{R}^n)$ with the sc-structure given by $\overline{E}_m = H^{m+2,-\delta_m}(\mathbb{R})$, and the space $\overline{F} = H^1(\mathbb{R})$ with the sc-structure defined by $\overline{F}_m = H^{m+1,-\delta_m}(\mathbb{R})$. On these spaces we have the sc-linear continuous operator $\overline{L} : \overline{E} \to \overline{F}$ defined by $\overline{L}w = \dot{w}+Aw$. In view of restriction (6.1) one verifies the following statement.

Lemma 6.17. *The sc-linear operators $\widehat{L}$ and $\overline{L}$ are sc-isomorphisms.*

The above lemmata 6.16 and 6.17 are well-known and we refer, for example to [**19**] and [**26**]. The aim of this subsection is the proof of the following a priori estimate.

Theorem 6.18. *For every $m \geq 0$ there exists a constant $c_m > 0$ so that*

$$\|L_r(u, v)\|_m + |u(0)| + |v(0)| \geq c_m \cdot \|(u, v)\|_m$$

for all $r \in [0, \frac{1}{2}]$ and $(u, v) \in E^m$.

Proof. Arguing indirectly we assume that on some level m there are sequences $r_k \to r_0$ and $(u_k, v_k) \in E^m$ satisfying $\|(u_k, v_k)\|_m = 1$ and

$$\|(L_0 + F_{R_k})(u_k, v_k)\|_m + |u_k(0)| + |v_k(0)| \to 0 \tag{6.8}$$

as $k \to \infty$, where $R_k = \varphi(r_k)$.

We consider two cases depending on whether the limit r_0 is equal to 0 or different from 0. We start with the case $r_0 > 0$. In this case $R_k \to R_0$. The operator L_0 is Fredholm with index 0 and F_{R_0} is a compact operator, hence $L_0 + F_{R_0}$ is Fredholm with index 0. Denote by X the finite dimensional kernel of $L_0+F_{R_0}$ and by Y the complementary subspace in E_m so that $E_m = X \oplus Y$. The restriction $(L_0 + F_{R_0})|Y \to F_m$ is injective. We split the sequence (u_k, v_k) accordingly into the sum $(u_k, v_k) = (\overline{u}_k, \overline{v}_k) + (\widehat{u}_k, \widehat{v}_k)$ with $(\overline{u}_k, \overline{v}_k) \in X$ and $(\widehat{u}_k, \widehat{v}_k) \in Y$. It follows from the definition of the operator F_R that $\|F_{R_k}(u_k, v_k) - F_{R_0}(u_k, v_k)\|_m \to 0$ and since $(L_0 + F_{R_0})(\overline{u}_k, \overline{v}_k) = 0$ we conclude from (6.8) that

$$\|(L_0 + F_{R_0})(\widehat{u}_k, \widehat{v}_k)\|_m \to 0. \tag{6.9}$$

The injectivity of $(L_0 + F_{r_0})|Y \to F_m$ implies $(\widehat{u}_k, \widehat{v}_k) \to 0$ in E_m. Since the kernel X of $L_0 + F_{r_0}$ is finite dimensional and the sequence $(\overline{u}_k, \overline{v}_k)$ bounded, we may assume without loss of generality that $(\overline{u}_k, \overline{v}_k)$ converges in E_m to $(u, v) \in X$. Consequently,

$$L_0(u, v) + F_{R_0}(u, v) = 0.$$

This together with $(u, v)(0) = 0$ imply, in view of the uniqueness of initial value problem for differential equations, that $(u, v) = 0$. Hence we have proved $(u_k, v_k) \to 0$ in E_m contradicting $\|(u, v)\|_m = 1$.

We next turn to the case $r_0 = 0$. Again we shall reach a contradiction by proving $(u_k, v_k) \to 0$. We first show that we may assume $u_k \equiv 0$ and $v_k \equiv 0$ on the intervals $[0, 1]$ and $[-1, 0]$ respectively. Indeed, the sequence u_k considered over the interval $[0, 2]$ is bounded in $H^{m+2}([0, 2])$. Thus, in view of Rellich's compact embedding theorem, we may assume without loss of generality that u_k converges to u in $H^{m+1}([0, 2])$. In particular, $\|u_k - u\|_{H^1([0,2])} \to 0$. From (6.8) we conclude that $\|\dot{u}_k + Au_k\|_{H^0([0,2])} \to 0$ so that $\dot{u} + Au = 0$ on $[0, 2]$. This together with $u(0) = 0$ imply $u = 0$. Thus, $\|u_k\|_{H^{m+1}([0,2])} \to 0$, hence

$$\|\dot{u}_k\|_{H^{m+1}([0,2])} \le \|\dot{u}_k + Au_k\|_{H^{m+1}([0,2])} + \|Au_k\|_{H^{m+1}([0,2])} \to 0,$$

showing $\|u_k\|_{H^{m+2}([0,2])} \to 0$. By the same argument, $\|v_k\|_{H^{2+m}([-2,0])} \to 0$. Set $\lambda(s) = \beta(s - 2)$ where β is the cut-off function introduced in (2.5). Then, $\|\lambda u_k\|_{H^{m+2}([0,2])} \to 0$ so that $\|(1-\lambda)u_k, v_k)\|_m \ge \frac{3}{4}$ for large k. Because the support of $F^+_{R_k}(u_k, v_k)$ is contained in $[\frac{R_k}{2} - 1, \frac{R_k}{2} + 1] \times [-\frac{R_k}{2} - 1, -\frac{R_k}{2} + 1]$ we have $F^+_{R_k}((1 - \lambda)u_k, v_k) = F^+_{R_k}(u_k, v_k)$ so that

$$\begin{aligned}
\|\tfrac{d}{ds}((1 - \lambda)u_k) + &A(1 - \lambda)u_k + F^+_{R_k}((1 - \lambda)u_k, v_k)\|_m \\
&= \|(1 - \lambda)[\dot{u}_k + Au_k + F^+_{R_k}(u_k, v_k)] - \dot{\lambda}u_k\|_m \\
&\le C \cdot \left[\|L^+_{R_k}(u_k, v_k)\|_m + \|\dot{\lambda}u_k\|_{H^{m+2}([0,2])}\right] \to 0.
\end{aligned}$$

Similar reasoning applies for the sequence v_k so that indeed we may assume that (u_k, v_k) satisfies $\|(u_k, v_k)\|_m \ge \frac{1}{2}$ and that the supports of u_k and v_k are contained in $[1, \infty)$ and $(-\infty, -1]$.

To proceed further we need some preparation. For the rest of the proof we only consider pairs $(u, v) \in E$ of functions for which there exists $s_0 > 0$ such that $u(s) = 0$ if $s \in [0, s_0]$ and $v(s) = 0$ if $s \in [-s_0, 0]$. Setting $u(s) = 0$ for $s \le s_0$ and $v(s) = 0$ for $s \ge -s_0$ we consider these functions as defined on all of $\mathbb{R}$. Hence also the glued and anti-glued functions are defined on all of $\mathbb{R}$. We shall abbreviate them by

$$(q, p) = (\oplus_r(u, v), \ominus_r(u, v)) \tag{6.10}$$

and introduce the shifted functions

$$(q_R(s), p_R(s)) = (q(s+\tfrac{R}{2}), p(s+\tfrac{R}{2})) \tag{6.11}$$

defined for all $s \in \mathbb{R}$. Equation (6.11) becomes

$$\begin{bmatrix} q_R \\ p_R \end{bmatrix} = \begin{bmatrix} \beta & 1-\beta \\ \beta-1 & \beta \end{bmatrix} \cdot \begin{bmatrix} u(s+\frac{R}{2}) \\ v(s-\frac{R}{2}) \end{bmatrix} \tag{6.12}$$

where q_R, p_R and β are evaluated at s. For convenience we introduce the following abbreviations for the components of the map $L_r : E \to F$ defined in (6.7).

$$\begin{aligned} L_R^+(u,v) &= \dot{u} + Au + F_R^+(u,v) \\ L_R^-(u,v) &= \dot{v} + Au + F_R^-(u,v). \end{aligned} \tag{6.13}$$

By an easy calculations one verifies the formula

$$\begin{bmatrix} \dot{q}_R + Aq_R \\ \dot{p}_R + Ap_R \end{bmatrix} = \begin{bmatrix} \beta & 1-\beta \\ \beta-1 & \beta \end{bmatrix} \cdot \begin{bmatrix} L_R^+(u,v)(s+\frac{R}{2}) \\ L_R^-(u,v)(s-\frac{R}{2}) \end{bmatrix} \tag{6.14}$$

where q_R, p_R and β are evaluated at s. In the following we abbreviate $\delta = \delta_m$ and write $\|w\|_I$ for the L^2-norm of a function w over the interval $I \subset \mathbb{R}$.

LEMMA 6.19. *There exists a constant C_m depending only on the cut-off function β such that*

$$\begin{aligned} \|\dot{q}_R + Aq_R\|^2_{H^{m+2,-\delta}(\mathbb{R})} &\leq Ce^{-\delta R}\big[\ \|L_R^+(u,v)\|_m^2 + \|L_R^-(u,v)\|_m^2\ \big] \\ \|\dot{p}_R + Ap_R\|^2_{H^{m+2,\delta}(\mathbb{R})} &\leq Ce^{-\delta R}\big[\ \|L_R^+(u,v)\|_m^2 + \|L_R^-(u,v)\|_m^2\ \big]. \end{aligned}$$

PROOF. We only consider the first estimate since the proof of the second is similar. To simplify the notation we drop the functions u and v together the index R in $L_R^+(u,v)(s)$ and $L_R^-(u,v)(s)$, writing $L^+(s)$ and $L^-(s)$ instead. Using (6.14) we have the following pointwise inequality for $0 \leq j \leq m+2$,

$$\begin{aligned} &|D^j(\dot{q}_R + Aq_R)|^2 \\ &\qquad \leq C \cdot [\ |D^j(\beta \cdot L^+(s+\tfrac{R}{2}))|^2 + |D^j((1-\beta)\cdot L^-(s-\tfrac{R}{2}))|^2\ ]. \end{aligned}$$

In view of the properties of β, and since $L^+(s+\frac{R}{2}) = 0$ if $s \leq -\frac{R}{2}$ and $L^-(s-\frac{R}{2}) = 0$ if $s \geq \frac{R}{2}$, the first summand on the right hand side above is supported in the interval $[-\frac{R}{2}, 1]$ and the second in the interval

$[-1, \frac{R}{2}]$. After multiplying both sides above by $e^{-2\delta|s|}$ and integrating over $\mathbb{R}$ we find

$$
\begin{aligned}
&\|[D^j(\dot{q}_R + Aq_R)]e^{-\delta|s|}\|^2_{\mathbb{R}} \\
&\qquad \leq C \cdot \Big[\ \|[D^j L^+(s+\tfrac{R}{2})]e^{\delta s}\|^2_{[-\frac{R}{2},0]} + \|[D^j L^+(s+\tfrac{R}{2})]e^{-\delta s}\|^2_{[0,1]} \\
&\qquad\qquad + \|[D^j L^-(s-\tfrac{R}{2})]e^{\delta s}\|^2_{[-1,0]} + \|[D^j L^-(s-\tfrac{R}{2})]e^{-\delta s}\|^2_{[0,\frac{R}{2}]} \ \Big].
\end{aligned}
$$

We estimate each of the terms on the right hand side.

$$
\begin{aligned}
\|[D^j L^+(s+\tfrac{R}{2})]e^{\delta s}\|_{[-\frac{R}{2},0]} &= e^{-\delta\frac{R}{2}}\|[D^j L^+(s+\tfrac{R}{2})]e^{\delta(s+\frac{R}{2})}\|_{[-\frac{R}{2},0]} \\
&\leq e^{-\delta\frac{R}{2}}\|[D^j L^+(s)]e^{\delta s}\|_{R_+},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|[D^j L^+(s+\tfrac{R}{2})]e^{-\delta s}\|_{[0,1]} = e^{-\delta\frac{R}{2}} \cdot \|[D^j L^+(s+\tfrac{R}{2})]e^{\delta(s+\frac{R}{2})}e^{-2\delta s}\|_{[0,1]} \\
&\leq e^{-\delta\frac{R}{2}} \cdot \|L^+(s+\tfrac{R}{2})e^{\delta(s+\frac{R}{2})}\|_{[0,1]} \leq e^{-\delta\frac{R}{2}} \cdot \|L^+(s)e^{\delta s}\|_{\mathbb{R}_+}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\|[D^j L^-(s-\tfrac{R}{2})]e^{\delta s}\|_{[-1,0]} &\leq e^{-\delta\frac{R}{2}}\|[D^j L^-(s)]e^{-\delta s}\|_{R_-} \\
\|[D^j L^-(s-\tfrac{R}{2})]e^{-\delta s}\|_{[0,\frac{R}{2}]} &\leq e^{-\delta\frac{R}{2}} \cdot \|[D^j L^-(s)]e^{-\delta s}\|_{\mathbb{R}_-}.
\end{aligned}
$$

Summing from $j=0$ to $j=m+1$ we obtain

$$
\|\dot{q}_R + Aq_R\|^2_{H^{m+2,-\delta}(\mathbb{R})} \leq Ce^{-\delta R}\big[\ \|L^+\|^2_m + \|L^-\|^2_m \ \big].
$$

as claimed. The proof of the lemma is complete. ∎

Multiplying the matrix equation (6.12) by the inverse matrix one obtains the formula

$$
\begin{bmatrix} u(s+\frac{R}{2}) \\ v(s-\frac{R}{2}) \end{bmatrix} = \frac{1}{\alpha} \cdot \begin{bmatrix} \beta & \beta-1 \\ 1-\beta & \beta \end{bmatrix} \cdot \begin{bmatrix} q_R \\ p_R \end{bmatrix} \tag{6.15}
$$

where β, q_R and p_R are evaluated at s and where $\alpha = \beta^2 + (1-\beta)^2$. Proceeding as in the proof of Lemma 6.19, using (6.15) and the properties of β as well as the facts that $q_R(s) = 0$ for $s \leq -\frac{R}{2}$ and $p_R(s) = 0$ for $s \geq \frac{R}{2}$, one proves the following lemma.

Lemma 6.20. *There exists a constant $C_m > 0$ depending only on the cut-off function β such that*

$$
\begin{aligned}
\|u\|^2_m &\leq Ce^{\delta R}\big[\|q_R\|^2_{H^{m+2,-\delta}(\mathbb{R})} + \|p_R\|^2_{H^{m+2,\delta}(\mathbb{R})}\big] \\
\|v\|^2_m &\leq Ce^{\delta R}\big[\|q_R\|^2_{H^{m+2,-\delta}(\mathbb{R})} + \|p_R\|^2_{H^{m+2,\delta}(\mathbb{R})}\big].
\end{aligned}
$$

Having Lemma 6.19 and Lemma 6.20 at hand we can finish the proof of Theorem 6.18. We set

$$q_{R_k}(s) = \oplus_{R_k}(u_k, v_k)(s + \tfrac{R_k}{2}) \quad \text{and} \quad p_{R_k}(s) = \ominus_{R_k}(u_k, v_k)(s + \tfrac{R_k}{2}).$$

Recalling the operators $\overline{L} : \overline{E}_m \to \overline{F}_m$ and $\widehat{L} : \widehat{E}_m \to \widehat{F}_m$ from Lemma 6.17, we conclude from Lemma 6.19

$$\|\overline{L}(e^{\delta \frac{R_k}{2}} q_{R_k})\|_{H^{m+2,-\delta}(\mathbb{R})} \to 0 \quad \text{and} \quad \|\widehat{L}(e^{\delta \frac{R_k}{2}} p_{R_k})\|_{H^{m+2,\delta}(\mathbb{R})} \to 0.$$

as $k \to \infty$. The operators $\overline{L}$ and $\widehat{L}$ are isomorphisms, hence

$$\|e^{\delta \frac{R_k}{2}} q_{R_k}\|_{H^{m+2,-\delta}(\mathbb{R})} \to 0 \quad \text{and} \quad \|e^{\delta \frac{R_k}{2}} p_{R_k}\|_{H^{m+2,\delta}(\mathbb{R})} \to 0.$$

Applying Lemma 6.20,

$$\|u_k\|_m \to 0 \quad \text{and} \quad \|v_k\|_m \to 0.$$

Recalling $\delta = \delta_m$ this means that $(u_k, v_k) \to 0$ in E_m in contradiction to $\|(u_k, v_k)\|_m \geq \frac{1}{2}$. This finishes the proof of Theorem 6.18.

■

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